Efficient Algorithms for Frequently Asked Questions

4. Hypertree Decompositions

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DaST

Data • (Systems+Theory)

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FAQ Computation Time Depends on the Structure of the Hypergraph

Consider the following FAQ expression over the Boolean semiring:

$$\Phi() = \bigvee_{(x_1,\ldots,x_5)\in\prod_{i\in[5]}\mathsf{Dom}(X_i)} \psi_{12}(x_1,x_2) \wedge \psi_{23}(x_2,x_3) \wedge \psi_{34}(x_3,x_4) \wedge \psi_{15}(x_1,x_5)$$

Φ asks whether there is a tuple $(x_1,\ldots,x_5)$ such that $\psi_{ij}(x_i,x_j) = \text{true}$

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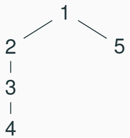
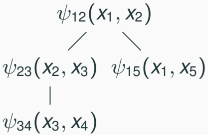


Hypergraph of Φ, all edges are binary



Possible bottom-up evaluation strategy

Evaluation strategy known for decades under different names:

- Message passing (in AI literature; Pearl'83)
- Semi-join reduction (in DB literature; Yannakakis'82; discussed in course)

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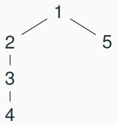

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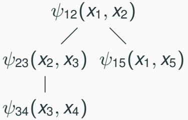

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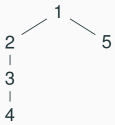

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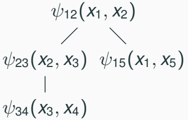

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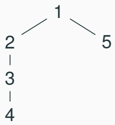

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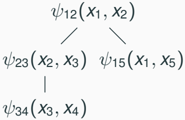

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$$V_{23\to 12}(x_2) = \bigvee_{x_3} \psi_{23}(x_2, x_3) \wedge V_{34\to 23}(x_3)$$

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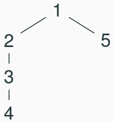

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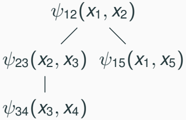

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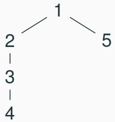

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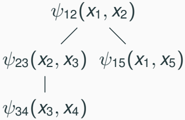

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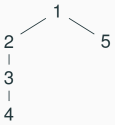
$$\Phi() = \bigvee_{x_1, x_2} \psi_{12}(x_1, x_2) \wedge V_{15\to 12}(x_1) \wedge V_{23\to 12}(x_2)$$

Computation Time



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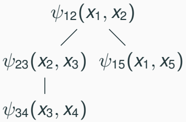


Possible bottom-up evaluation strategy

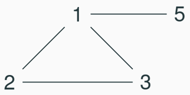
All computation steps are local and their cost upper bounded by the factor sizes

- Typical assumption: $|\psi_{ij}| \leq N$ for some value N
- We pass along at most N values between factors
- Local computation is just filtering local values with incoming values
- Overall: linear computation time – This is the best in worst case

FAQ Computation for a Different Hypergraph

Now, consider a slightly different FAQ Φ': Same as Φ but $X_4 = X_1$

$$\Phi'() = \bigvee_{x_1, x_2, x_3, x_5} \psi_{12}(x_1, x_2) \wedge \psi_{23}(x_2, x_3) \wedge \psi_{13}(x_1, x_3) \wedge \psi_{15}(x_1, x_5)$$



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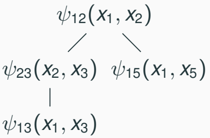

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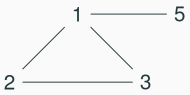

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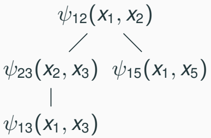

Possible bottom-up evaluation strategy

Computation not anymore local!

- x_1 needs to be propagated from ψ_{13} through ψ_{23} to ψ_{12}
- ψ_{23} does not have x_1, so it receives it and forwards it further
- This incurs the cost of carrying x_1 values along two computation steps

$\Rightarrow O(N^2)$ complexity (we will later learn how to do it in $O(N^{1.5})$)

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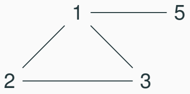

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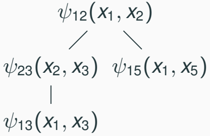

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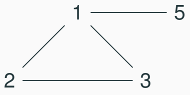

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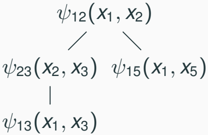

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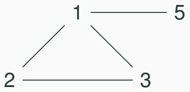

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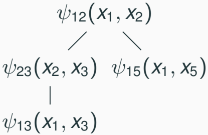

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Cost: $O(N^2)$

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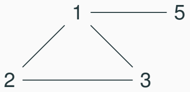

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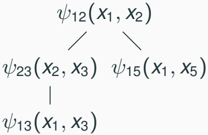

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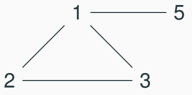

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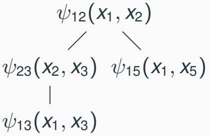

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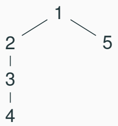
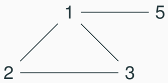


Acyclic Hypergraph



Cyclic Hypergraph

- Left: Only push up information of size $< N$ that is local at factor
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- The difference is reflected in the computational complexity: $O(N)$ vs $O(N^2)$

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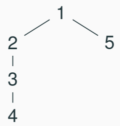

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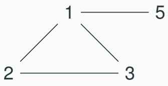

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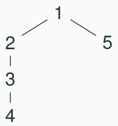

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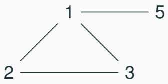

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2. Can we "transform" cyclic hypergraphs into acyclic ones?

Answer to Question 1: Acyclic Hypergraphs (Overview)

Several acyclicity notions exist. Studied in this course: α-acyclic & β-acyclic

FAQs without free variables can be computed in:

- Linear time in the size of input factors if its hypergraph is α-acyclic

  Assumption: Each factor ψ_S represented as list of tuples $\mathbf{x}_S$ with $\psi_S(\mathbf{x}_S) \neq \mathbf{0}$

- Linear time in the size of input factors if its hypergraph is β-acyclic

  Assumption: Each factor represented compactly as box, e.g., for (#)SAT

FAQs with free variables:

- *In principle* as above, BUT hypergraph is also free-connex
- Linear time for precomputation
- Then output the answer in constant time per tuple (enumeration delay)
- $\Rightarrow$ Linear time in input size plus output size

Answer to Question 2: Hypertree Decompositions (Overview)

Hypertree decompositions

- Transform an arbitrary hypergraph into a hypertree
- Measure of how close the hypergraph is to a hypertree: width
- Complexity of transformation is $\mathcal{O}(N^w)$, where
  - N is the maximal size of an input factor
  - w is the width of the hypergraph
- Once we have a hypertree $\longrightarrow$ see answer to Question 1

Hypertree Decompositions

Hypertree Decompositions: Definition

A hypertree decomposition $\mathcal{T}$ of a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is a pair (T, χ) with:

- T is a tree
- χ is a function mapping each node in T to a subset of $\mathcal{V}$ called *bag*

Properties of a decomposition $\mathcal{T} = (T, \chi)$:

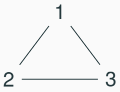
- *Coverage*: $\forall e \in \mathcal{E}$, there is a node $t \in T$ such that $e \subseteq \chi(t)$
- *Connectivity*: $\forall v \in \mathcal{V} : \{t \mid t \in T, v \in \chi(t)\}$ forms a connected subtree in $\mathcal{T}$

Observation: Each node $t \in T$ of the hypertree decomposition $\mathcal{T}$ represents the sub-hypergraph $\mathcal{H}'$ of $\mathcal{H}$ induced by the nodes $\chi(t)$ of $\mathcal{H}$

- The nodes of $\mathcal{H}'$ are $\chi(t)$
- The hyperedges of $\mathcal{H}'$ are $\mathcal{H}$'s hyperedges restricted to the nodes $\chi(t)$ of $\mathcal{H}$

Hypertree Decompositions for the Triangle Hypergraph

Triangle query: $\Phi(x_1, x_2, x_3) = \psi_{12}(x_1, x_2) \otimes \psi_{23}(x_2, x_3) \otimes \psi_{13}(x_1, x_3)$



Task: Construct a hypertree decomposition
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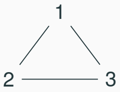
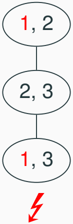


Task: Construct a hypertree decomposition with one bag per edge

First trial



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Connectivity violated!

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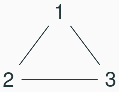

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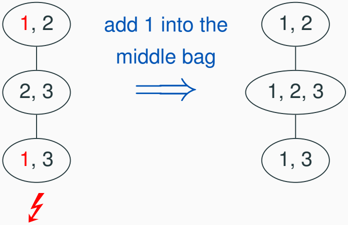

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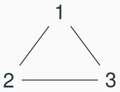
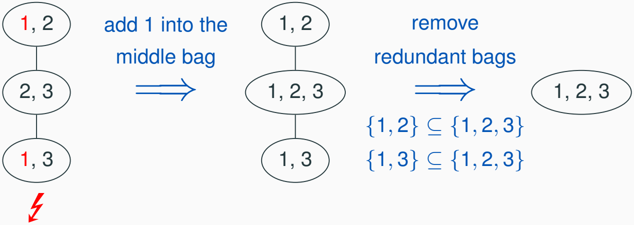
Hypertree Decompositions for the Triangle Hypergraph

Triangle query: $\Phi(x_1, x_2, x_3) = \psi_{12}(x_1, x_2) \otimes \psi_{23}(x_2, x_3) \otimes \psi_{13}(x_1, x_3)$



Task: Construct a hypertree decomposition
with one bag per edge

First trial



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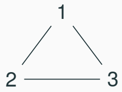

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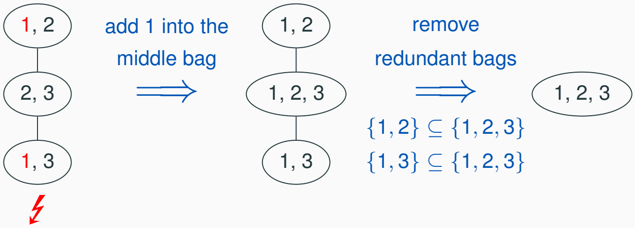


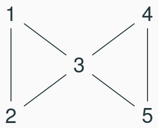
1 not included in the middle bag

Connectivity violated!

There are five other possibilities. All violate the connectivity condition.

The only hypertree decomposition without redundant bags has only one bag

Hypertree Decompositions for the Bowtie Hypergraph



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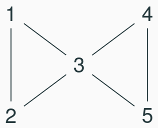
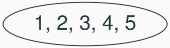


Possible hypertree decompositions



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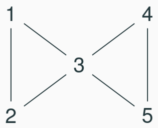

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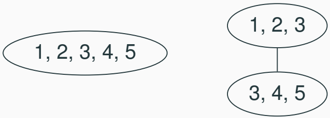

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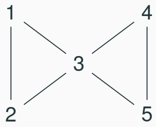
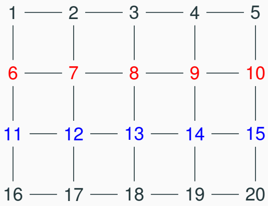


Possible hypertree decompositions

The bag $\{4, 5\}$ is redundant since it is included in bag $\{3, 4, 5\}$.

Redundant bags need not be considered as they add no extra information.

Hypertree Decompositions for the Grid Hypergraph



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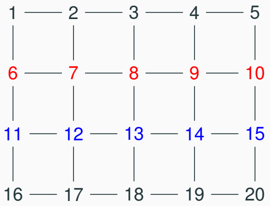

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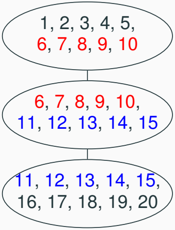
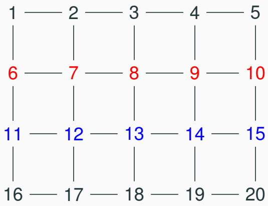
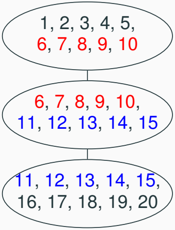
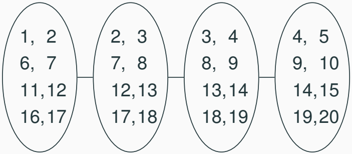


Hypertree Decompositions for the Grid Hypergraph



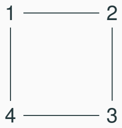
Possible hypertree decompositions





Hypertree Decompositions for 4-Cycle and 6-Cycle Hypergraphs

4-Cycle



Hypertree decompositions

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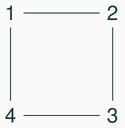
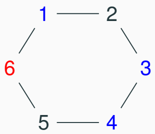
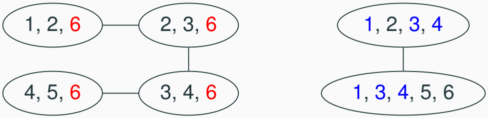


Hypertree decompositions

6-Cycle



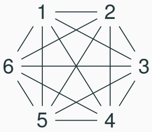


Hypertree Decompositions for Clique and Loomis-Whitney Hypergraphs

Clique of degree n :

Hypergraph $\left([n], \binom{[n]}{2}\right)$

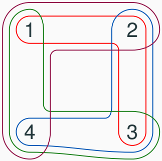
Clique-6



Loomis-Whitney of degree n:

Hypergraph $\left([n], \binom{[n]}{n-1}\right)$

Loomis-Whitney-4



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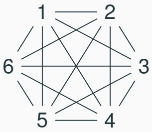

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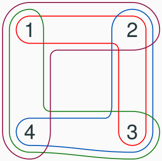


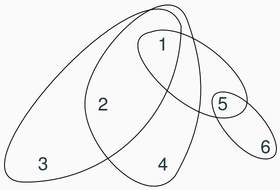
Both hypergraphs only admit the trivial decomposition with one bag

Join Trees: Hypertree Decompositions with One Bag per Hyperedge

α-acyclic hypergraphs admit hypertree decompositions with one bag per hyperdge

- Best decompositions, as no merging of factors in a bag is necessary
- Such decompositions are called Join Trees
- Hypergraphs are α-acyclic precisely when they admit join trees

Hypergraph



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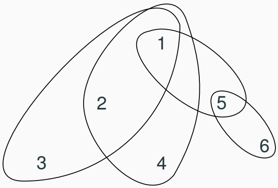
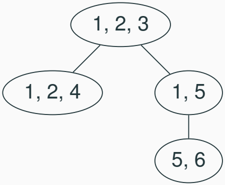


Possible hypertree decomposition



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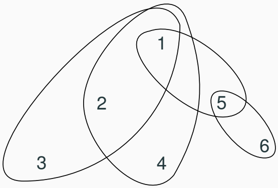

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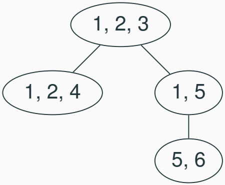


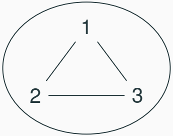
Question: Are α-acyclic precisely those hypergraphs without cycles?

α-acyclic Hypergraphs Can Have Cycles *IF* Covered by Hyperedges

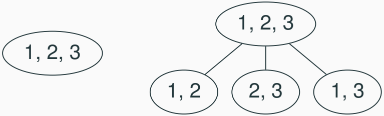
Consider the FAQ:

$$\Phi() = \bigoplus_{x_1, x_2, x_3} \psi_{123}(x_1, x_2, x_3) \otimes \psi_{12}(x_1, x_2) \otimes \psi_{13}(x_1, x_3) \otimes \psi_{23}(x_2, x_3)$$

Hypergraph



Hypertree decompositions



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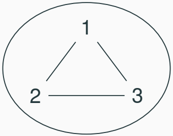

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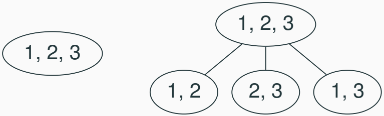


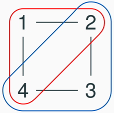
- Cycle formed by the factors ψ_{12}, ψ_{13}, and ψ_{23}
- BUT covered by the factor ψ_{123}
- We can evaluate Φ efficiently by absorbing each other factor into ψ_{123}
  $\psi_{123}(x_1, x_2, x_3) := \psi_{123}(x_1, x_2, x_3) \otimes \psi_{ij}(x_i, x_j)$, $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$

Non-trivial α-Acyclicity Example

$$\Phi() = \bigoplus_{x_1,x_2,x_3,x_4} \psi_{124}(x_1, x_2, x_4) \otimes \psi_{234}(x_2, x_3, x_4) \otimes$$
$$\psi_{12}(x_1, x_2) \otimes \psi_{23}(x_2, x_3) \otimes \psi_{34}(x_3, x_4) \otimes \psi_{14}(x_1, x_4)$$

- Cycle formed by the factors ψ_{12}, ψ_{23}, ψ_{34}, and ψ_{14}
- BUT covered by the factors ψ_{124} and ψ_{234}
- We can evaluate Φ efficiently:
  - Absorb the factors ψ_{12} and ψ_{14} into the factor ψ_{124}
  - Absorb the factors ψ_{23} and ψ_{34} into the factor ψ_{234}
  - Multiply the factors ψ_{124} and ψ_{234} and aggregate away the variables

Hypergraph



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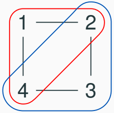
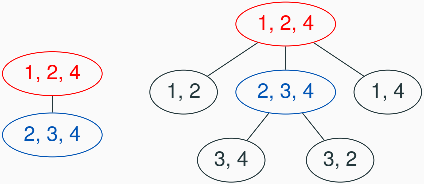
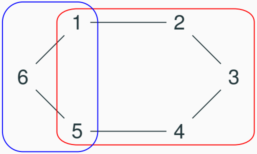


Hypertree decompositions (Second is join tree)



Further α-acyclicity Examples

Hypergraph



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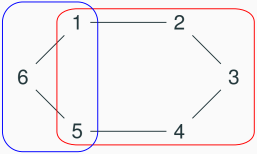
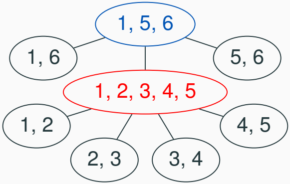
Hypergraph



Hypertree decompositions (Join trees)



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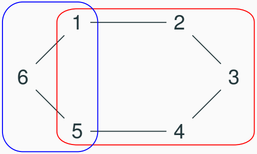

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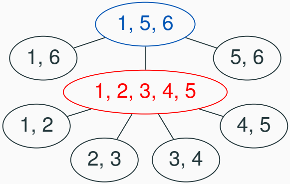

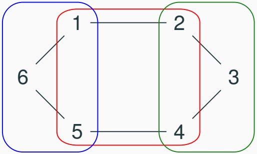

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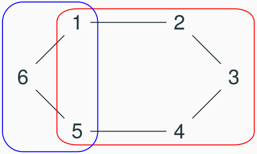

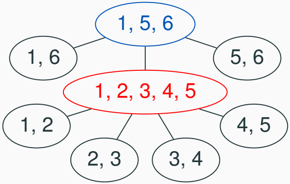

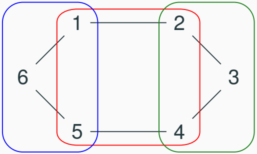

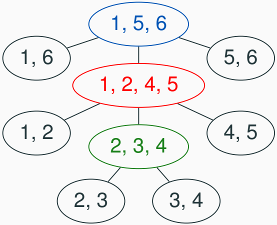

The GYO Algorithm

The GYO (Graham, Yu, Ozsoyoglu) algorithm is used to decide α-acyclicity:

Input: Hypergraph $\mathcal{H}$

Output: Hypergraph obtained by repeating the following rules as long as possible:

- Eliminate a node that is contained in only one hyperedge
- Eliminate a hyperedge that is contained in another hyperedge

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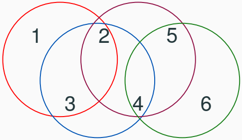
$$\boxed{\mathcal{H} \text{ is } \alpha\text{-acyclic if and only if } \mathrm{GYO}(\mathcal{H}) = (\emptyset, \{\emptyset\})}$$

In words: $\mathcal{H}$ is α-acyclic if and only if the application of GYO to $\mathcal{H}$ returns a hypergraph with no vertices and one empty hyperedge

The GYO Algorithm: Example 1/3

initial hypergraph $\mathcal{H}$



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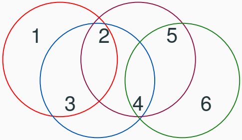
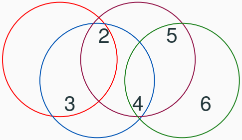


node 1 removed



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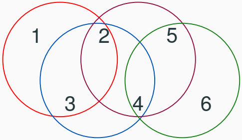

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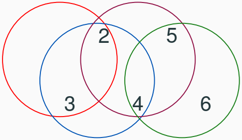
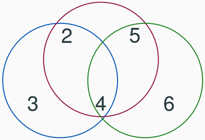


edge $\{2, 3\}$ removed



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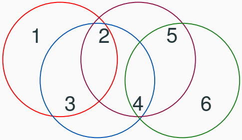

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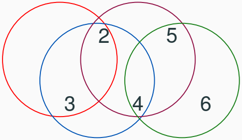

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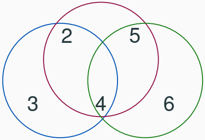
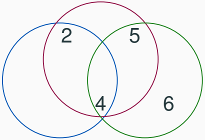
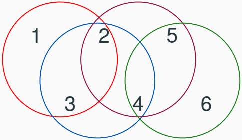


node 3 removed



The GYO Algorithm: Example 1/3

initial hypergraph $\mathcal{H}$



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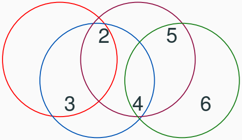

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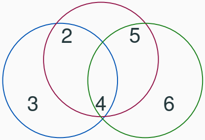

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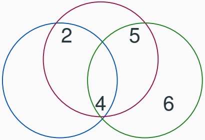

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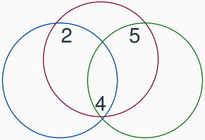
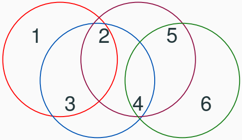


The GYO Algorithm: Example 1/3

initial hypergraph $\mathcal{H}$



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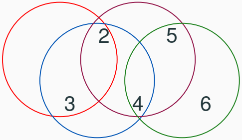

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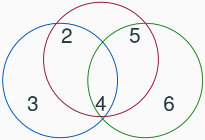

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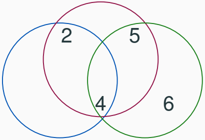

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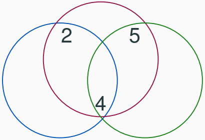

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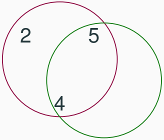
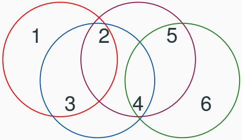


The GYO Algorithm: Example 1/3

initial hypergraph $\mathcal{H}$



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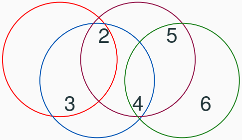

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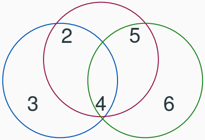

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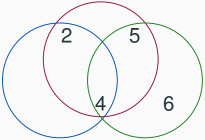

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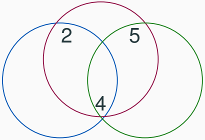

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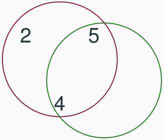

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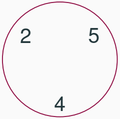
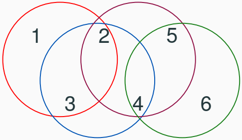
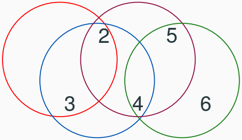


The GYO Algorithm: Example 1/3

initial hypergraph $\mathcal{H}$



node 1 removed



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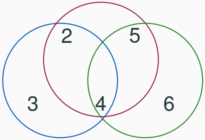

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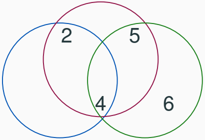

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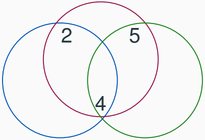

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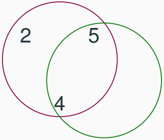

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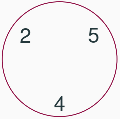

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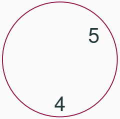
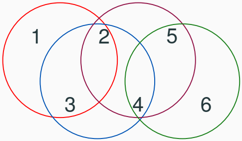


The GYO Algorithm: Example 1/3

initial hypergraph $\mathcal{H}$



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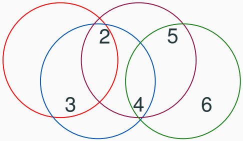

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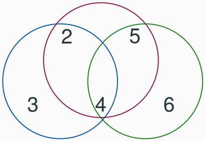

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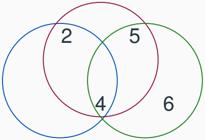

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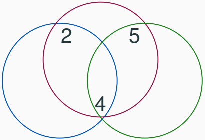

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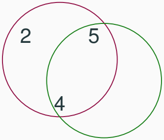

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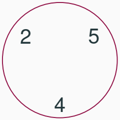

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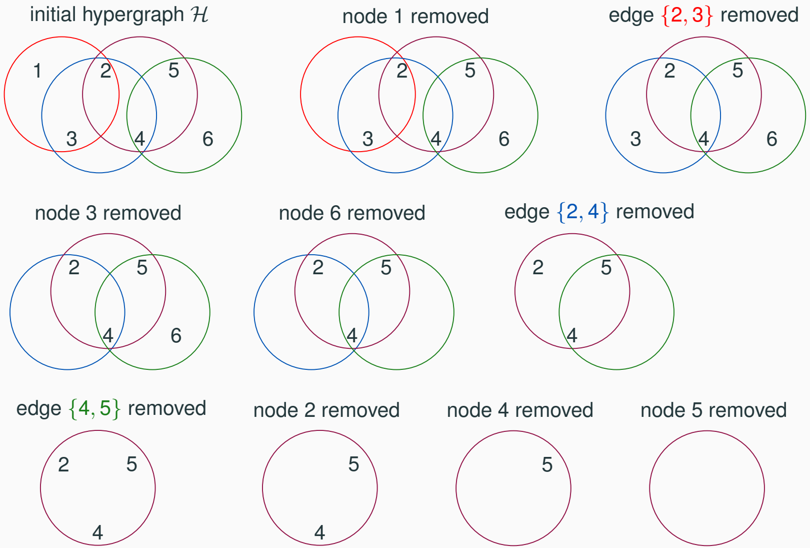
node 4 removed

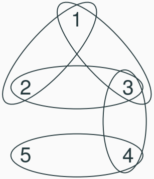
The GYO Algorithm: Example 1/3



$$\Longrightarrow \text{GYO}(\mathcal{H}) = (\emptyset, \{\emptyset\}) \Longrightarrow \mathcal{H} \text{ is } \alpha\text{-acyclic}$$

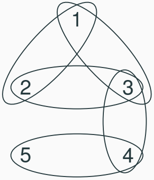
The GYO Algorithm: Example 2/3

initial hypergraph $\mathcal{H}$



The GYO Algorithm: Example 2/3

initial hypergraph $\mathcal{H}$



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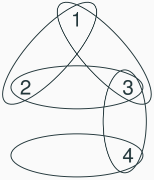
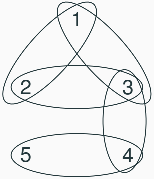


The GYO Algorithm: Example 2/3

initial hypergraph $\mathcal{H}$



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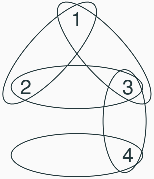

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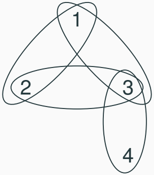
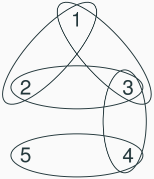


The GYO Algorithm: Example 2/3

initial hypergraph $\mathcal{H}$



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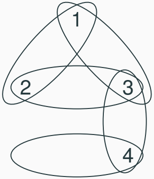

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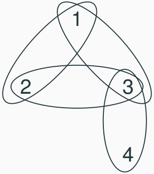

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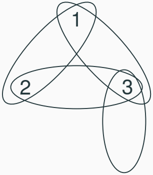
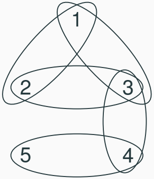


The GYO Algorithm: Example 2/3

initial hypergraph $\mathcal{H}$



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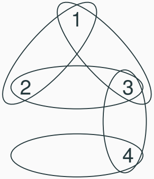

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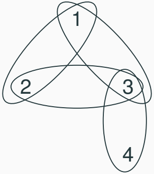

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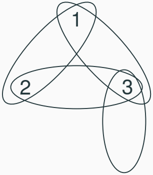

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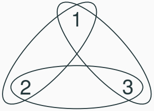
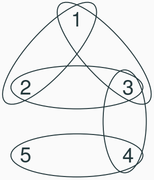


The GYO Algorithm: Example 2/3

initial hypergraph $\mathcal{H}$



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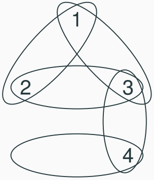

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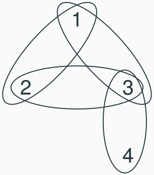

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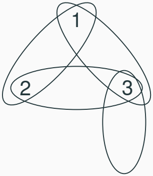

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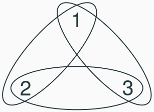
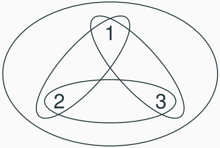


no more rule applicable

$\Longrightarrow$ GYO($\mathcal{H}$) $\neq (\emptyset, \{\emptyset\}) \Longrightarrow \mathcal{H}$ is not α-acyclic

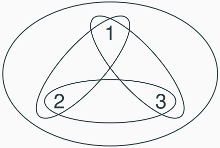
The GYO Algorithm: Example 3/3

initial hypergraph $\mathcal{H}$

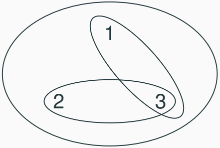


The GYO Algorithm: Example 3/3

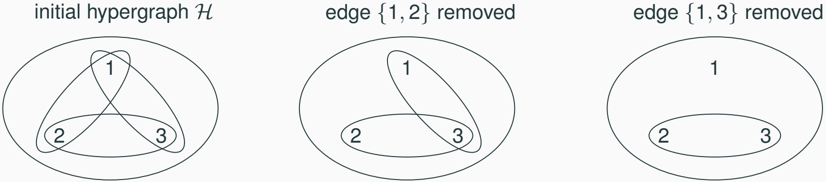
initial hypergraph $\mathcal{H}$



edge $\{1, 2\}$ removed

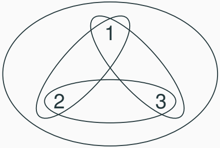


The GYO Algorithm: Example 3/3



The GYO Algorithm: Example 3/3

initial hypergraph $\mathcal{H}$



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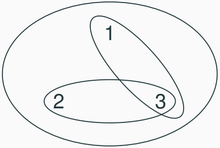
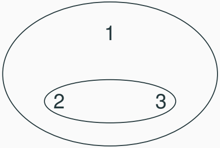
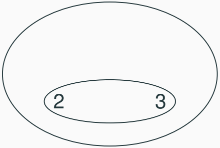


edge $\{1, 3\}$ removed

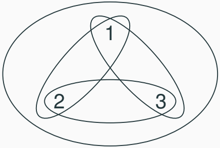


node 1 removed



The GYO Algorithm: Example 3/3

initial hypergraph $\mathcal{H}$



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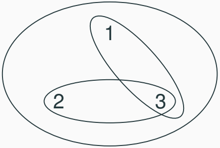

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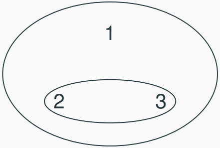

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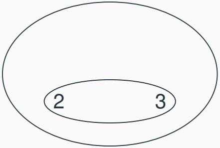
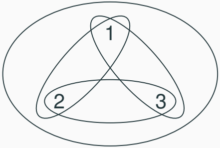


edge $\{2, 3\}$ removed

The GYO Algorithm: Example 3/3

initial hypergraph $\mathcal{H}$



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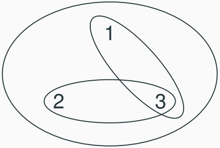
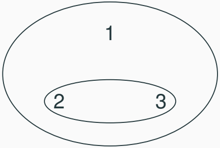
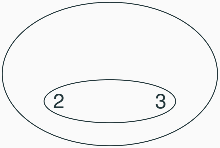


edge $\{1, 3\}$ removed



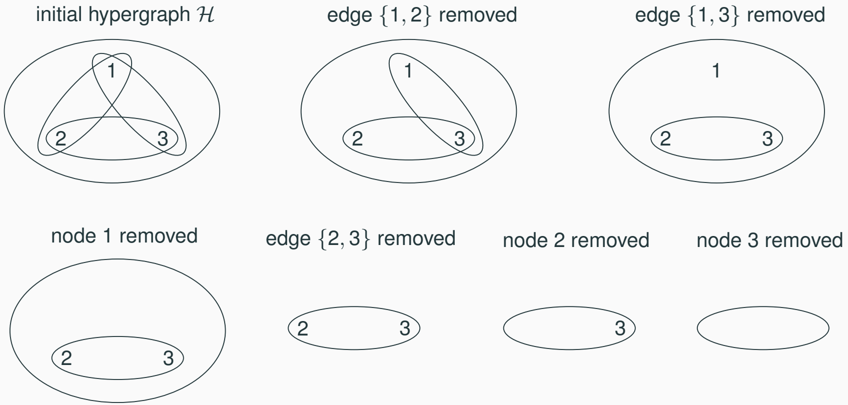
node 1 removed



edge $\{2, 3\}$ removed

node 2 removed

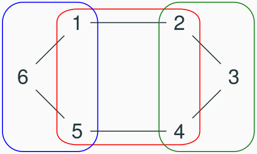
The GYO Algorithm: Example 3/3



$\Longrightarrow$ GYO($\mathcal{H}$) $= (\emptyset, \{\emptyset\}) \Longrightarrow \mathcal{H}$ is α-acyclic

Computing a Join Tree for an α-Acyclic Hypergraph

Hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$



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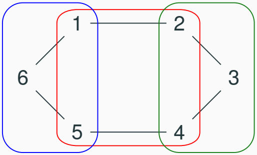

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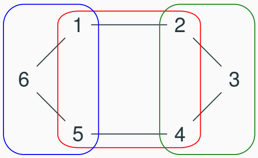

Algorithm

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   - vertex set: $\mathcal{E}$

Weighted graph for $\mathcal{H}$:

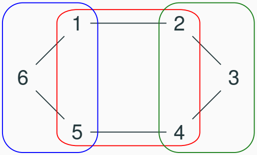
$\{1, 6\}$ $\{1, 2\}$

$\{1, 5, 6\}$ $\{5, 6\}$ $\{1, 2, 4, 5\}$ $\{2, 3\}$ $\{2, 3, 4\}$

$\{4, 5\}$ $\{3, 4\}$

Computing a Join Tree for an α-Acyclic Hypergraph

Hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$



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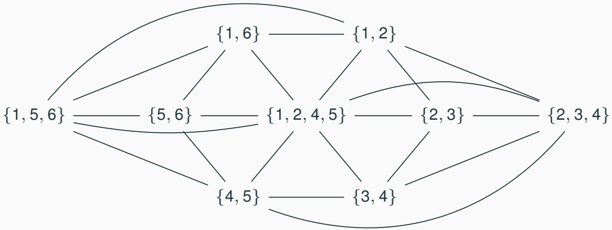

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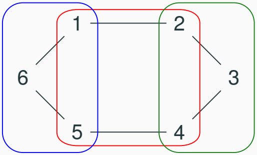

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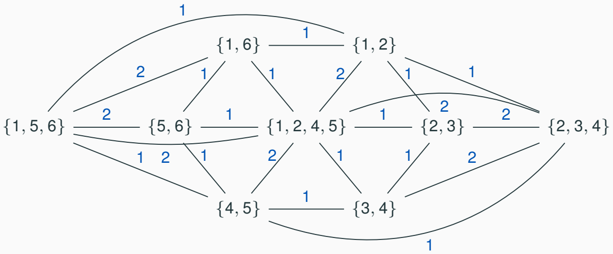

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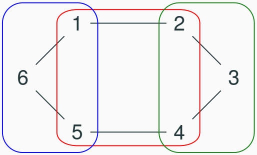

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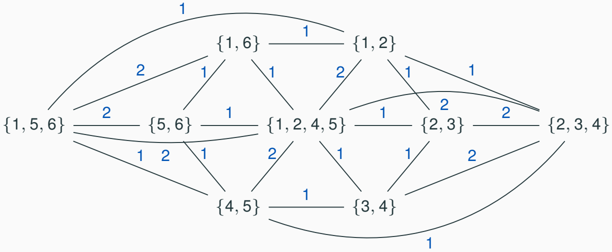

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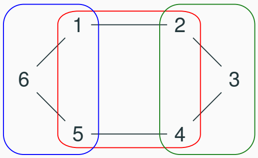

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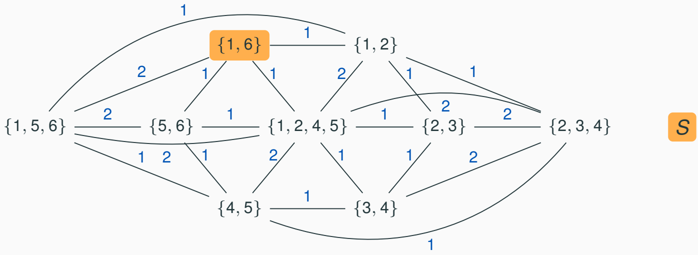

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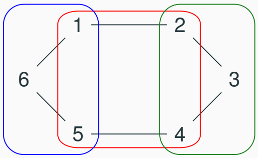

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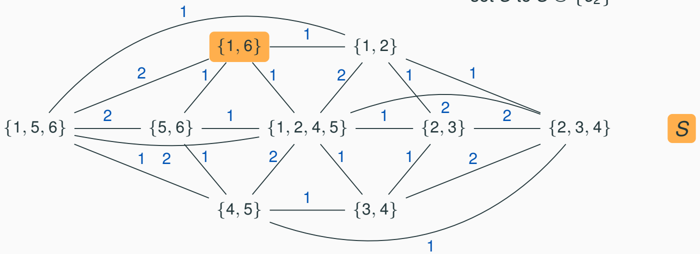

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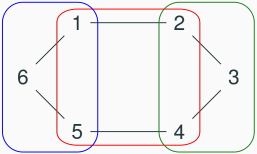

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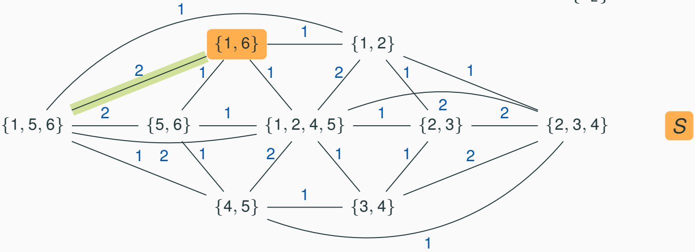

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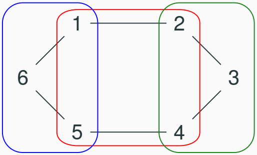

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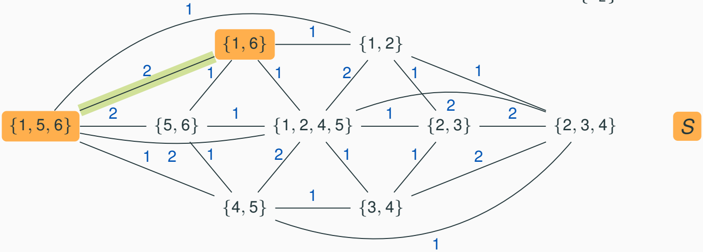

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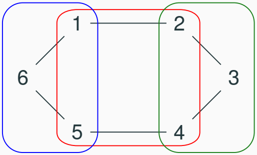

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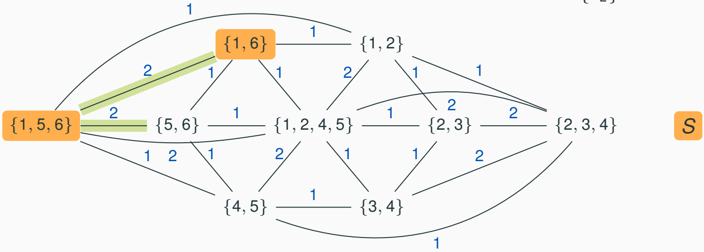

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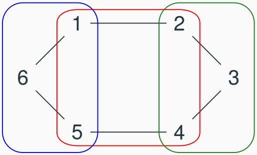

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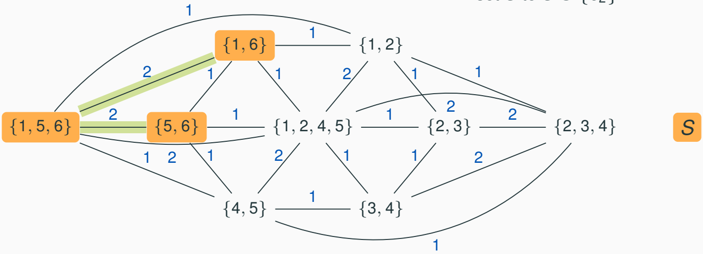

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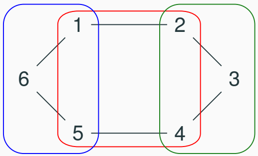

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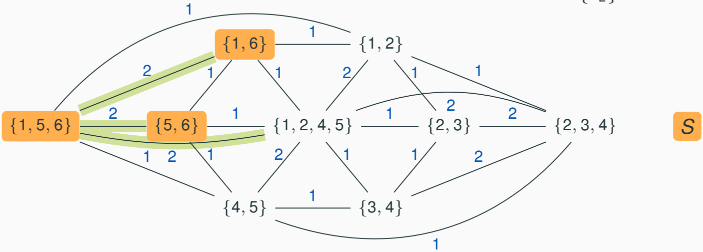

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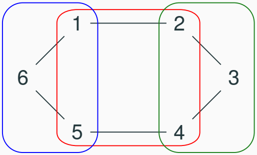

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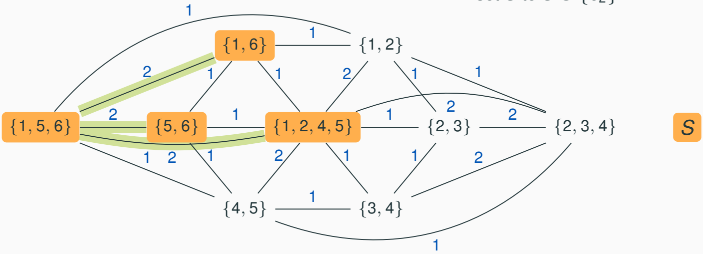

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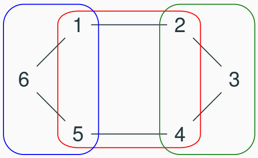

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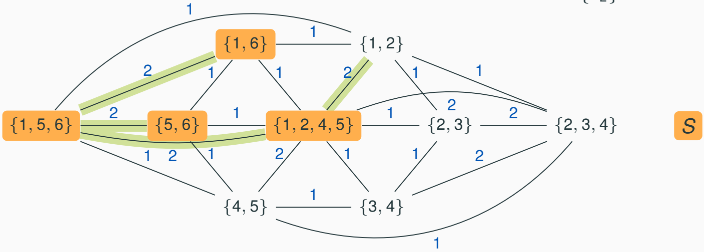

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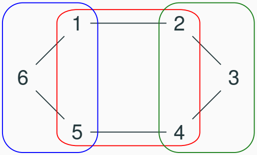

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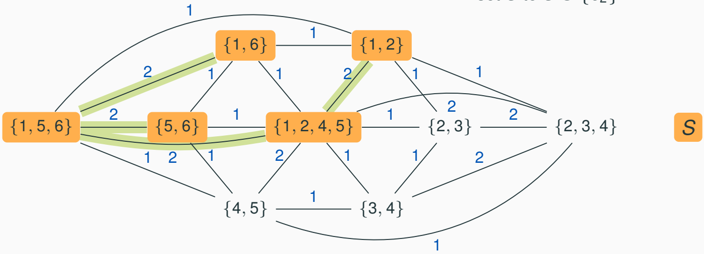

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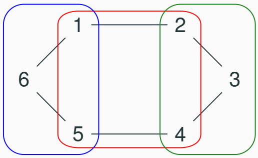

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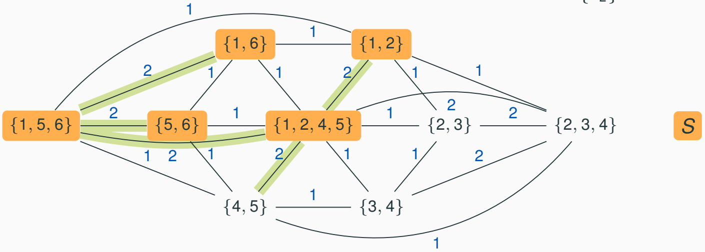

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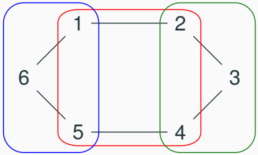

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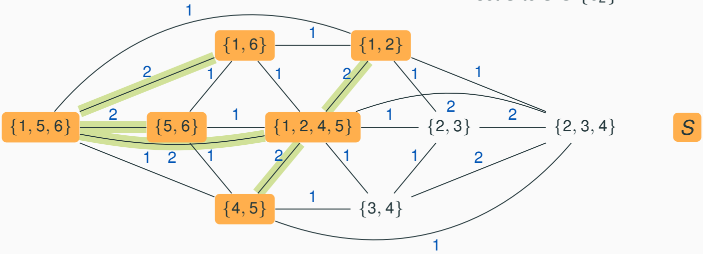

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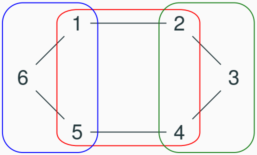

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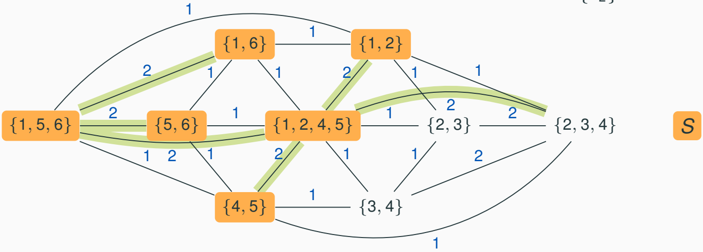
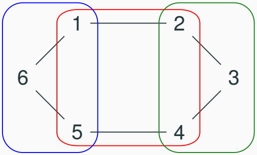


Computing a Join Tree for an α-Acyclic Hypergraph

Hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$



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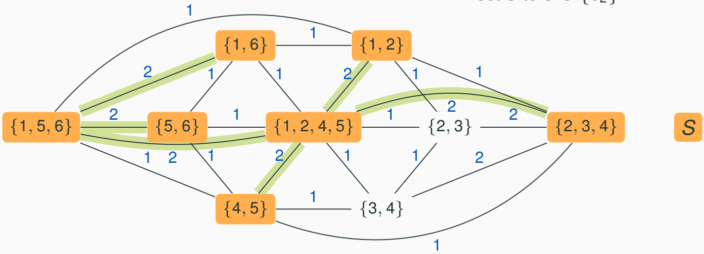

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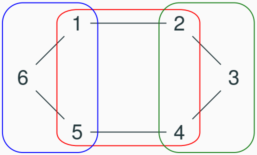

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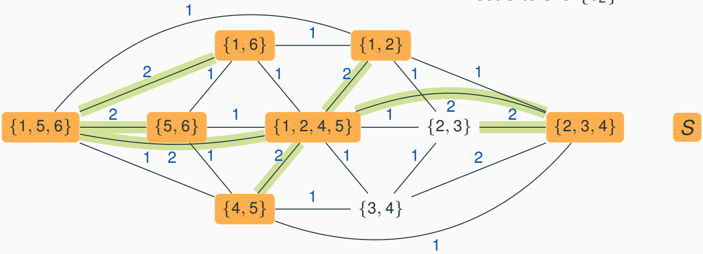

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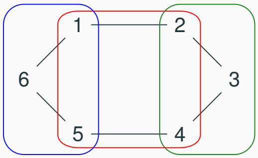

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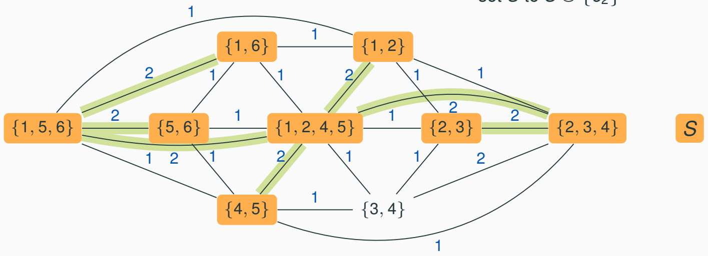

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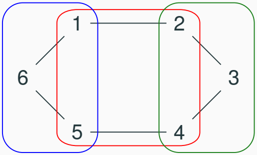

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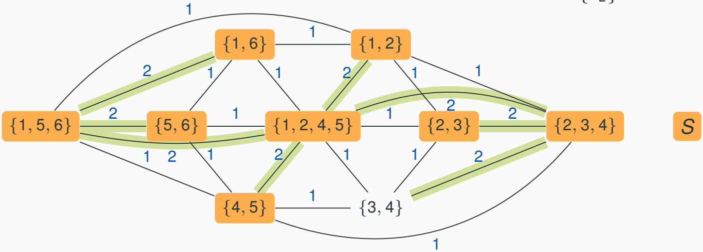

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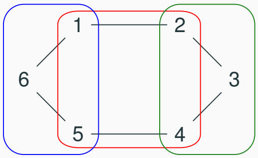

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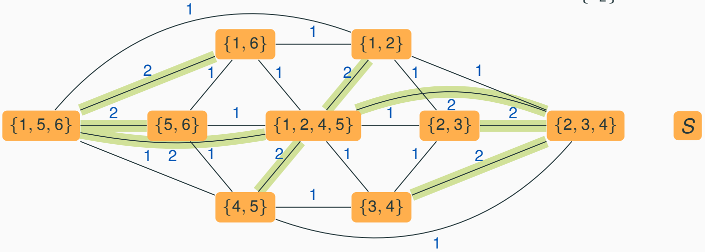

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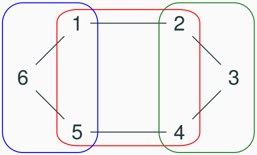

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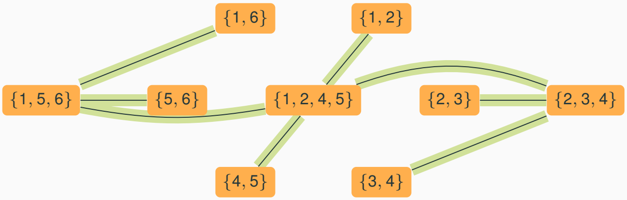

Computing a Join Tree for an α-Acyclic Hypergraph

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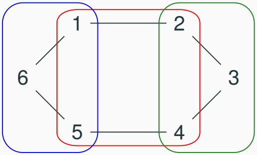
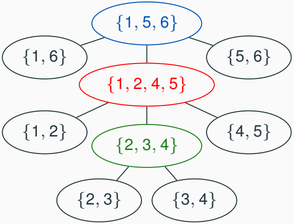


Join tree for $\mathcal{H}$:



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β-Acyclicity: Every Subgraph is Acyclic

$\boxed{\text{$\beta$-acyclicity is closed under hyperedge removal, α-acyclicity is not closed}}$

- α-acyclic hypergraphs may have cycles covered by hyperedges
  - The hypergraph without such hyperedges is not α-acyclic
- β-acyclic hypergraphs cannot have cycles
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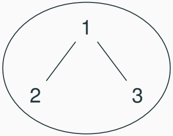

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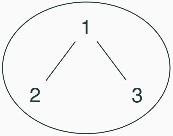
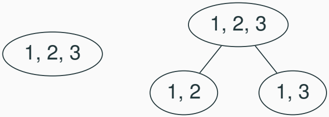


Hypertree decompositions



Free-connex α-Acyclicity

- α-acyclicity is prerequisite to efficient computation
- FAQ compute time also depends on free variables $X_1, \ldots, X_f$

$$\Phi(\mathbf{x}_{[f]}) = \bigoplus_{x_{f+1}}^{(f+1)} \cdots \bigoplus_{x_n}^{(n)} \bigotimes_{S \in \mathcal{E}} \psi_S(\mathbf{x}_S)$$

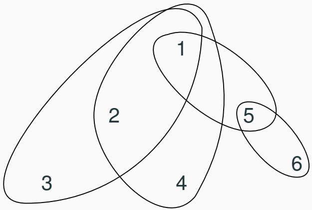
- Free-connex property: There is a join tree for the hypergraph of Φ, where
  - All free variables $[f]$ appear in nodes that form a connected subtree
  - Each bound variable either occurs in nodes without free variables or only in one node with free variables
  - $\to$ All bound variables can be aggregated away in linear time

When is an FAQ with hypergraph $\mathcal{H}$ and free variables $[f]$ free-connex α-acyclic?

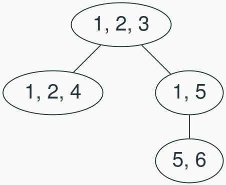
$\mathcal{H}$ remains α-acyclic even after adding the hyperedge $[f]$ over the free variables

Free-connex α-Acyclicity Example 1/2

α-acyclic hypergraph $\mathcal{H}$



Possible join tree for $\mathcal{H}$



Free-connex α-Acyclicity Example 1/2

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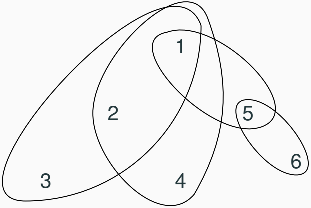

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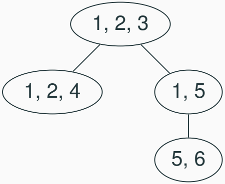
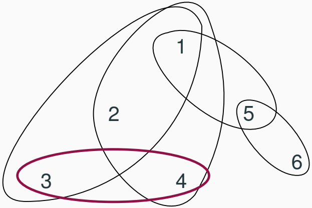


choose $\{3, 4\}$
as free variables



Free-connex α-Acyclicity Example 1/2

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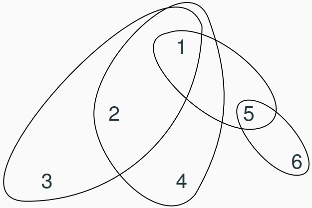

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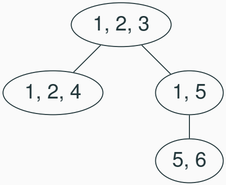

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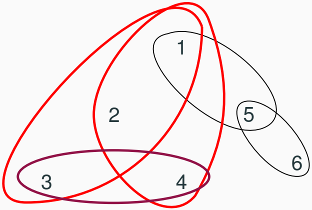


$\Longrightarrow$ Add the hyperedge $\{3, 4\}$ to the hypergraph

$\Longrightarrow$ There is a cycle not covered by a hyperedge

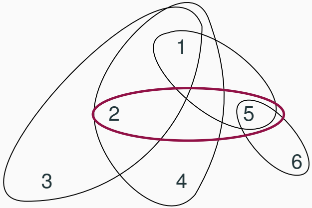
$\Longrightarrow$ Extended hypergraph not α-acyclic

$\Longrightarrow$ No join tree for extended hypergraph

$\Longrightarrow$ Original hypergraph not free-connex α-acyclic

Free-connex α-Acyclicity Example 2/2

choose $\{2, 5\}$
as free variables



Free-connex α-Acyclicity Example 2/2

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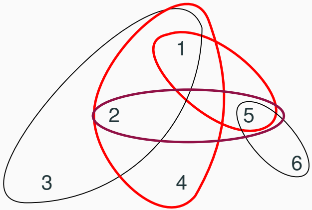

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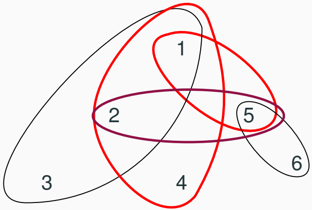

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choose $\{1, 2, 5\}$
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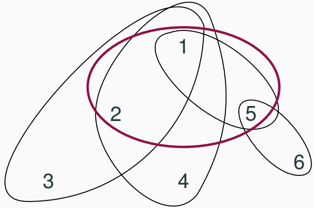

Free-connex α-Acyclicity Example 2/2

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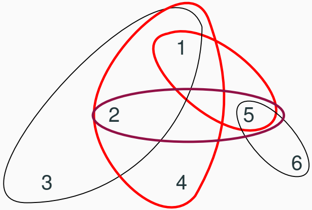

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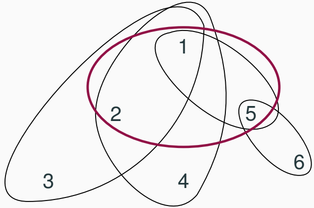
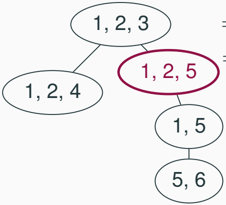


Possible join tree for the extended hypergraph



$\Longrightarrow$ Extended hypergraph α-acyclic

$\Longrightarrow$ Original hypergraph
free-connex α-acyclic

Landscape of Hypergraph Types