

# Efficient Algorithms, Spring 2021

## 5. Width Measures

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**DaST**   
Data • (Systems+Theory)

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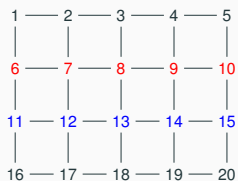


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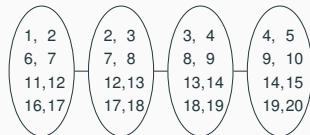
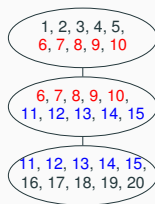
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# Choosing a Good Decomposition

Hypergraphs

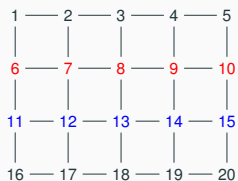


Choose the left or the right decomposition?

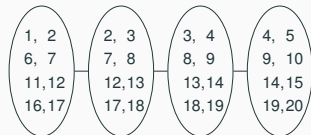
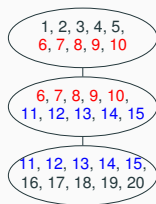


# Choosing a Good Decomposition

Hypergraphs



Choose the left or the right decomposition?



TL;DR: We need to measure the goodness of a hypertree decomposition

# Overview of Width Measures Covered in This Lecture

We will look at a variety of measures of goodness typically called **widths**

- **Treewidth**  $\text{tw}$
- **Edge cover** number  $\rho$  and **fractional edge cover** number  $\rho^*$
- **Hypertree width**  $\text{hw}$  and **fractional hypertree width**  $\text{fhw}$

$$\rho(\mathcal{H}) \underbrace{\geq}_{|\mathcal{E}|-1} \text{hw}(\mathcal{H}) \quad \text{and} \quad \rho^*(\mathcal{H}) \underbrace{\geq}_{|\mathcal{E}|-1} \text{fhw}(\mathcal{H})$$

Given FAQ  $\Phi$  with hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  and factors of size  $N$

- The size of the listing representation of  $\Phi$  is  $O(N^{\rho^*(\mathcal{H})})$
- The time to compute  $\Phi$ :  $O(N^{w(\mathcal{H})})$ , where  $w$  is any of:  $\rho$ ;  $\rho^*$ ;  $\text{hw}$ ;  $\text{fhw}$

## Treewidth: Counting the Number of Vertices in a Bag

Given: Hypergraph  $\mathcal{H}$  with set  $\mathbf{T}(\mathcal{H})$  of decompositions

- Treewidth  $\text{tw}$  of hypertree decomposition  $\mathcal{T} = (T, \chi)$ :

$$\text{tw}(\mathcal{T}) = \max_{t \in T} |\chi(t)| - 1$$

In words: The size of the largest bag in  $\mathcal{T}$  minus one

- Treewidth  $\text{tw}(\mathcal{H})$  of hypergraph  $\mathcal{H}$ :

$$\text{tw}(\mathcal{H}) = \min_{\mathcal{T} \in \mathbf{T}(\mathcal{H})} \text{tw}(\mathcal{T})$$

In words: The smallest treewidth of any of its decompositions

The computation time is  $O(N^{\text{tw}(\mathcal{H})+1})$ , where  $N$  is max domain size for variables

**Quiz:** What is the treewidth for the grid and 4-cycle hypergraphs?

## Width Measures Based on Factor Sizes

Use factor sizes instead of domain sizes

- Treewidth assumes factors  $\approx$  the products of the domains of their variables
- However, factors are typically much smaller than the products of the domains of their variables in many domains (e.g., DB, CSP, SAT)

## Measures Based on Factor Sizes: The Triangle Example

Triangle query:  $\Phi(x_1, x_2, x_3) = \psi_{12}(x_1, x_2) \otimes \psi_{23}(x_2, x_3) \otimes \psi_{13}(x_1, x_3)$

- Assumption:  $|\psi_{ij}| < N$
- Upper bound on computation time:  $|\psi_{12}| \cdot |\psi_{23}| = N^2$ 
  - Iterate over all combinations of tuples in  $\psi_{12}$  and  $\psi_{23}$
  - Ensure the combinations use the same  $x_2$  in both  $\psi_{12}$  and  $\psi_{23}$  and also use  $(x_1, x_3)$  that occur in  $\psi_{13}$

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  - Ensure the combinations use the same  $x_2$  in both  $\psi_{12}$  and  $\psi_{23}$  and also use  $(x_1, x_3)$  that occur in  $\psi_{13}$
- Lower bound: There are factors for which the computation time is at least  $N$ 
  - $\psi_{12} = \{1\} \times [N]$ ,  $\psi_{23} = [N] \times \{1\}$ ,  $\psi_{13} \supseteq \{(1, 1)\}$

Can we close the gap between lower and upper bounds on computation time?



## Edge Covers and Independent Sets

We can generalise the analysis of the triangle query

For the upper bound:

- Cover all nodes (variables) by  $k$  edges (factors)  $\Rightarrow$  size  $\leq N^k$ .
- Pick the **smallest  $k$**  so to reduce the complexity
- The **smallest  $k$**  is the **edge cover number** denoted  $\rho$
- **This is an edge cover of the query hypergraph!**

For the lower bound:

- $m$  independent nodes (variables)  $\Rightarrow$  construct factors such that size  $\geq N^m$ .
- Pick the **largest  $m$**  so to get a realistic/higher lower bound on the complexity
- **This is an independent set of the query hypergraph!**

# Edge Covers and Independent Sets Expressed using Integer Programming

**Minimum Edge Cover** of hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ :

$$\begin{aligned} & \text{minimise} && \sum_{S \in \mathcal{E}} w_S \\ & \text{subject to} && \sum_{S \in \mathcal{E}: v \in S} w_S \geq 1 \quad \forall v \in \mathcal{V}, \\ & && w_S \in \{0, 1\} \quad \forall S \in \mathcal{E} \end{aligned}$$

The cost of an optimal feasible solution is the edge cover number  $\rho$

**Maximum Independent Set** of hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ :

$$\begin{aligned} & \text{maximise} && \sum_{v \in \mathcal{V}} w_v \\ & \text{subject to} && w_{v_1} + w_{v_2} \leq 1 \quad \forall v_1, v_2 \in S, S \in \mathcal{E} \\ & && w_v \in \{0, 1\} \quad \forall v \in \mathcal{V} \end{aligned}$$

The costs of feasible solutions for the two optimisations programs may differ

## Fractional Edge Covers and Fractional Independent Sets

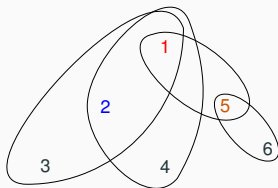
Relaxation of the two optimisation problems as linear programs:

- Variables can now range between 0 and 1 instead of being either 0 or 1
- By **duality of linear programming**, the two linear programs have the same cost of feasible solutions
- This cost  $\rho^*(\mathcal{H})$  is called the **fractional edge cover number**

$$\begin{aligned} & \text{minimise} && \sum_{S \in \mathcal{E}} w_S \\ & \text{subject to} && \sum_{S \in \mathcal{E}: v \in S} w_S \geq 1 \quad \forall v \in \mathcal{V}, \\ & && 0 \leq w_S \leq 1 \quad \forall S \in \mathcal{E} \end{aligned}$$

## Example (1/3): Fractional Edge Cover for an Acyclic Hypergraph

$$\Phi(x_1, \dots, x_5) = \psi_{123}(x_1, x_2, x_3) \otimes \psi_{124}(x_1, x_2, x_4) \otimes \psi_{15}(x_1, x_5) \otimes \psi_{56}(x_5, x_6)$$



- The three edges (factors)  $\psi_{123}, \psi_{124}, \psi_{56}$  can cover all nodes (variables)

$$\rho^*(\mathcal{H}) = \rho(\mathcal{H}) \leq 3$$

- Each node (variable) 3, 4, and 6 must be covered by a distinct edge (factor)

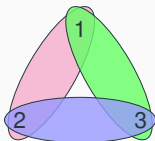
$$(\text{Fractional})\text{IndependentSet}(\mathcal{H}) \geq 3$$

$$\Rightarrow \rho^*(\mathcal{H}) = \rho(\mathcal{H}) = 3$$

$\Rightarrow$  For input size  $N$ ,  $\Phi$  takes time  $O(N^3)$  and for some inputs this is  $\Theta(N^3)$

## Example (2/3): Fractional Edge Cover for the Triangle Hypergraph

$$\Phi(x_1, x_2, x_3) = \psi_{12}(x_1, x_2) \otimes \psi_{23}(x_2, x_3) \otimes \psi_{13}(x_1, x_3)$$



$$\text{minimise } w_{\psi_{12}} + w_{\psi_{13}} + w_{\psi_{23}}$$

subject to

$$1 : w_{\psi_{12}} + w_{\psi_{13}} \geq 1$$

$$2 : w_{\psi_{12}} + w_{\psi_{23}} \geq 1$$

$$3 : w_{\psi_{13}} + w_{\psi_{23}} \geq 1$$

$$w_{\psi_{12}} \geq 0 \quad w_{\psi_{13}} \geq 0 \quad w_{\psi_{23}} \geq 0$$

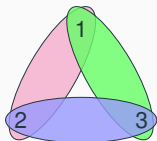
Our previous upper bound was  $O(N^2)$ , since the edge cover number is 2:

- Set any two of  $w_{\psi_{12}}, w_{\psi_{13}}, w_{\psi_{23}}$  to 1

What is the **fractional edge cover number** for the triangle hypergraph  $\mathcal{H}$ ?

## Example (2/3): Fractional Edge Cover for the Triangle Hypergraph

$$\Phi(x_1, x_2, x_3) = \psi_{12}(x_1, x_2) \otimes \psi_{23}(x_2, x_3) \otimes \psi_{13}(x_1, x_3)$$



$$\text{minimise } w_{\psi_{12}} + w_{\psi_{13}} + w_{\psi_{23}}$$

subject to

$$1 : w_{\psi_{12}} + w_{\psi_{13}} \geq 1$$

$$2 : w_{\psi_{12}} + w_{\psi_{23}} \geq 1$$

$$3 : w_{\psi_{13}} + w_{\psi_{23}} \geq 1$$

$$w_{\psi_{12}} \geq 0 \quad w_{\psi_{13}} \geq 0 \quad w_{\psi_{23}} \geq 0$$

Our previous upper bound was  $O(N^2)$ , since the edge cover number is 2:

- Set any two of  $w_{\psi_{12}}, w_{\psi_{13}}, w_{\psi_{23}}$  to 1

What is the fractional edge cover number for the triangle hypergraph  $\mathcal{H}$ ?

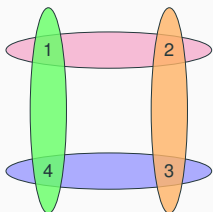
We can do better:  $w_{\psi_{12}} = w_{\psi_{13}} = w_{\psi_{23}} = 1/2$ . Then,  $\rho^*(\mathcal{H}) = 3/2$

Lower bound reaches  $N^{3/2}$  for  $\psi_{12} = \psi_{13} = \psi_{23} = [N^{1/2}] \times [N^{1/2}]$

## Example (3/3): Fractional Edge Cover for 4-Cycle Hypergraph

$$\Phi(x_1, \dots, x_4) = \psi_{12}(x_1, x_2) \otimes \psi_{23}(x_2, x_3) \otimes \psi_{34}(x_3, x_4) \otimes \psi_{14}(x_1, x_4)$$

The linear program for the fractional edge cover number:



$$\text{minimise } w_{\psi_{12}} + w_{\psi_{23}} + w_{\psi_{34}} + w_{\psi_{14}}$$

subject to

$$1 : w_{\psi_{12}} + w_{\psi_{14}} \geq 1$$

$$2 : w_{\psi_{12}} + w_{\psi_{23}} \geq 1$$

$$3 : w_{\psi_{23}} + w_{\psi_{34}} \geq 1$$

$$4 : w_{\psi_{34}} + w_{\psi_{14}} \geq 1$$

$$w_{\psi_{12}} \geq 0 \quad w_{\psi_{23}} \geq 0 \quad w_{\psi_{34}} \geq 0 \quad w_{\psi_{14}} \geq 0$$

Solution:  $w_{\psi_{12}} = w_{\psi_{34}} = 1$ . Alternative:  $w_{\psi_{23}} = w_{\psi_{14}} = 1$ . Then,  $\rho^*(\mathcal{H}) = 2$ .

Lower bound reaches  $N^2$  for  $\psi_{12} = \psi_{34} = [N] \times \{1\}$  and  $\psi_{23} = \psi_{14} = \{1\} \times [N]$ .

## Refinement under Cardinality Constraints

Recall the linear program for computing the fractional edge cover number  $\rho^*$

$$\text{minimise } \sum_{S \in \mathcal{E}} w_S$$

$$\text{subject to } \sum_{S \in \mathcal{E}: v \in S} w_S \geq 1 \quad \forall v \in \mathcal{V},$$

$$0 \leq w_S \leq 1 \quad \forall S \in \mathcal{E}$$

Equivalent formulation of minimisation objective assuming each factor has size  $N$ :

$$N^{\sum_{S \in \mathcal{E}} w_S} = \prod_{S \in \mathcal{E}} N^{w_S}$$

[We show here the refinement for  $\rho^*$ ; it works similarly for  $\rho$ ]



## Refinement under Cardinality Constraints

Recall the linear program for computing the fractional edge cover number  $\rho^*$

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Equivalent formulation of minimisation objective assuming each factor has size  $N$ :

$$N^{\sum_{S \in \mathcal{E}} w_S} = \prod_{S \in \mathcal{E}} N^{w_S}$$

In practice, factors have different sizes, i.e.,  $\psi_S$  has size  $N_S$ :

$$\prod_{S \in \mathcal{E}} N_S^{w_S} \quad \text{or by taking the log} \quad \sum_{S \in \mathcal{E}} w_S \log N_S$$

[We show here the refinement for  $\rho^*$ ; it works similarly for  $\rho$ ]

## Hypertree Width and Fractional Hypertree Width

- So far, we applied  $\rho$  or  $\rho^*$  to the entire hypergraph  $\mathcal{H}$  of an FAQ
- We can also apply them to each bag of a hypertree decomposition  $\mathcal{T}$  of  $\mathcal{H}$
- This leads to **(fractional) hypertree width** of  $\mathcal{T} = (T, \chi)$  and of  $\mathcal{H}$

$$\text{htw}(\mathcal{T}) = \max_{t \in T} \rho(\chi(t)) \quad \text{and} \quad \text{fhtw}(\mathcal{T}) = \max_{t \in T} \rho^*(\chi(t))$$

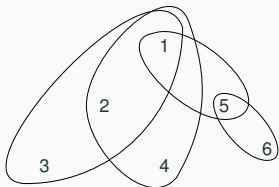
In words: The (fractional) hypertree width of a hypertree decomposition is the maximum (fractional) edge cover number of any of its bags

$$\text{htw}(\mathcal{H}) = \min_{\mathcal{T} \in \mathbf{T}(\mathcal{H})} \text{htw}(\mathcal{T}) \quad \text{and} \quad \text{fhtw}(\mathcal{H}) = \min_{\mathcal{T} \in \mathbf{T}(\mathcal{H})} \text{fhtw}(\mathcal{T})$$

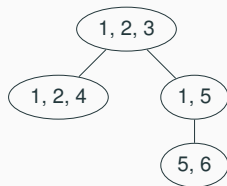
In words: The (fractional) hypertree width of a hypergraph is the minimum (fractional) hypertree width of any of its hypertree decompositions

## Examples (1/3): (Fractional) Hypertree Width for Acyclic Hypergraph

$\alpha$ -acyclic hypergraph



hypertree decomposition



- Each bag of the decomposition only has variables from one hyperedge
- $\Rightarrow$  the (fractional) edge cover number is 1 for each bag
- $\Rightarrow$  the (fractional) hypertree width is 1 for the decomposition
- $\Rightarrow$  the (fractional) hypertree width is 1 for the hypergraph

The (fractional) hypertree width of any  $\alpha$ -acyclic hypergraph is one

## Examples (2/3): Hypertree Width for Bowtie Hypergraph

Consider the bowtie hypergraph  $\mathcal{H}$  and two possible decompositions  $\mathcal{T}_1$  and  $\mathcal{T}_2$



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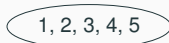
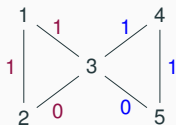


**Left hypertree decomposition**  $\mathcal{T}_1$  is one bag for the entire hypergraph  $\mathcal{H}$

- With the above weight assignments:  $\rho(\mathcal{H}) = 3$
- $\text{htw}(\mathcal{T}_1) = \rho(\mathcal{H}) = 3$ , since we only have one bag

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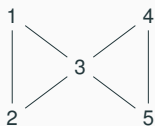
**Right hypertree decomposition**  $\mathcal{T}_2$  has one bag per triangle in the hypergraph  $\mathcal{H}$

- We treat each bag independently:  $\rho(\{1, 2, 3\}) = 2$ ,  $\rho(\{3, 4, 5\}) = 2$
- $\text{htw}(\mathcal{T}_2) = \max\{\rho(\{1, 2, 3\}), \rho(\{3, 4, 5\})\} = 2$

Overall,  $\text{htw}(\mathcal{H}) = \text{htw}(\mathcal{T}_2) = 2$ , while  $\rho(\mathcal{H}) = 3$

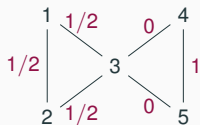
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Consider again the bowtie hypergraph  $\mathcal{H}$  and two decompositions  $\mathcal{T}_1$  and  $\mathcal{T}_2$



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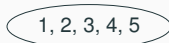
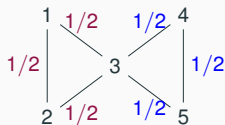
**Left hypertree decomposition**  $\mathcal{T}_1$  is one bag for the entire hypergraph  $\mathcal{H}$

- With the above weight assignments:  $\rho^*(\mathcal{H}) = 5/2$
- $\text{fhtw}(\mathcal{T}_1) = \rho^*(\mathcal{H}) = 5/2$ , since we only have one bag



## Examples (2/3): Fractional Hypertree Width for Bowtie Hypergraph

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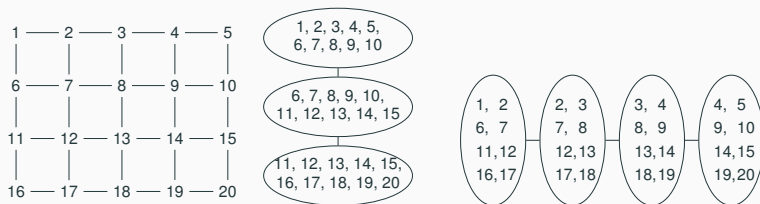
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- $\text{fhtw}(\mathcal{T}_2) = \max\{\rho^*({1, 2, 3}), \rho^*({3, 4, 5})\} = 3/2$

Overall,  $\text{fhtw}(\mathcal{H}) = \text{fhtw}(\mathcal{T}_2) = 3/2$ , while  $\rho^*(\mathcal{H}) = 5/2$

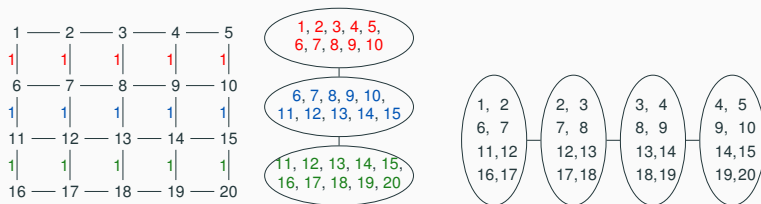
## Examples (3/3): (Fractional) Hypertree Width for the Grid Hypergraph

Consider the grid hypergraph  $\mathcal{H}$  and two possible decompositions  $\mathcal{T}_1$  and  $\mathcal{T}_2$



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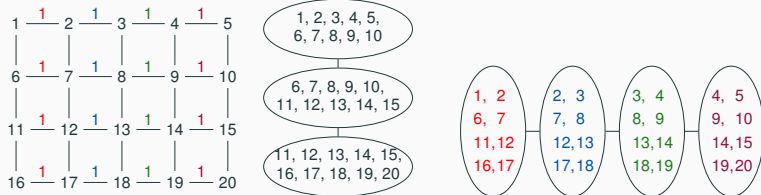


Left hypertree decomposition  $\mathcal{T}_1$  has one bag for each two consecutive rows

- With the above weight assignments:  $\rho^*(b) = \rho(b) = 5$  for each bag  $b$  in  $\mathcal{T}_1$
- This means  $\text{htw}(\mathcal{T}_1) \leq 5$  and  $\text{fhtw}(\mathcal{T}_1) \leq 5$

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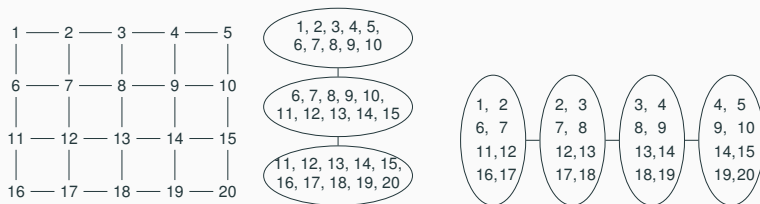
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**Right hypertree decomposition**  $\mathcal{T}_2$  has one bag for each two consecutive columns

- With the above weight assignments:  $\rho^*(b) = \rho(b) = 4$  for each bag  $b$  in  $\mathcal{T}_2$
- This means  $\text{htw}(\mathcal{T}_2) \leq 4$  and  $\text{fhtw}(\mathcal{T}_2) \leq 4$

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**Right hypertree decomposition  $\mathcal{T}_2$**  has one bag for each two consecutive columns

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- This means  $\text{htw}(\mathcal{T}_2) \leq 4$  and  $\text{fhtw}(\mathcal{T}_2) \leq 4$

Overall,  $\text{htw}(\mathcal{H}) \leq \text{htw}(\mathcal{T}_2) \leq 4$  and  $\text{fhtw}(\mathcal{H}) \leq \text{fhtw}(\mathcal{T}_2) \leq 4$

## (Fractional) Hypertree Width in the Presence of Free Variables

- So far, we considered widths for hypergraphs for FAQs without free variables
- Consider now: FAQs with hypergraph  $\mathcal{H}$  and free variables  $[f]$
- Good hypertree decompositions have the free variables forming a connected subtree – no need to remember their values beyond child-parent messages
- Let  $\mathbf{T}_{[f]}(\mathcal{H}) \subseteq \mathbf{T}(\mathcal{H})$  be the set of hypertree decompositions of  $\mathcal{H}$  that are good for the free variables  $[f]$
- The (fractional) hypertree width of  $\mathcal{T}_{[f]} = (T, \chi) \in \mathbf{T}_{[f]}(\mathcal{H})$

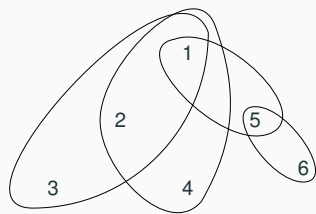
$$\text{htw}(\mathcal{T}_{[f]}) = \max_{t \in T} \rho(\chi(t)) \quad \text{and} \quad \text{fhtw}(\mathcal{T}_{[f]}) = \max_{t \in T} \rho^*(\chi(t))$$

- The (fractional) hypertree width of hypergraph  $\mathcal{H}$  and free variables  $[f]$

$$\text{htw}(\mathcal{H}) = \min_{\mathcal{T}_{[f]} \in \mathbf{T}_{[f]}(\mathcal{H})} \text{htw}(\mathcal{T}_{[f]}) \quad \text{and} \quad \text{fhtw}(\mathcal{H}) = \min_{\mathcal{T}_{[f]} \in \mathbf{T}_{[f]}(\mathcal{H})} \text{fhtw}(\mathcal{T}_{[f]})$$

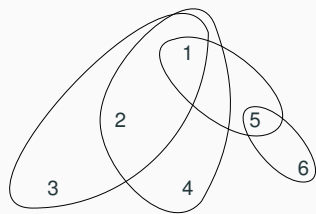
## Example: Hypertree Width for Acyclic Hypergraph with Free Variables

$\alpha$ -acyclic hypergraph  $\mathcal{H}$

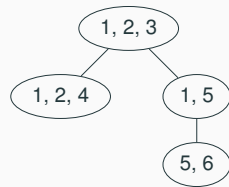


## Example: Hypertree Width for Acyclic Hypergraph with Free Variables

$\alpha$ -acyclic hypergraph  $\mathcal{H}$



Possible join tree for  $\mathcal{H}$

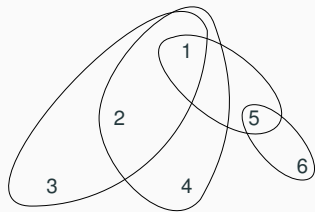


$$\text{htw}(\mathcal{T}) = 1, \text{fhtw}(\mathcal{T}) = 1$$

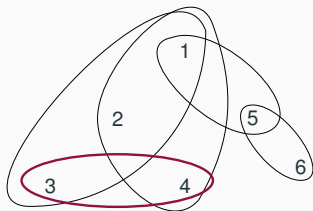


# Example: Hypertree Width for Acyclic Hypergraph with Free Variables

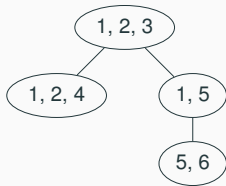
$\alpha$ -acyclic hypergraph  $\mathcal{H}$



choose  $\{3, 4\}$   
as free variables



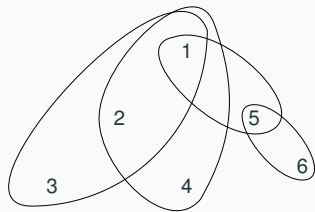
Possible join tree for  $\mathcal{H}$



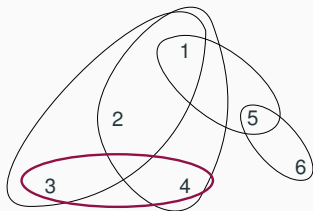
$$\text{htw}(\mathcal{T}) = 1, \text{fhtw}(\mathcal{T}) = 1$$

# Example: Hypertree Width for Acyclic Hypergraph with Free Variables

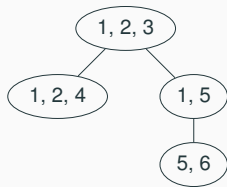
$\alpha$ -acyclic hypergraph  $\mathcal{H}$



choose  $\{3, 4\}$   
as free variables



Possible join tree for  $\mathcal{H}$



Possible decomposition



$$\text{htw}(\mathcal{T}) = 1, \text{fhtw}(\mathcal{T}) = 1$$

$$\text{htw}(\mathcal{T}_{3,4}) = 2, \text{fhtw}(\mathcal{T}_{3,4}) = 3/2$$