Efficient Algorithms, Spring 2021

7. Worst-Case Optimal Size Bounds for Joins

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Agenda for This Lecture

Worst-case optimal size bounds for joins

- Key parameter: The fractional edge cover number ρ^*
- Mentioned it several times in the previous lectures

Upper bound via an information-theoretic argument

- Warm-up: Triangle join
- General Case using Shearer's Lemma

Lower bound

- Warm-up: Triangle join
- General case via dual linear program for fractional edge cover number

The effect of the size of input factors: Same size vs different sizes

The Upper Bound Argument

Upper Bound on Join Output Size

Consider the join (all variables free, no marginalisation)

$$\Phi(\mathbf{x}) = \bigotimes_{S \in \mathcal{E}} \psi_S(\mathbf{x}_S)$$

with hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and input factor sizes $|\psi_S| = N_S$ for $S \in \mathcal{E}$

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- Let $(w_S)_{S \in \mathcal{E}}$ be any feasible solution to the linear program computing $\rho^*(\mathcal{H})$ with minimisation objective $\prod_{S \in \mathcal{E}} N_S^{w_S}$
- We will show that the output size $|\Phi|$ is upper-bounded by $\prod_{S \in \mathcal{E}} N_S^{w_S}$

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- By choosing $N = \max_{S \in \mathcal{E}} N_S$, this implies

$$|\Phi| \leq \prod_{S \in \mathcal{E}} N_S^{w_S} \leq \prod_{S \in \mathcal{E}} N^{w_S} = N^{\sum_{S \in \mathcal{E}} w_S} = N^{\rho^*(\mathcal{H})}$$

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- We will sketch a proof based on information theory
- Warm-up first: Triangle join with input factor sizes N

Warm-Up: Size Bound for Triangle Join

Upper Bound on Triangle Join Output Size

$$\Phi(x_1, x_2, x_3) = \psi_{12}(x_1, x_2) \otimes \psi_{23}(x_2, x_3) \otimes \psi_{13}(x_1, x_3)$$

with input factor sizes $|\psi_{12}| = |\psi_{23}| = |\psi_{13}| = N$

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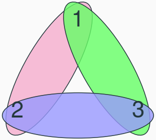
Hypergraph $\mathcal{H}$



Linear program computing $\rho^*(\mathcal{H})$

minimise $w_{12} + w_{23} + w_{13}$

subject to

$$\begin{array}{llllll}
1: & w_{12} & + & w_{23} & & \geq 1 \\
2: & w_{12} & & & + \; w_{13} & \geq 1 \\
3: & & & w_{23} & + \; w_{13} & \geq 1 \\
& w_{12} \geq 0 & & w_{23} \geq 0 & w_{13} \geq 0 &
\end{array}$$

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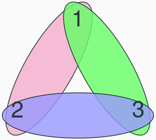

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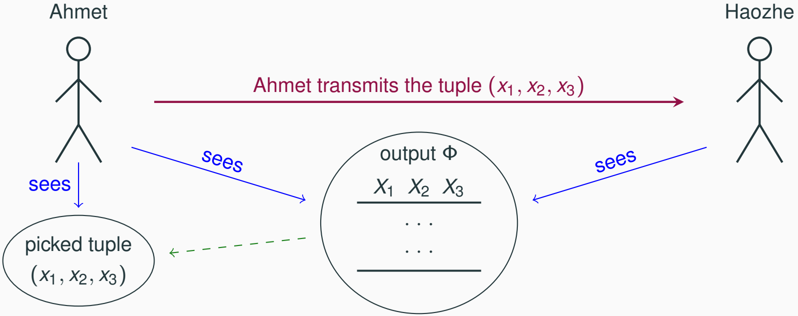
$$\begin{array}{llllll} 1: & w_{12} & + & w_{23} & & \geq 1 \\ 2: & w_{12} & & + & w_{13} & \geq 1 \\ 3: & & & w_{23} + & w_{13} & \geq 1 \\ & w_{12} \geq 0 & & w_{23} \geq 0 & w_{13} \geq 0 & \end{array}$$

- The optimal solution to the above program is $w_{12} = w_{23} = w_{13} = \frac{1}{2}$
- We will show that $|\Phi| \leq N^{\frac{3}{2}}$

A Two-Player Game

Consider a two-player game between Ahmet and Haozhe

- Both players know the output of the triangle query
- Ahmet picks an arbitrary tuple from the output and transmits it to Haozhe



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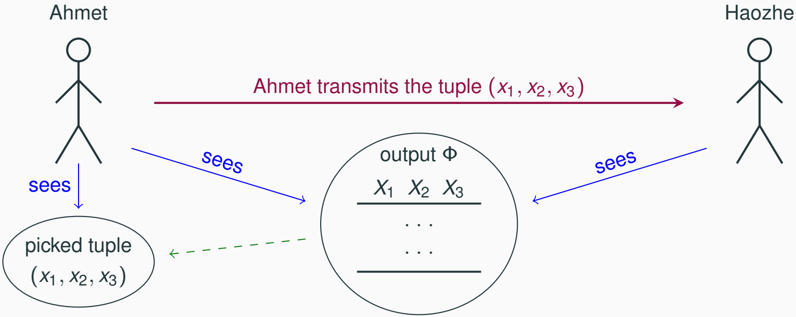
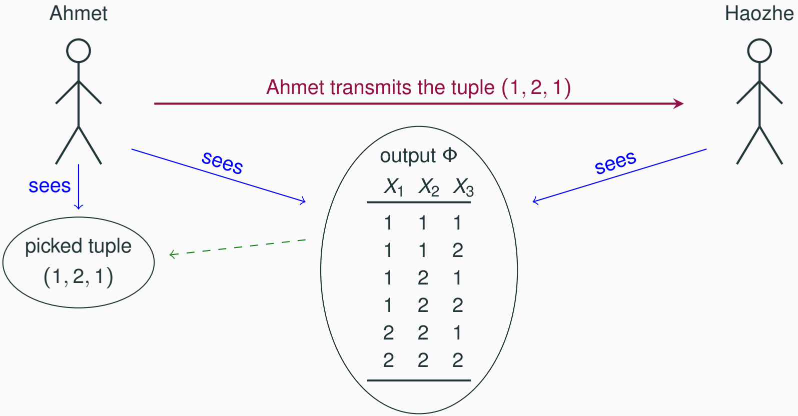


- Assume that the players have agreed on a binary coding system

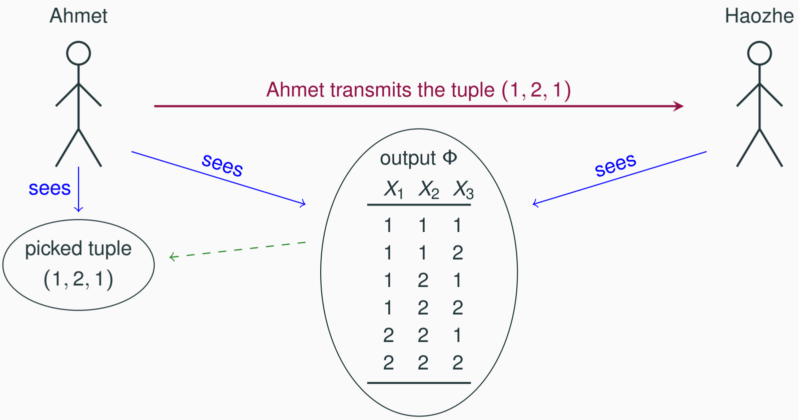
How many bits does Ahmet need on avg to inform Haozhe which tuple he picked?

Two-Player Game Example



X_1	X_2	X_3
1	1	1
1	1	2
1	2	1
1	2	2
2	2	1
2	2	2

Two-Player Game Example



X_1	X_2	X_3
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The best Ahmet and Haozhe can do is:

- Assign to each of the N tuples an index from 0 to $N - 1$
- Ahmet transmits to Haozhe the index of the picked tuple in binary

In the above example: $\log |\Phi| = \log 6$ bits are needed

Information Theoretic Perspective

- Ahmet picking an arbitrary tuple can be considered an experiment with random variable O
- The values of O are the output tuples in Φ
- The avg number of bits needed to transmit tuples depends on the uncertainty about O

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Special cases:

- If O takes on a tuple with probability 1 (there is only one tuple), then there is no uncertainty and the avg number of needed bits is 0
- If the tuples are uniformly distributed, then the uncertainty is maximal and the avg number of needed bits is $\log|\Phi|$

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The avg number of needed bits is the entropy $H(O)$ of O

Quick Recap: Entropy

The entropy of a random variable O with n possible outcomes $v_1, \ldots, v_n$:

$$H(O) = -\sum_{i \in [n]} \mathrm{P}(v_i) \cdot \log \mathrm{P}(v_i)$$

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- Special case 2: If the tuples are uniformly distributed, then the uncertainty is maximal and the avg number of needed bits is $\log |\Phi|$

  Uniform distribution means $P(v_i) = \frac{1}{n}, \forall i \in [n]$. Then,

  $$H(O) = -\sum_{i \in [n]} \mathrm{P}(v_i) \cdot \log \mathrm{P}(v_i) = -n \cdot (\frac{1}{n} \cdot \log \frac{1}{n}) = -\log \frac{1}{n} = -(\log 1 - \log n) = \log n$$

Our Goal

- We assume that Ahmet picks a tuple from the output uniformly at random
  $\implies H(O) = \log |\Phi|$

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- Assume that I_{12}, I_{23}, and I_{13} are random variables where each I_{ij} takes on a tuple from ψ_{ij} uniformly at random
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This implies:

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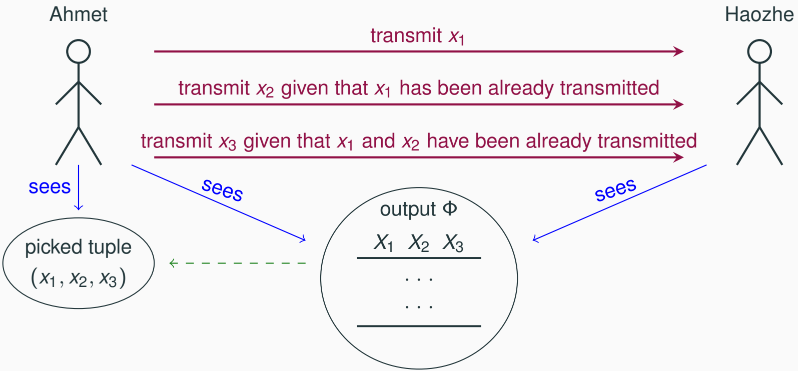
Next: a strategy for Ahmet that helps to express $H(O)$ in terms of $H(I_{12})$, $H(I_{23})$, and $H(I_{13})$

Alternative Strategy

Ahmet transmits the picked tuple in three steps



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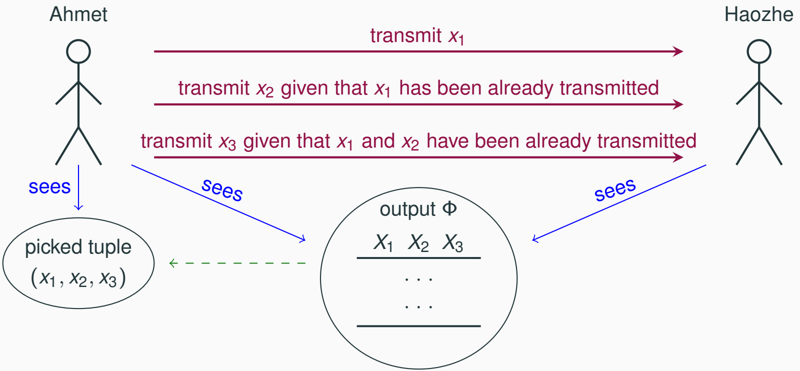

- In each step, Ahmet uses an optimal encoding given that Haozhe knows the values transmitted before

How many bits does Ahmet need on avg at each step?

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$H(O_1)$

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$H(O_1)$	$H(O_2 \mid O_1)$

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- We have $H(O_i, O_j) = H(O_i) + H(O_j \mid O_i)$

Next, we look closer at the relationship between $H(O_i, O_j)$ and $H(I_{ij})$

Observation 1

Transmitting (x_1, x_2) such that there is an x_3 with $(x_1, x_2, x_3) \in \Phi$ does not require more bits than transmitting $(x_1, x_2) \in \psi_{12}$ chosen uniformly at random

$$H(O_1) + H(O_2 \mid O_1) = H(O_1, O_2) \leq H(I_{12})$$

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$$H(O_1) + H(O_2 \mid O_1) = H(O_1, O_2) \leq H(I_{12})$$

Example

input ψ_{12}

X_1	X_2
1	1
1	2
2	2
2	5
2	6

input ψ_{23}

X_2	X_3
1	1
1	2
2	1
2	2
3	1

input ψ_{13}

X_1	X_3
1	1
1	2
2	1
2	2
1	5

output Φ

X_1	X_2	X_3
1	1	1
1	1	2
1	2	1
1	2	2
2	2	1
2	2	2

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Transmitting (x_1, x_2) such that there is an x_3 with $(x_1, x_2, x_3) \in \Phi$ does not require more bits than transmitting $(x_1, x_2) \in \psi_{12}$ chosen uniformly at random

$$H(O_1) + H(O_2 \mid O_1) = H(O_1, O_2) \leq H(I_{12})$$

Example

input ψ_{12}

X_1	X_2	
1	1	1/5
1	2	1/5
2	2	1/5
2	5	1/5
2	6	1/5

input ψ_{23}

X_2	X_3
1	1
1	2
2	1
2	2
3	1

input ψ_{13}

X_1	X_3
1	1
1	2
2	1
2	2
1	5

output Φ

X_1	X_2	X_3	
1	1	1	1/6
1	1	2	1/6
1	2	1	1/6
1	2	2	1/6
2	2	1	1/6
2	2	2	1/6

marginalised output $\bigoplus_{x_3} \Phi$

X_1	X_2	
1	1	1/3
1	2	1/3
2	2	1/3

$$H(O_1, O_2) = \log 3 \leq \log 5 = H(I_{12})$$

Observation 2

Transmitting (x_2, x_3) such that there is an x_1 with $(x_1, x_2, x_3) \in \Phi$ does not require more bits than transmitting $(x_2, x_3) \in \psi_{23}$ chosen uniformly at random

$$H(O_2) + H(O_3 \mid O_2) = H(O_2, O_3) \leq H(I_{23})$$

Observation 2

Transmitting (x_2, x_3) such that there is an x_1 with $(x_1, x_2, x_3) \in \Phi$ does not require more bits than transmitting $(x_2, x_3) \in \psi_{23}$ chosen uniformly at random

$$H(O_2) + H(O_3 \mid O_2) = H(O_2, O_3) \leq H(I_{23})$$

Example

input ψ_{12}

X_1	X_2
1	1
1	2
2	2
2	5
2	6

input ψ_{23}

X_2	X_3
1	1
1	2
2	1
2	2
3	1

input ψ_{13}

X_1	X_3
1	1
1	2
2	1
2	2
1	5

output Φ

X_1	X_2	X_3
1	1	1
1	1	2
1	2	1
1	2	2
2	2	1
2	2	2

Observation 2

Transmitting (x_2, x_3) such that there is an x_1 with $(x_1, x_2, x_3) \in \Phi$ does not require more bits than transmitting $(x_2, x_3) \in \psi_{23}$ chosen uniformly at random

$$H(O_2) + H(O_3 \mid O_2) = H(O_2, O_3) \leq H(I_{23})$$

Example

input ψ_{12}

X_1	X_2
1	1
1	2
2	2
2	5
2	6

input ψ_{23}

X_2	X_3	
1	1	1/5
1	2	1/5
2	1	1/5
2	2	1/5
3	1	1/5

input ψ_{13}

X_1	X_3
1	1
1	2
2	1
2	2
1	5

output Φ

X_1	X_2	X_3	
1	1	1	1/6
1	1	2	1/6
1	2	1	1/6
1	2	2	1/6
2	2	1	1/6
2	2	2	1/6

marginalised output $\bigoplus_{x_1} \Phi$

X_2	X_3	
1	1	1/6
1	2	1/6
2	1	1/3
2	2	1/3

$$H(O_2, O_3) = \frac{2}{6} \log 6 + \frac{2}{3} \log 3 \leq \log 5 = H(I_{23})$$

Observation 3

Similar to the other Observations

$$H(O_1) + H(O_3 \mid O_1) = H(O_1, O_3) \leq H(I_{13})$$

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Example

input ψ_{12}

X_1	X_2
1	1
1	2
2	2
2	5
2	6

input ψ_{23}

X_2	X_3
1	1
1	2
2	1
2	2
3	1

input ψ_{13}

X_1	X_3
1	1
1	2
2	1
2	2
1	5

output Φ

X_1	X_2	X_3
1	1	1
1	1	2
1	2	1
1	2	2
2	2	1
2	2	2

Observation 3

Similar to the other Observations

$$H(O_1) + H(O_3 \mid O_1) = H(O_1, O_3) \leq H(I_{13})$$

Example

input ψ_{12}

X_1	X_2
1	1
1	2
2	2
2	5
2	6

input ψ_{23}

X_2	X_3
1	1
1	2
2	1
2	2
3	1

input ψ_{13}

X_1	X_3	
1	1	1/5
1	2	1/5
2	1	1/5
2	2	1/5
1	5	1/5

output Φ

X_1	X_2	X_3	
1	1	1	1/6
1	1	2	1/6
1	2	1	1/6
1	2	2	1/6
2	2	1	1/6
2	2	2	1/6

marginalised output $\bigoplus_{x_2} \Phi$

X_1	X_3	
1	1	1/3
1	2	1/3
2	1	1/6
2	2	1/6

$$H(O_1, O_3) = \frac{2}{3}\log 3 + \frac{2}{6}\log 6 \leq \log 5 = H(I_{13})$$

Putting Things Together

$2 \log |\Phi| = 2H(O)$ output tuples uniformly distributed

Putting Things Together

$$2\log|\Phi| = 2H(O) \qquad \text{output tuples uniformly distributed}$$

$$= 2\left[H(O_1) + H(O_2 \mid O_1) + H(O_3 \mid O_1, O_2)\right]$$

Putting Things Together

$$2\log|\Phi| = 2H(O) \quad \text{output tuples uniformly distributed}$$

$$= 2\Big[H(O_1) + H(O_2 \mid O_1) + H(O_3 \mid O_1, O_2)\Big]$$

$$= \Big[H(O_1) + H(O_2 \mid O_1)\Big] + \Big[H(O_2 \mid O_1) + H(O_3 \mid O_1, O_2)\Big] +$$

$$\Big[H(O_1) + H(O_3 \mid O_1, O_2)\Big]$$

Putting Things Together

$$
\begin{aligned}
2\log|\Phi| &= 2H(O) && \text{output tuples uniformly distributed}\\
&= 2\Big[H(O_1) + H(O_2 \mid O_1) + H(O_3 \mid O_1, O_2)\Big]\\
&= \Big[H(O_1) + H(O_2 \mid O_1)\Big] + \Big[H(O_2 \mid O_1) + H(O_3 \mid O_1, O_2)\Big] +\\
&\quad \Big[H(O_1) + H(O_3 \mid O_1, O_2)\Big]\\
&\le \Big[H(O_1) + H(O_2 \mid O_1)\Big] + \Big[H(O_2) + H(O_3 \mid O_2)\Big] +\\
&\quad \Big[H(O_1) + H(O_3 \mid O_1)\Big] && \text{dropping information cannot decrease entropy}
\end{aligned}
$$

Putting Things Together

$$2\log|\Phi| = 2H(O) \qquad \text{output tuples uniformly distributed}$$

$$= 2\Big[H(O_1) + H(O_2 \mid O_1) + H(O_3 \mid O_1, O_2)\Big]$$

$$= \Big[H(O_1) + H(O_2 \mid O_1)\Big] + \Big[H(O_2 \mid O_1) + H(O_3 \mid O_1, O_2)\Big] + \Big[H(O_1) + H(O_3 \mid O_1, O_2)\Big]$$

$$\leq \Big[H(O_1) + H(O_2 \mid O_1)\Big] + \Big[H(O_2) + H(O_3 \mid O_2)\Big] + \Big[H(O_1) + H(O_3 \mid O_1)\Big] \qquad \text{dropping information cannot decrease entropy}$$

$$= H(O_1, O_2) + H(O_2, O_3) + H(O_1, O_3) \qquad \text{conditional entropies}$$

Putting Things Together

$$2\log|\Phi| = 2H(O) \qquad \text{output tuples uniformly distributed}$$

$$= 2\Big[H(O_1) + H(O_2 \mid O_1) + H(O_3 \mid O_1, O_2)\Big]$$

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$$\leq \Big[H(O_1) + H(O_2 \mid O_1)\Big] + \Big[H(O_2) + H(O_3 \mid O_2)\Big] + \Big[H(O_1) + H(O_3 \mid O_1)\Big] \qquad \text{dropping information cannot decrease entropy}$$

$$= H(O_1, O_2) + H(O_2, O_3) + H(O_1, O_3) \qquad \text{conditional entropies}$$

$$\leq H(I_{12}) + H(I_{23}) + H(I_{13}) \qquad \text{Observations 1, 2, and 3}$$

Putting Things Together

$$
\begin{aligned}
2\log|\Phi| &= 2H(O) && \text{output tuples uniformly distributed} \\
&= 2\Big[H(O_1) + H(O_2 \mid O_1) + H(O_3 \mid O_1, O_2)\Big] \\
&= \Big[H(O_1) + H(O_2 \mid O_1)\Big] + \Big[H(O_2 \mid O_1) + H(O_3 \mid O_1, O_2)\Big] + \\
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&\le \Big[H(O_1) + H(O_2 \mid O_1)\Big] + \Big[H(O_2) + H(O_3 \mid O_2)\Big] + \\
&\quad \Big[H(O_1) + H(O_3 \mid O_1)\Big] && \text{dropping information cannot decrease entropy} \\
&= H(O_1, O_2) + H(O_2, O_3) + H(O_1, O_3) && \text{conditional entropies} \\
&\le H(I_{12}) + H(I_{23}) + H(I_{13}) && \text{Observations 1, 2, and 3} \\
&= \log N + \log N + \log N && \text{input tuples uniformly distributed}
\end{aligned}
$$

Putting Things Together

$$
\begin{aligned}
& 2\log|\Phi| = 2H(O) && \text{output tuples uniformly distributed} \\
&= 2\Big[H(O_1) + H(O_2 \mid O_1) + H(O_3 \mid O_1, O_2)\Big] \\
&= \Big[H(O_1) + H(O_2 \mid O_1)\Big] + \Big[H(O_2 \mid O_1) + H(O_3 \mid O_1, O_2)\Big] + \\
&\quad \Big[H(O_1) + H(O_3 \mid O_1, O_2)\Big] \\
&\leq \Big[H(O_1) + H(O_2 \mid O_1)\Big] + \Big[H(O_2) + H(O_3 \mid O_2)\Big] + \\
&\quad \Big[H(O_1) + H(O_3 \mid O_1)\Big] && \text{dropping information cannot decrease entropy} \\
&= H(O_1, O_2) + H(O_2, O_3) + H(O_1, O_3) && \text{conditional entropies} \\
&\leq H(I_{12}) + H(I_{23}) + H(I_{13}) && \text{Observations 1, 2, and 3} \\
&= \log N + \log N + \log N && \text{input tuples uniformly distributed}
\end{aligned}
$$

$$\implies |\Phi| \leq N^{\frac{3}{2}} \qquad \text{as explained before}$$

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$$
\begin{aligned}
& 2\log|\Phi| = 2H(O) && \text{output tuples uniformly distributed} \\
&= 2\Big[H(O_1) + H(O_2 \mid O_1) + H(O_3 \mid O_1, O_2)\Big] \\
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&\quad \Big[H(O_1) + H(O_3 \mid O_1, O_2)\Big] \\
&\leq \Big[H(O_1) + H(O_2 \mid O_1)\Big] + \Big[H(O_2) + H(O_3 \mid O_2)\Big] + \\
&\quad \Big[H(O_1) + H(O_3 \mid O_1)\Big] && \text{dropping information cannot decrease entropy} \\
&= H(O_1, O_2) + H(O_2, O_3) + H(O_1, O_3) && \text{conditional entropies} \\
&\leq H(I_{12}) + H(I_{23}) + H(I_{13}) && \text{Observations 1, 2, and 3} \\
&= \log N + \log N + \log N && \text{input tuples uniformly distributed}
\end{aligned}
$$

$$\implies |\Phi| \leq N^{\frac{3}{2}} \qquad \text{as explained before}$$

We next generalise the approach taken in this example to arbitrary joins

General Case: Size Bound for Any Join

Quick Recap on Random Variables over Discrete Domains

- $\text{Dom}(X)$ is the domain of variable X
- For each $x \in \text{Dom}(X)$, we have a probability $P(X = x)$
- Joint Probability of random variables X and Y:

  Let $x \in \text{Dom}(X)$, $y \in \text{Dom}(Y)$.

  $P(X = x, Y = y)$ gives the joint probability of $X = x$ and $Y = y$

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  Let $x \in \text{Dom}(X)$, $y \in \text{Dom}(Y)$.

  $P(X = x, Y = y)$ gives the joint probability of $X = x$ and $Y = y$
- Marginalised probability:

$$P(X = x) = \sum_{y} P(X = x, Y = y)$$

- Conditional probability: Assuming $P(Y = y) \neq 0$,

$$P(X = x | Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Entropy of Random Variable

- Entropy of a random variable X:

$$H(X) = -\sum_x P(X = x) \cdot \log P(X = x)$$

Intuitively: $H(X)$ measures the uncertainty about X

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  Intuitively: $H(X)$ measures the uncertainty about X

- Joint entropy:

$$H(X, Y) = -\sum_{x,y} P(X = x, Y = y) \cdot \log P(X = x, Y = y)$$

- Conditional entropy: Assuming $P(Y = y) \neq 0$,

$$H(X|Y = y) = -\sum_{x} P(X = x|Y = y) \cdot \log P(X = x|Y = y)$$

$$H(X|Y) = \sum_{y} P(Y = y) \cdot H(X|Y = y)$$

Observations

Observation 1: The joint entropy of $\mathbf{X}_{[n]} = (X_1, \ldots, X_n)$ can be expressed as the the sum of the entropies of each X_i conditioned on $\mathbf{X}_{[i-1]} = (X_1, \ldots, X_{i-1})$

$$H(\mathbf{X}_{[n]}) = H(X_1) + H(X_2|X_1) + \ldots + H(X_n \mid \mathbf{X}_{[n-1]})$$

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$$H(\mathbf{X}_{[n]}) = H(X_1) + H(X_2|X_1) + \ldots + H(X_n \mid \mathbf{X}_{[n-1]})$$

Observation 2: The entropy of X conditioned on $\mathbf{X}_{[n]} = (X_1, \ldots, X_n)$ is not larger than the entropy of X conditioned on a subset $\mathbf{X}_J$ of $\mathbf{X}_{[n]}$

$$H(X \mid \mathbf{X}_{[n]}) \leq H(X \mid \mathbf{X}_J) \text{ for all } J \subseteq [n]$$

Shearer's Lemma

$$q \cdot H(\mathbf{X}_{[n]}) \leq \sum_{J \in \mathcal{J}} H(\mathbf{X}_J)$$

where

- $\mathbf{X}_{[n]} = (X_1, \ldots, X_n)$ are random variables
- $\mathcal{J} \subseteq 2^{[n]}$ is multiset such that each $i \in [n]$ is in at least q members of $\mathcal{J}$
  - $2^{[n]}$ is the set of all possible subsets of $[n] = \{1, \ldots, n\}$
  - $\mathcal{J}$ is a subset of $2^{[n]}$, but possibly with repetitions (hence, multiset)
  - $\mathcal{J}$ is like the set of hyperedges of a multi-hypergraph whose set of nodes is $[n]$

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  - $\mathcal{J}$ is like the set of hyperedges of a multi-hypergraph whose set of nodes is $[n]$

Intuition (for $q = 1$): Consider any $i \in J$ for some $J \subseteq [n]$.

- X_i is conditioned on more variables in the joint distribution $\mathbf{X}_{[n]}$ than in the joint distribution $\mathbf{X}_J$, since $J \subseteq [n]$.
- So the uncertainty about X_i within $\mathbf{X}_{[n]}$ is not more than within $\mathbf{X}_J$.

Example

Triangle Query $\Phi(x_1, x_2, x_3) = \psi_{12}(x_1, x_2) \otimes \psi_{23}(x_2, x_3) \otimes \psi_{13}(x_1, x_3)$

with hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and output:

output Φ

X_1	X_2	X_3
1	1	1
1	1	2
1	2	1
1	2	2
2	2	1
2	2	2

Example

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output Φ

X_1	X_2	X_3
1	1	1
1	1	2
1	2	1
1	2	2
2	2	1
2	2	2

- Choose $\mathcal{J} = \mathcal{E} = \{\{1,2\},\{2,3\},\{1,3\}\}$
- Each $i \in [3]$ occurs in at least two members of $\mathcal{J}$

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with hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and output:

output Φ

X_1	X_2	X_3	
1	1	1	$1/6$
1	1	2	$1/6$
1	2	1	$1/6$
1	2	2	$1/6$
2	2	1	$1/6$
2	2	2	$1/6$

- Choose $\mathcal{J} = \mathcal{E} = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$
- Each $i \in [3]$ occurs in at least two members of $\mathcal{J}$

$2H(O) = 2 \log 6 \approx 1.56$

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output Φ

X_1	X_2	X_3	
1	1	1	$1/6$
1	1	2	$1/6$
1	2	1	$1/6$
1	2	2	$1/6$
2	2	1	$1/6$
2	2	2	$1/6$

marginalised output $\bigoplus_{x_3} \Phi$

X_1	X_2	
1	1	$1/3$
1	2	$1/3$
2	2	$1/3$

- Choose $\mathcal{J} = \mathcal{E} = \{\{1,2\},\{2,3\},\{1,3\}\}$
- Each $i \in [3]$ occurs in at least two members of $\mathcal{J}$

$2H(O) = 2\log 6 \approx 1.56$ $\quad\quad$ $\log 3$

$H(O_1, O_2)$

Example

Triangle Query $\Phi(x_1, x_2, x_3) = \psi_{12}(x_1, x_2) \otimes \psi_{23}(x_2, x_3) \otimes \psi_{13}(x_1, x_3)$

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output Φ

X_1	X_2	X_3	
1	1	1	$1/6$
1	1	2	$1/6$
1	2	1	$1/6$
1	2	2	$1/6$
2	2	1	$1/6$
2	2	2	$1/6$

marginalised output $\bigoplus_{x_3} \Phi$

X_1	X_2	
1	1	$1/3$
1	2	$1/3$
2	2	$1/3$

marginalised output $\bigoplus_{x_1} \Phi$

X_2	X_3	
1	1	$1/6$
1	2	$1/6$
2	1	$1/3$
2	2	$1/3$

- Choose $\mathcal{J} = \mathcal{E} = \{\{1,2\},\{2,3\},\{1,3\}\}$
- Each $i \in [3]$ occurs in at least two members of $\mathcal{J}$

$2H(O) = 2\log 6 \approx 1.56$

$$\log 3 + \frac{2}{6}\log 6 + \frac{2}{3}\log 3$$

$H(O_1, O_2)$ $\quad H(O_2, O_3)$

Example

Triangle Query $\Phi(x_1, x_2, x_3) = \psi_{12}(x_1, x_2) \otimes \psi_{23}(x_2, x_3) \otimes \psi_{13}(x_1, x_3)$

with hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and output:

output Φ

X_1	X_2	X_3	
1	1	1	1/6
1	1	2	1/6
1	2	1	1/6
1	2	2	1/6
2	2	1	1/6
2	2	2	1/6

marginalised output $\bigoplus_{x_3} \Phi$

X_1	X_2	
1	1	1/3
1	2	1/3
2	2	1/3

marginalised output $\bigoplus_{x_1} \Phi$

X_2	X_3	
1	1	1/6
1	2	1/6
2	1	1/3
2	2	1/3

marginalised output $\bigoplus_{x_2} \Phi$

X_1	X_3	
1	1	1/3
1	2	1/3
2	1	1/6
2	2	1/6

- Choose $\mathcal{J} = \mathcal{E} = \{\{1,2\},\{2,3\},\{1,3\}\}$
- Each $i \in [3]$ occurs in at least two members of $\mathcal{J}$

$2H(O) = 2\log 6 \approx 1.56$

$$\underbrace{\log 3}_{H(O_1,O_2)} + \underbrace{\frac{2}{6}\log 6 + \frac{2}{3}\log 3}_{H(O_2,O_3)} + \underbrace{\frac{2}{6}\log 6 + \frac{2}{3}\log 3}_{H(O_1,O_3)}$$

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Triangle Query $\Phi(x_1, x_2, x_3) = \psi_{12}(x_1, x_2) \otimes \psi_{23}(x_2, x_3) \otimes \psi_{13}(x_1, x_3)$

with hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and output:

output Φ

X_1	X_2	X_3	
1	1	1	$1/6$
1	1	2	$1/6$
1	2	1	$1/6$
1	2	2	$1/6$
2	2	1	$1/6$
2	2	2	$1/6$

marginalised output $\bigoplus_{x_3} \Phi$

X_1	X_2	
1	1	$1/3$
1	2	$1/3$
2	2	$1/3$

marginalised output $\bigoplus_{x_1} \Phi$

X_2	X_3	
1	1	$1/6$
1	2	$1/6$
2	1	$1/3$
2	2	$1/3$

marginalised output $\bigoplus_{x_2} \Phi$

X_1	X_3	
1	1	$1/3$
1	2	$1/3$
2	1	$1/6$
2	2	$1/6$

- Choose $\mathcal{J} = \mathcal{E} = \{\{1,2\}, \{2,3\}, \{1,3\}\}$
- Each $i \in [3]$ occurs in at least two members of $\mathcal{J}$

$$2H(O) = 2\log 6 \approx 1.56 \leq 1.63 \approx \underbrace{\log 3}_{H(O_1, O_2)} + \underbrace{\frac{2}{6}\log 6 + \frac{2}{3}\log 3}_{H(O_2, O_3)} + \underbrace{\frac{2}{6}\log 6 + \frac{2}{3}\log 3}_{H(O_1, O_3)}$$

Proof of Shearer's Lemma

$$q \cdot H(\mathbf{X}_{[n]})$$

$$= q \cdot \sum_{i \in [n]} H(X_i \mid \mathbf{X}_{[i-1]}) \quad \text{Observation 1 on chain rule for joint entropy}$$

Proof of Shearer's Lemma

$$q \cdot H(\mathbf{X}_{[n]})$$

$$= q \cdot \sum_{i \in [n]} H(X_i \mid \mathbf{X}_{[i-1]}) \quad \text{Observation 1 on chain rule for joint entropy}$$

$$= \quad q \cdot H(X_1) \; + \quad q \cdot H(X_2 \mid X_1) \; + \ldots + \quad q \cdot H(X_n \mid \mathbf{X}_{[n-1]})$$

Proof of Shearer's Lemma

$$
\begin{aligned}
& q \cdot H(\mathbf{X}_{[n]}) \\
= {} & q \cdot \sum_{i \in [n]} H(X_i \mid \mathbf{X}_{[i-1]}) \quad \text{Observation 1 on chain rule for joint entropy} \\
= {} & \quad q \cdot H(X_1) \quad + \quad q \cdot H(X_2 \mid X_1) \quad + \ldots + \quad q \cdot H(X_n \mid \mathbf{X}_{[n-1]}) \\
& \qquad\quad \text{|}\wedge \qquad\qquad\qquad\quad \text{|}\wedge \qquad\qquad\qquad\qquad\qquad \text{|}\wedge \\
\leq {} & \sum_{J \in \mathcal{J} : 1 \in J} H(X_1) + \sum_{J \in \mathcal{J} : 2 \in J} H(X_2 \mid X_1) + \ldots + \sum_{J \in \mathcal{J} : n \in J} H(X_n \mid \mathbf{X}_{[n-1]})
\end{aligned}
$$

Since each i appears in at least q sets

Proof of Shearer's Lemma

$$q \cdot H(\mathbf{X}_{[n]})$$

$$= q \cdot \sum_{i \in [n]} H(X_i \mid \mathbf{X}_{[i-1]})$$ Observation 1 on chain rule for joint entropy

$$= \quad q \cdot H(X_1) \quad + \quad q \cdot H(X_2 \mid X_1) \quad + \ldots + \quad q \cdot H(X_n \mid \mathbf{X}_{[n-1]})$$

$$\leq \sum_{J \in \mathcal{J}: 1 \in J} H(X_1) + \sum_{J \in \mathcal{J}: 2 \in J} H(X_2 \mid X_1) + \ldots + \sum_{J \in \mathcal{J}: n \in J} H(X_n \mid \mathbf{X}_{[n-1]})$$

Since each i appears in at least q sets

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Observation 2: Conditioning on less variables does not decrease entropy

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$$= \sum_{J \in \mathcal{J}} \sum_{i \in J} H(X_i \mid \mathbf{X}_{[i-1] \cap J}) = \sum_{J \in \mathcal{J}} H(\mathbf{X}_J)$$ Observation 1 on chain rule

Connection to Join Output Size

FAQ $\Phi(\mathbf{x}) = \bigotimes_{S \in \mathcal{E}} \psi_S(\mathbf{x}_S)$ with hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and factor sizes $(N_S)_{S \in \mathcal{E}}$

- Let $(w_S)_{S \in \mathcal{E}}$ be any feasible solution to the linear program computing $\rho^*(\mathcal{H})$ with minimisation objective $\prod_{S \in \mathcal{E}} N_S^{w_S}$

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  - Choose natural numbers q and $(p_S)_{S\in\mathcal{E}}$ such that $w_S = \frac{p_S}{q}$ for all $S \in \mathcal{E}$
  - Let $\mathcal{J} \subseteq 2^{[n]}$ be a multiset that consists of p_S copies of each $S \in \mathcal{E}$
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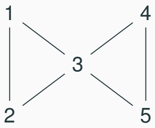
  This holds because the number of sets containing i is:

$$\sum_{S\in\mathcal{J}:i\in S} p_S = \sum_{S\in\mathcal{J}:i\in S} q \cdot w_S = q \cdot \underbrace{\sum_{S\in\mathcal{J}:i\in S} w_S}_{\geq 1 \text{ due to linear program}} \geq q$$

Example Connecting Shearer Setup with Feasible Solution for ρ^*

Hypergraph $\mathcal{H}$



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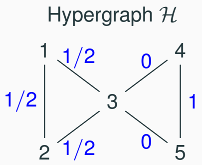

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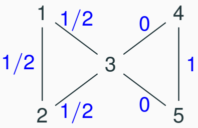

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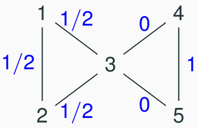

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- Then, $\mathcal{J} = \{\{1,2\},\{2,3\},\{1,3\},\{4,5\},\{4,5\}\}$

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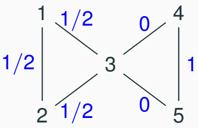

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- Then, $\mathcal{J} = \{\{1,2\},\{2,3\},\{1,3\},\{4,5\},\{4,5\}\}$

$\Longrightarrow$ Every $i \in [5]$ occurs in 2 sets in $\mathcal{J}$.

Putting Things Together

W.l.o.g assume $|\Phi| \neq 0$, otherwise the size bound trivially holds

Let $X = (\mathbf{X}_{[n]})$ be uniformly distributed over the output Φ

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This implies:

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The Lower Bound Argument

Lower Bound for Join Output Size

Consider an FAQ join $\Phi(\mathbf{x}) = \bigotimes_{S \in \mathcal{E}} \psi_S(\mathbf{x}_S)$ with hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$

We have shown:

- If input factors ψ_S are of size N, then $|\Phi| \leq N^{\rho^*(\mathcal{H})}$

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What we would like to show in the ideal case:

- If input factors ψ_S are of size N, then $|\Phi| \geq N^{\rho^*(\mathcal{H})}$
- This is not always possible

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We can however show:

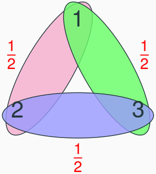
- For every N_0, we construct factors of size $N \geq N_0$ such that $|\Phi| \geq N^{\rho^*(\mathcal{H})}$
- This lower bound extends to factors of different sizes

Warm-Up: Size Bound for Triangle Join

Lower Bound on Triangle Join Output Size (1/2)

$$\Phi(x_1, x_2, x_3) = \psi_{12}(x_1, x_2) \otimes \psi_{23}(x_2, x_3) \otimes \psi_{13}(x_1, x_3)$$

Hypergraph $\mathcal{H}$



$$\rho^*(\mathcal{H}) = \frac{3}{2}$$

Lower Bound on Triangle Join Output Size (2/2)

$$\Phi(x_1, x_2, x_3) = \psi_{12}(x_1, x_2) \otimes \psi_{23}(x_2, x_3) \otimes \psi_{13}(x_1, x_3)$$

- We can construct input factors ψ_{ij} of size 4 with $|\Phi| = 4^{\frac{3}{2}} = 8$.

input ψ_{12}

X_1	X_2
1	1
1	2
2	1
2	2

= [2] $\times$ [2]

input ψ_{23}

X_2	X_3
1	1
1	2
2	1
2	2

= [2] $\times$ [2]

input ψ_{13}

X_1	X_3
1	1
1	2
2	1
2	2

= [2] $\times$ [2]

output Φ

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input ψ_{13}

X_1	X_3
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1	2
2	1
2	2

= [2] $\times$ [2]

output Φ

X_1	X_2	X_3
1	1	1
1	1	2
1	2	1
1	2	2
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- We next generalise the idea of this construction

Dual Linear Program

The dual of the linear program computing the fractional edge cover number ρ^*

LP for $\rho^*(\mathcal{H})$			Dual LP for $D(\mathcal{H})$		
minimise	$\sum_{S\in\mathcal{E}} w_S$		maximise	$\sum_{i\in[n]} v_i$	
subject to	$\sum_{S\in\mathcal{E}:v\in S} w_S \geq 1$	$\forall v \in \mathcal{V}$,	subject to	$\sum_{i\in S} v_i \leq 1$	$\forall S \in \mathcal{E}$,
	$0 \leq w_S \leq 1$	$\forall S \in \mathcal{E}$		$0 \leq v_i \leq 1$	$\forall i \in [n]$

- Left: Weights w_S assigned to hyperedges
- Right: Weights v_i assigned to nodes

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- Left: Weights w_S assigned to hyperedges
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By linear program duality: $\rho^*(\mathcal{H}) = D(\mathcal{H})$

Dual LP for Triangle Join

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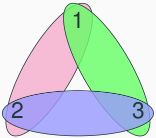

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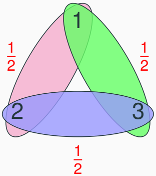

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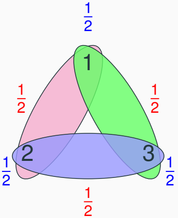

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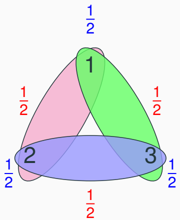

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$$D(\mathcal{H}) = \frac{3}{2}$$

For factors size N_0, take $N \geq N_0$ a power of 2.

Choose $p, q \in \mathbb{R}$ such that $\frac{1}{2} \cdot \log N = \frac{p}{q}$.

We construct $\psi_{12} = \psi_{13} = \psi_{23} = [2^p] \times [2^p]$ and then

- $|\psi_{12}| = |\psi_{13}| = |\psi_{23}| = 2^{2p} = 2^{q \log N} = (2^{\log N})^q = N^q$
- $|\Phi| = 2^{3p} = 2^{3q\frac{1}{2} \log N} = (2^{\log N})^{q\frac{3}{2}} = N^{q\frac{3}{2}} = (N^q)^{\frac{3}{2}}$

Size Lower Bound for Any Join

Construction of Input Factors

- Consider an optimal solution $(v_i)_{i \in [n]}$ to the linear program computing $D(\mathcal{H})$
- Choose natural numbers q, $(p_i)_{i \in [n]}$ such that $v_i \cdot \log N = \frac{p_i}{q}$
  - This works if $N \geq N_0$ is a power of 2, so $\log N$ is a natural number
- We construct in two steps input factors ψ_S of size N^q such that

$$|\Phi| \geq (N^q)^{\rho^*(\mathcal{H})}$$

Construction of Input Factors: Step 1

For each $S \in \mathcal{E}$, construct ψ'_S as the Cartesian product

$$\psi'_S = \times_{i \in S}[2^{p_i}]$$

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&= \prod_{i \in S} N^{q \cdot v_i} \\
&= \prod_{i \in S} (N^q)^{v_i} \\
&= (N^q)^{\sum_{i \in S} v_i} \\
&\leq N^q && \sum_{i \in S} v_i \leq 1
\end{aligned}$$

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For each $S \in \mathcal{E}$, construct an arbitrary ψ_S with $\psi_S \supseteq \psi'_S$ and $|\psi_S| = N^q$

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&= (N^q)^{D(\mathcal{H})} && (v_i)_{i\in[n]} \text{ is optimal solution to dual program}
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&= (N^q)^{\rho^*(\mathcal{H})} && \text{linear program duality}
\end{aligned}$$

Lower Bound in Case of Input Factors with Different Sizes

Given a join $\Phi(\mathbf{x}) = \bigotimes_{S \in \mathcal{E}} \psi_S(\mathbf{x}_S)$ with hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and input factor sizes N_S for $S \in \mathcal{E}$, the dual linear program extends to

$$
\begin{aligned}
\text{maximise} \quad & \textstyle\sum_{i \in [n]} v_i & \\
\text{subject to} \quad & \textstyle\sum_{i \in S} v_i \leq \log N_S & \forall S \in \mathcal{E}, \\
& v_i \geq 0 & \forall i \in [n]
\end{aligned}
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- Given an optimal solution $(v_i)_{i \in [n]}$ to the above program, we choose natural numbers q, $(p_i)_{i \in [n]}$ such that $v_i = \frac{p_i}{q}$

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- Let $(w_S)_{S \in \mathcal{E}}$ be an optimal solution to the linear program computing $\rho^*(\mathcal{H})$ with minimisation objective $\prod_{S \in \mathcal{E}} (N_S^q)^{w_S}$

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- We construct input factors $\psi_S \supseteq \times_{i \in S} [2^{p_i}]$ of sizes N_S^q
- Let $(w_S)_{S \in \mathcal{E}}$ be an optimal solution to the linear program computing $\rho^*(\mathcal{H})$ with minimisation objective $\prod_{S \in \mathcal{E}} (N_S^q)^{w_s}$
- We can show $|\Phi| \geq \prod_{S \in \mathcal{E}} (N_S^q)^{w_s}$