Efficient Algorithms, Spring 2021

2. Semirings

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Common Operations Needed by Computational Problems

Key observation: Computational problems commonly use

- sequences of two binary operations
- applied on a finite set of values from a given domain (e.g., numbers).

Typical operations: sum-product, or-and, min-product, min-sum, max-product.

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The answers lie in the mathematical notion of (semi)ring.

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  - $\oplus$ is associative: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
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Additional condition for ring: $(\mathbf{D}, \oplus, \mathbf{0})$ is a group, i.e.,

each element a has an additive inverse $-a$: $a \oplus -a = \mathbf{0}$

Examples of Semirings

$\mathbf{D}$	$\oplus$	$\otimes$	$\mathbf{0}$	$\mathbf{1}$	Name
$\{\text{true}, \text{false}\}$	$\vee$	$\wedge$	false	true	Boolean
$\mathbb{N}$	$+$	$*$	0	1	natural sum-product
$\mathbb{Z}$	$+$	$*$	0	1	integer sum-product
$(0, \infty]$	min	$*$	∞	1	min-product
$[0, \infty)$	max	$*$	0	1	max-product
$(-\infty, \infty]$	min	$+$	∞	0	min-sum
$[-\infty, \infty)$	max	$+$	$-\infty$	0	max-sum
$[-\infty, \infty]$	max	min	$-\infty$	∞	max-min
$\mathbb{N}[\mathbf{X}]$	$+$	$*$	0	1	polynomials over **X**
$(\mathbb{R}^{m\times n}, \mathbb{R}^{n\times n})$	$+_{\text{i}}$	$*_{\text{i}}$	$(\mathbf{0}_{0\times n}, \mathbf{0}_{n\times n})$	$(\mathbf{0}_{m\times n}, \mathbf{0}_{n\times n})$	inner-product
$(\mathbb{R}^{m\times n}, \mathbb{R}^{m\times m})$	$+_{\text{o}}$	$*_{\text{o}}$	$(\mathbf{0}_{0\times n}, \mathbf{0}_{0\times 0})$	$(\mathbf{0}_{m\times n}, \mathbf{0}_{m\times m})$	outer-product

Boolean Semiring

$(\{\text{true}, \text{false}\}, \vee, \wedge, \text{false}, \text{true})$ is the Boolean semiring

- Two elements: true and false; $\vee$ is the logical OR, $\wedge$ is the logical AND
- No ring since **1** (true) has no additive inverse: $\nexists x : \text{true} \ \vee \ x = \text{false}$

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$$\text{true} \ \wedge (\text{false} \vee \text{true})$$

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Where is it used?

- Constraint satisfaction problems
- Boolean conjunctive queries
- SAT

Natural Sum-Product Semiring

$(\mathbb{N}, +, *, 0, 1)$ is the natural sum-product semiring

- Domain: natural numbers including 0
- $+$ is arithmetic addition, $*$ is arithmetic multiplication
- No ring since no element besides 0 has an additive inverse:
  e.g., 1 has no inverse: $\nexists x \in \mathbb{N} : 1 + x = 0$

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Where is it used?

- Counting the number of tuples in answers to queries over relational data

Variations of the Sum-Product Semiring

Integer sum-product semiring $(\mathbb{Z}, +, *, 0, 1)$

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- Where is it used?
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Real sum-product semiring $(\mathbb{R}, +, *, 0, 1)$

- Domain: nonnegative reals
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- Ring since no element has an additive inverse
- Where is it used?
  - Inference in probabilistic graphical models
  - Matrix operations: Matrix chain multiplication, Permanent, DFT

Max-Product Semiring

Max-product semiring $([0, \infty), \max, *, 0, 1)$

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Where is it used?

- Maximum a-posteriori in probabilistic state machines and graphical models
- Maximum likelihood decoder for linear codes

Polynomial Semiring

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- $\mathbb{N}[\mathbf{X}]$ is the set of polynomials over variables in **X** and coefficients in $\mathbb{N}$
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Example with polynomials $a = 2x + 3y$ and $b = x + 2z$:

$$a + b = 2x + 3y + x + 2z = 3x + 3y + 2z$$

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Where is it used?

- Provenance information, where variables are identifiers of tuples in relations
- If variables are random: Probabilistic databases
- If variables are multiplicities: Bag semantics for relations

Inner-Product Semiring

Matrix $\mathbf{X} = [\mathbf{x}_1 \cdots \mathbf{x}_n] \in \mathbb{R}^{m \times n}$.

Define semiring for inner-product matrix $\mathbf{X}^\mathsf{T}\mathbf{X} = \begin{pmatrix} \mathbf{x}_1^\mathsf{T}\mathbf{x}_1 & \cdots & \mathbf{x}_1^\mathsf{T}\mathbf{x}_n \\ \vdots & & \\ \mathbf{x}_n^\mathsf{T}\mathbf{x}_1 & \cdots & \mathbf{x}_n^\mathsf{T}\mathbf{x}_n \end{pmatrix}$.

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- $a +_\mathrm{i} b = \left(\begin{pmatrix} \mathbf{X}_a \\ \mathbf{X}_b \end{pmatrix}, \mathbf{Q}_a + \mathbf{Q}_b \right)$
- $a *_\mathrm{i} b = \begin{cases} (\mathbf{0}_{0 \times n}, \mathbf{0}_{n \times n}) & \text{if } a \text{ or } b \text{ are } (\mathbf{0}_{0 \times n}, \mathbf{0}_{n \times n}) \\ (\mathbf{X}_a + \mathbf{X}_b, \mathbf{X}_a^\mathsf{T}\mathbf{X}_b + \mathbf{X}_b^\mathsf{T}\mathbf{X}_a + \mathbf{Q}_a + \mathbf{Q}_b) & \text{otherwise} \end{cases}$

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Where is it used?

- Covariance matrix used, e.g., for learning regression models

Inner-Product Semiring: Multiplication Example

Let $\mathbf{X} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3]$. Then $\mathbf{X}^\mathsf{T}\mathbf{X} = \begin{pmatrix} \mathbf{x}_1^\mathsf{T}\mathbf{x}_1 & \mathbf{x}_1^\mathsf{T}\mathbf{x}_2 & \mathbf{x}_1^\mathsf{T}\mathbf{x}_3 \\ \mathbf{x}_2^\mathsf{T}\mathbf{x}_1 & \mathbf{x}_2^\mathsf{T}\mathbf{x}_2 & \mathbf{x}_2^\mathsf{T}\mathbf{x}_3 \\ \mathbf{x}_3^\mathsf{T}\mathbf{x}_1 & \mathbf{x}_3^\mathsf{T}\mathbf{x}_2 & \mathbf{x}_3^\mathsf{T}\mathbf{x}_3 \end{pmatrix}$

Inner-Product Semiring: Multiplication Example

Let $\mathbf{X} = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3]$. Then $\mathbf{X}^\mathsf{T}\mathbf{X} = \begin{pmatrix} \mathbf{x}_1^\mathsf{T}\mathbf{x}_1 & \mathbf{x}_1^\mathsf{T}\mathbf{x}_2 & \mathbf{x}_1^\mathsf{T}\mathbf{x}_3 \\ \mathbf{x}_2^\mathsf{T}\mathbf{x}_1 & \mathbf{x}_2^\mathsf{T}\mathbf{x}_2 & \mathbf{x}_2^\mathsf{T}\mathbf{x}_3 \\ \mathbf{x}_3^\mathsf{T}\mathbf{x}_1 & \mathbf{x}_3^\mathsf{T}\mathbf{x}_2 & \mathbf{x}_3^\mathsf{T}\mathbf{x}_3 \end{pmatrix}$

Consider two elements $a = (\underbrace{[\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{0}]}_{\mathbf{X}_a}, \underbrace{\mathbf{X}_a^\mathsf{T}\mathbf{X}_a}_{\mathbf{Q}_a})$ and $b = (\underbrace{[\mathbf{0}\ \mathbf{0}\ \mathbf{x}_3]}_{\mathbf{X}_b}, \underbrace{\mathbf{X}_b^\mathsf{T}\mathbf{X}_b}_{\mathbf{Q}_b})$.

$\mathbf{X}_a$ and $\mathbf{X}_b$ have the same dimensions!

Then, $a *_\mathrm{i} b \stackrel{def}{=} (\underbrace{\mathbf{X}_a + \mathbf{X}_b}_{\mathbf{X}}, \underbrace{\mathbf{Q}_a + \mathbf{Q}_b + \mathbf{X}_a^\mathsf{T}\mathbf{X}_b + \mathbf{X}_b^\mathsf{T}\mathbf{X}_a}_{\mathbf{Q}=\mathbf{X}^\mathsf{T}\mathbf{X}})$

Inner-Product Semiring: Multiplication Example

Let $\mathbf{X} = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3]$. Then $\mathbf{X}^\top\mathbf{X} = \begin{pmatrix} \mathbf{x}_1^\top\mathbf{x}_1 & \mathbf{x}_1^\top\mathbf{x}_2 & \mathbf{x}_1^\top\mathbf{x}_3 \\ \mathbf{x}_2^\top\mathbf{x}_1 & \mathbf{x}_2^\top\mathbf{x}_2 & \mathbf{x}_2^\top\mathbf{x}_3 \\ \mathbf{x}_3^\top\mathbf{x}_1 & \mathbf{x}_3^\top\mathbf{x}_2 & \mathbf{x}_3^\top\mathbf{x}_3 \end{pmatrix}$

Consider two elements $a = (\underbrace{[\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{0}]}_{\mathbf{X}_a}, \underbrace{\mathbf{X}_a^\top\mathbf{X}_a}_{\mathbf{Q}_a})$ and $b = (\underbrace{[\mathbf{0}\ \mathbf{0}\ \mathbf{x}_3]}_{\mathbf{X}_b}, \underbrace{\mathbf{X}_b^\top\mathbf{X}_b}_{\mathbf{Q}_b})$.

$\mathbf{X}_a$ and $\mathbf{X}_b$ have the same dimensions!

Then, $a *_{\text{i}} b \stackrel{def}{=} (\underbrace{\mathbf{X}_a + \mathbf{X}_b}_{\mathbf{X}}, \underbrace{\mathbf{Q}_a + \mathbf{Q}_b + \mathbf{X}_a^\top\mathbf{X}_b + \mathbf{X}_b^\top\mathbf{X}_a}_{\mathbf{Q}=\mathbf{X}^\top\mathbf{X}})$

$$\underbrace{\underbrace{\begin{pmatrix} \mathbf{x}_1^\top\mathbf{x}_1 & \mathbf{x}_1^\top\mathbf{x}_2 & \mathbf{0} \\ \mathbf{x}_2^\top\mathbf{x}_1 & \mathbf{x}_2^\top\mathbf{x}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}}_{\mathbf{Q}_a} + \underbrace{\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{x}_3^\top\mathbf{x}_3 \end{pmatrix}}_{\mathbf{Q}_b} + \underbrace{\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{x}_1^\top\mathbf{x}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{x}_2^\top\mathbf{x}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}}_{\mathbf{X}_a^\top\mathbf{X}_b} + \underbrace{\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{x}_1^\top\mathbf{x}_3 & \mathbf{x}_2^\top\mathbf{x}_3 & \mathbf{0} \end{pmatrix}}_{\mathbf{X}_b^\top\mathbf{X}_a}}_{\mathbf{X}^\top\mathbf{X}}$$

Inner-Product Semiring: Summation Example

Let $\mathbf{X} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \\ \mathbf{y}_1 & \mathbf{y}_2 \end{pmatrix}$, $\mathbf{X}_a = [\mathbf{x}_1\ \mathbf{x}_2] \in \mathbb{R}^{m_a \times 2}$ and $\mathbf{X}_b = [\mathbf{y}_1\ \mathbf{y}_2] \in \mathbb{R}^{m_b \times 2}$.

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Consider two elements $a = (\mathbf{X}_a, \underbrace{\mathbf{X}_a^\top \mathbf{X}_a}_{\mathbf{Q}_a})$ and $b = (\mathbf{X}_b, \underbrace{\mathbf{X}_b^\top \mathbf{X}_b}_{\mathbf{Q}_b})$.

$\mathbf{X}_a$ and $\mathbf{X}_b$ have the same number of columns (in this case two)!

Then,

$$a +_{\mathrm{i}} b \stackrel{def}{=} \left(\underbrace{\begin{pmatrix} \mathbf{X}_a \\ \mathbf{X}_b \end{pmatrix}}_{\mathbf{X}}, \underbrace{\mathbf{Q}_a + \mathbf{Q}_b}_{\mathbf{X}^\top \mathbf{X}} \right)$$

Inner-Product Semiring: Summation Example

Let $\mathbf{X} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \\ \mathbf{y}_1 & \mathbf{y}_2 \end{pmatrix}$, $\mathbf{X}_a = [\mathbf{x}_1 \ \mathbf{x}_2] \in \mathbb{R}^{m_a \times 2}$ and $\mathbf{X}_b = [\mathbf{y}_1 \ \mathbf{y}_2] \in \mathbb{R}^{m_b \times 2}$.

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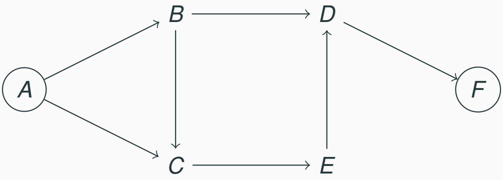
$$\mathbf{Q}_a + \mathbf{Q}_b = \underbrace{\begin{pmatrix} \mathbf{x}_1^\top \mathbf{x}_1 & \mathbf{x}_1^\top \mathbf{x}_2 \\ \mathbf{x}_2^\top \mathbf{x}_1 & \mathbf{x}_2^\top \mathbf{x}_2 \end{pmatrix}}_{\mathbf{X}_a^\top \mathbf{X}_a} + \underbrace{\begin{pmatrix} \mathbf{y}_1^\top \mathbf{y}_1 & \mathbf{y}_1^\top \mathbf{y}_2 \\ \mathbf{y}_2^\top \mathbf{y}_1 & \mathbf{y}_2^\top \mathbf{y}_2 \end{pmatrix}}_{\mathbf{X}_b^\top \mathbf{X}_b} = \underbrace{\begin{pmatrix} \mathbf{x}_1^\top \mathbf{x}_1 + \mathbf{y}_1^\top \mathbf{y}_1 & \mathbf{x}_1^\top \mathbf{x}_2 + \mathbf{y}_1^\top \mathbf{y}_2 \\ \mathbf{x}_2^\top \mathbf{x}_1 + \mathbf{y}_2^\top \mathbf{y}_1 & \mathbf{x}_2^\top \mathbf{x}_2 + \mathbf{y}_2^\top \mathbf{y}_2 \end{pmatrix}}_{\mathbf{X}^\top \mathbf{X}}$$

Problem 1: Algebraic Path Problem

The Algebraic Path Problem by Examples

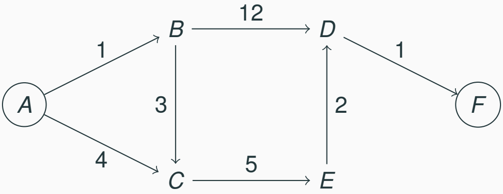
Consider the following directed graph with two distinguished nodes A and F



Next: Several graph path problems

- each solved by the same algorithm
- yet using a different semiring

Shortest Distance Using the Min-Sum Semiring



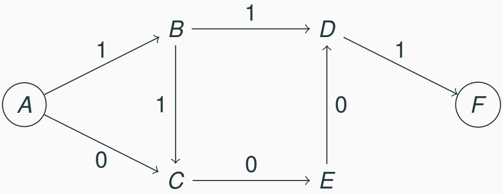
Compute the overall distance of each path from A to F

$$1 + 12 + 1 = 14$$
$$1 + 3 + 5 + 2 + 1 = 12$$
$$4 + 5 + 2 + 1 = 12$$

Then take the minimum distance of all these paths

$$\min\{14, 12, 12\} = 12$$

Connectivity Using the Max-Min Semiring



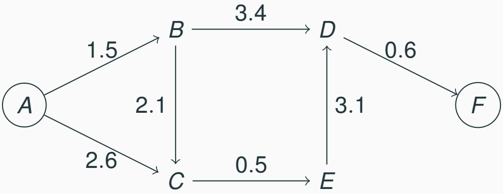
Compute whether each path connects A to F (0-edge means no connectivity)

$$\min\{1, 1, 1\} = 1$$
$$\min\{1, 1, 0, 0, 1\} = 0$$
$$\min\{0, 0, 0, 1\} = 0$$

Then compute whether there is at least a path connecting A to F only via 1-edges

$$\max\{1, 0, 0\} = 1$$

Largest Capacity Using the Max-Min Semiring



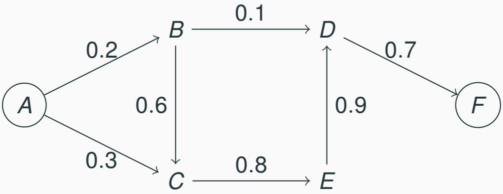
Compute the capacity along each path from A to F

$$\min\{1.5, 3.4, 0.6\} = 0.6$$
$$\min\{1.5, 2.1, 0.5, 3.1, 0.6\} = 0.5$$
$$\min\{2.6, 0.5, 3.1, 0.6\} = 0.5$$

Then compute the largest possible capacity of any path from A to F

$$\max\{0.6, 0.5, 0.5\} = 0.6$$

Maximum Reliability using the Max-Product Semiring



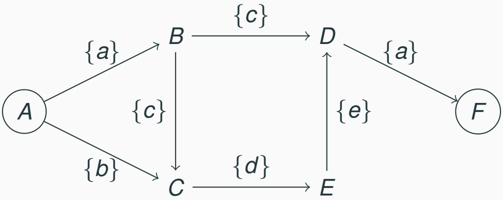
Compute the reliability along each path from A to F

$$0.2 \cdot 0.1 \cdot 0.7 = 0.014$$
$$0.2 \cdot 0.6 \cdot 0.8 \cdot 0.9 \cdot 0.7 = 0.06048$$
$$0.3 \cdot 0.8 \cdot 0.9 \cdot 0.7 = 0.1512$$

Then compute the maximum reliability from A to F

$$\max\{0.014, 0.06048, 0.1512\} = 0.1512$$

Language Accepted by Automaton using the ∪-∘ Semiring



Compute the string from start state *A* to final state *F*

$$\{a\} \circ \{c\} \circ \{a\} = \{aca\}$$
$$\{a\} \circ \{c\} \circ \{d\} \circ \{e\} \circ \{a\} = \{acdea\}$$
$$\{b\} \circ \{d\} \circ \{e\} \circ \{a\} = \{bdea\}$$

Then compute the set of all such possible strings

$$\bigcup\{\{aca\}, \{acdea\}, \{bdea\}\} = \{aca, acdea, bdea\}$$

Summing Up: The Algebraic Path Problem

- Previous slides: Path problems over different semirings
- Let $\mathbf{X}$ = matrix of edge weights
- Such path problems require computing

$$\mathbf{P} = \bigoplus_{r \geq 0} \mathbf{X}^r = \underbrace{\mathbf{I} + \mathbf{X} \oplus \mathbf{X}^2 \oplus \ldots}_{\substack{\text{possibly infinite series of semiring matrices} \\ \text{admits solution when series converges}}}$$

Summing Up: The Algebraic Path Problem

- Previous slides: Path problems over different semirings
- Let $\mathbf{X}$ = matrix of edge weights
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$$\mathbf{P} = \bigoplus_{r \geq 0} \mathbf{X}^r = \underbrace{\mathbf{I} + \mathbf{X} \oplus \mathbf{X}^2 \oplus \dots}_{\substack{\text{possibly infinite series of semiring matrices} \\ \text{admits solution when series converges}}}$$

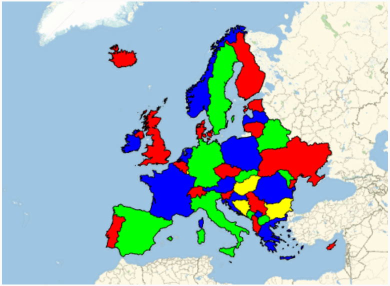
If the limit $\mathbf{P}$ exists, then it is to the least solution to the fixpoint equation

$$\mathbf{Y} = \mathbf{X}\,\mathbf{Y} + \mathbf{I}$$

- Path problems solved by one algorithm for a semiring fixpoint equation

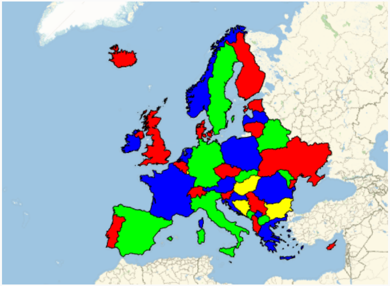
Problem 2: Satisfiability

Problem 1: Satisfiability 1/2



- Map colouring: Europe's countries can be coloured using 4 colours such that no neighbouring two countries have the same colour.
- The four colour map theorem says that this can be done for any map (without *exclaves*).

Problem 1: Satisfiability 1/2



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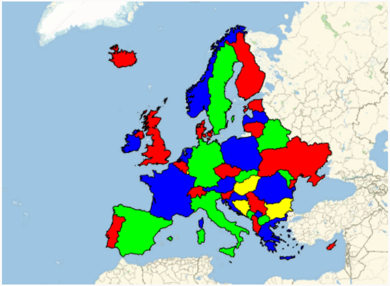
Question: Can we colour Europe's countries using only 3 colours?

Problem 1: Satisfiability 1/2



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Question: Can we colour Europe's countries using only 3 colours?

This question can be answered by modelling this 3-colorability problem by a propositional formula and checking its satisfiability.

This problem can be phrased in the Boolean semiring over Boolean variables

Problem 1: Satisfiability 2/2

- Say, we use the colours red, green, and blue.
- For each country (say, Switzerland) and each colour (say, red), we use a variable (say, R_{CH}) expressing that the country is coloured in that colour.

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Then, we can construct a formula Φ that is satisfiable iff Europe's map is 3-colourable:

$(R_{CH} \vee G_{CH} \vee B_{CH})$ "Switzerland has *at least* one colour."

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$\bigwedge$

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$\bigwedge$	
$(\neg R_{CH} \vee \neg G_{CH}) \wedge (\neg R_{CH} \vee \neg B_{CH}) \wedge (\neg G_{CH} \vee \neg B_{CH})$	"Switzerland has *at most* one colour."
$\bigwedge$	
$(\neg R_{CH} \vee \neg R_{DE}) \wedge (\neg G_{CH} \vee \neg G_{DE}) \wedge (\neg B_{CH} \vee \neg B_{DE})$	"Switzerland and Germany have different colours."
$\bigwedge$	
$\ldots$	$\ldots$

Problem 3: Database Queries

A Burgers & Hotdogs Use Case

Orders (O for short)

customer	day	dish
Elise	Monday	burger
Elise	Friday	burger
Steve	Friday	hotdog
Joe	Friday	hotdog

Dish (D for short)

dish	item
burger	patty
burger	onion
burger	bun
hotdog	bun
hotdog	onion
hotdog	sausage

Items (I for short)

item	price
patty	6
onion	2
bun	2
sausage	4

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Items (I for short)

item	price
patty	6
onion	2
bun	2
sausage	4

Consider the natural join of the above relations:

O(customer, day, dish) $\bowtie$ D(dish, item) $\bowtie$ I(item, price)

customer	day	dish	item	price
Elise	Monday	burger	patty	6
Elise	Monday	burger	onion	2
Elise	Monday	burger	bun	2
Elise	Friday	burger	patty	6
Elise	Friday	burger	onion	2
Elise	Friday	burger	bun	2
...	...	...	...	...

Burgers & Hotdogs in Relational Algebra

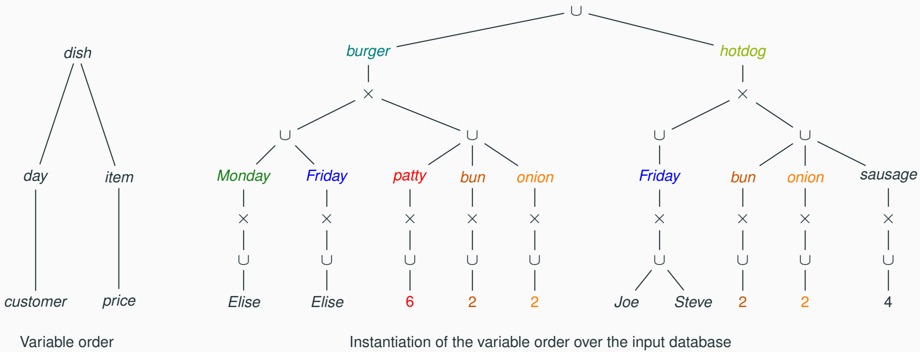
O(customer, day, dish) $\bowtie$ D(dish, item) $\bowtie$ I(item, price)

customer	day	dish	item	price
Elise	Monday	burger	patty	6
Elise	Monday	burger	onion	2
Elise	Monday	burger	bun	2
Elise	Friday	burger	patty	6
Elise	Friday	burger	onion	2
Elise	Friday	burger	bun	2
...	...	...	...	...

An algebraic encoding in the ∪-× semiring:

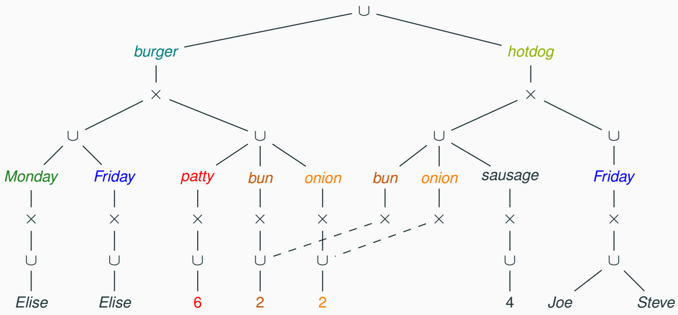
$$
\begin{aligned}
&\mathit{Elise} \times \mathit{Monday} \times \mathit{burger} \times \text{patty} \times 6 \cup \\
&\mathit{Elise} \times \mathit{Monday} \times \mathit{burger} \times \text{onion} \times 2 \cup \\
&\mathit{Elise} \times \mathit{Monday} \times \mathit{burger} \times \text{bun} \times 2 \cup \\
&\mathit{Elise} \times \mathit{Friday} \times \mathit{burger} \times \text{patty} \times 6 \cup \\
&\mathit{Elise} \times \mathit{Friday} \times \mathit{burger} \times \text{onion} \times 2 \cup \\
&\mathit{Elise} \times \mathit{Friday} \times \mathit{burger} \times \text{bun} \times 2 \cup \ldots
\end{aligned}
$$

The Union-Product Semiring Allows for Factorised Join Representation



There are several algebraically equivalent factorised joins defined by distributivity of Cartesian product $\times$ over union $\cup$ and their commutativity.

Factorised Aggregate Computation by Changing the Semiring

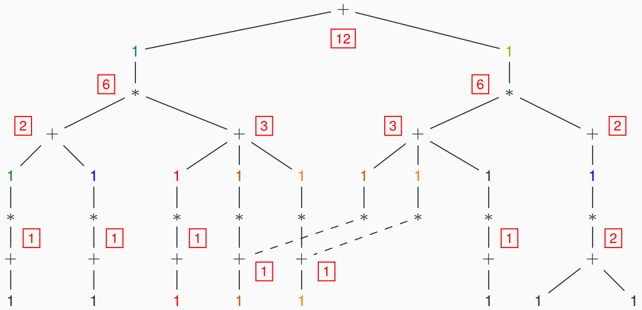


`COUNT`-ing the join size done in one pass over the factorisation:

- values $\mapsto$ 1,
- $\cup \mapsto +$, $\times \mapsto *$.

Effectively, we changed to the sum-product semiring

Factorised Aggregate Computation



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Effectively, we changed to the sum-product semiring

Problem 4: Medical Diagnosis with Probabilistic Models

Medical Diagnosis with Probabilistic Models

Patient, who recently returned from Asia, complains about shortness breath (Dyspnea). What is the probability that she suffers from Bronchitis?

Medical diagnosis using the joint probability distribution of random variables:

- Patient information: Visit to Asia (A), Smoking (S)
- Diseases: Tuberculosis (T), Lung Cancer (L), Bronchitis (B)
- Diagnostic Tests: X-Ray Result (X), Dyspnea (D)

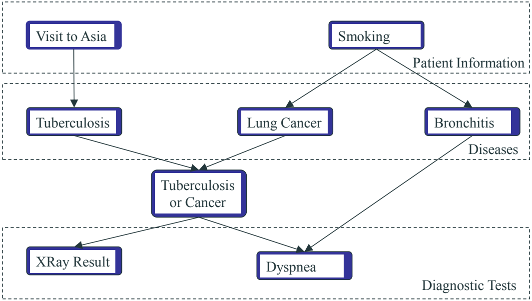
Key AI challenge: Learn such distributions, allow efficient inference over them

Much development on Bayesian Networks and Probabilistic Graphical Models

Bayesian Network for Our Medical Diagnosis

A	$P(A)$
T	.01
F	.99

S	$P(S)$
T	.4
F	.6



Network represents a knowledge structure that models the relationship between diseases, their causes and effects, patient information and diagnostic tests

A	T	$P(T\|A)$
T	T	.05
T	F	.95
F	T	.01
F	F	.99

S	B	$P(B\|S)$
T	T	.6
T	F	.4
F	T	.3
F	F	.7

S	L	$P(L\|S)$
T	T	.1
T	F	.9
F	T	.01
F	F	.99

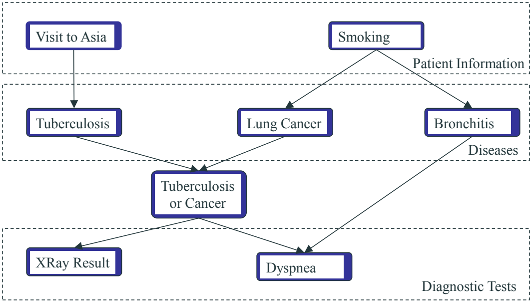
T	L	O	$P(O\|T, L)$
T	T	T	1
T	T	F	0
T	F	T	1
T	F	F	0
F	T	T	1
F	T	F	0
F	F	T	0
F	F	F	1

O	B	D	$P(D\|O, B)$
T	T	T	.9
T	T	F	.1
T	F	T	.7
T	F	F	.3
F	T	T	.8
F	T	F	.2
F	F	T	.1
F	F	F	.9

O	X	$P(X\|O)$
T	T	.98
T	F	.02
F	T	.05
F	F	.95

Variable O:
Tuberculosis or Cancer

Bayesian Network for Our Medical Diagnosis



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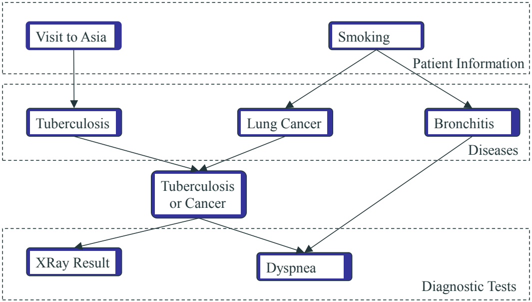
Bayesian Network for Our Medical Diagnosis

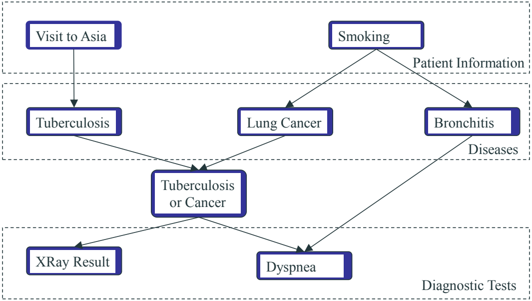


Network represents a knowledge structure that models the relationship between diseases, their causes and effects, patient information and diagnostic tests

The Bayesian Network structures the joint pdf using conditional independence:

$$P(A, T, S, L, B, O, X, D) = P(A) \cdot P(T|A) \cdot P(S) \cdot P(L|S) \cdot P(B|S) \cdot P(O|T, L) \cdot P(X|O) \cdot P(D|O, B)$$

Bayesian Network for Our Medical Diagnosis



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$$P(A, T, S, L, B, O, X, D) = P(A) \cdot P(T|A) \cdot P(S) \cdot P(L|S) \cdot P(B|S) \cdot P(O|T, L) \cdot P(X|O) \cdot P(D|O, B)$$

Our inference query is expressed as: $P(B|A, D) = \frac{P(A,B,D)}{P(A,D)}$ using Bayes rule.

Computing the Inference Query $P(A, B, D)$

We aggregate away all other variables not relevant to our query:

$$P(A, B, D) = \sum_{O,L,S,T,X} P(A, T, S, L, B, O, X, D)$$

How can we do this efficiently?

- $P(A, T, S, L, B, O, X, D)$ is a truth table with 2^8 rows
- In general, for n variables, we get 2^n rows!
- Quick Medical Reference has > 5000 variables

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Two main ideas (which are the pillars of our course):

- Exploit the factorised joint distribution
  - $P(A, T, S, L, B, O, X, D)$ factorised as the join of 8 conditional probability tables
- Apply the semiring's distributivity law of product over sum

Computing the Inference Query $P(A, B, D)$

$$P(A, B, D) = \sum_{O,L,S,T,X} P(A, T, S, L, B, O, X, D)$$

Computing the Inference Query $P(A, B, D)$

$$
\begin{aligned}
P(A, B, D) &= \sum_{O,L,S,T,X} P(A, T, S, L, B, O, X, D) \\
&= \sum_{O,L,S,T,X} P(A) \cdot P(D|O, B) \cdot P(X|O) \cdot P(T|A) \cdot P(O|T, L) \cdot P(S) \cdot P(L|S) \cdot P(B|S)
\end{aligned}
$$

Computing the Inference Query $P(A, B, D)$

$$
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&= \sum_{O,L,T,X} P(A) \cdot P(D|O, B) \cdot P(X|O) \cdot P(T|A) \cdot P(O|T, L) \cdot \underbrace{\sum_{S} P(S) \cdot P(L|S) \cdot P(B|S)}_{\phi_1(L,B)}
\end{aligned}
$$

Computing the Inference Query $P(A, B, D)$

$$
\begin{aligned}
P(A, B, D) &= \sum_{O,L,S,T,X} P(A, T, S, L, B, O, X, D) \\
&= \sum_{O,L,S,T,X} P(A) \cdot P(D|O, B) \cdot P(X|O) \cdot P(T|A) \cdot P(O|T, L) \cdot P(S) \cdot P(L|S) \cdot P(B|S) \\
&= \sum_{O,L,T,X} P(A) \cdot P(D|O, B) \cdot P(X|O) \cdot P(T|A) \cdot P(O|T, L) \cdot \underbrace{\sum_{S} P(S) \cdot P(L|S) \cdot P(B|S)}_{\phi_1(L,B)} \\
&= \sum_{O,T,X} P(A) \cdot P(D|O, B) \cdot P(X|O) \cdot P(T|A) \cdot \underbrace{\sum_{L} P(O|T, L) \cdot \phi_1(L, B)}_{\phi_2(T,O,B)}
\end{aligned}
$$

Computing the Inference Query $P(A, B, D)$

$$\begin{aligned}
P(A, B, D) &= \sum_{O,L,S,T,X} P(A, T, S, L, B, O, X, D) \\
&= \sum_{O,L,S,T,X} P(A) \cdot P(D|O, B) \cdot P(X|O) \cdot P(T|A) \cdot P(O|T, L) \cdot P(S) \cdot P(L|S) \cdot P(B|S) \\
&= \sum_{O,L,T,X} P(A) \cdot P(D|O, B) \cdot P(X|O) \cdot P(T|A) \cdot P(O|T, L) \cdot \underbrace{\sum_{S} P(S) \cdot P(L|S) \cdot P(B|S)}_{\phi_1(L,B)} \\
&= \sum_{O,T,X} P(A) \cdot P(D|O, B) \cdot P(X|O) \cdot P(T|A) \cdot \underbrace{\sum_{L} P(O|T, L) \cdot \phi_1(L, B)}_{\phi_2(T,O,B)} \\
&= \sum_{O,X} P(A) \cdot P(D|O, B) \cdot P(X|O) \cdot \underbrace{\sum_{T} P(T|A) \cdot \phi_2(T, O, B)}_{\phi_3(O,A,B)}
\end{aligned}$$

Computing the Inference Query $P(A, B, D)$

$$
\begin{aligned}
P(A, B, D) &= \sum_{O,L,S,T,X} P(A, T, S, L, B, O, X, D) \\
&= \sum_{O,L,S,T,X} P(A) \cdot P(D|O, B) \cdot P(X|O) \cdot P(T|A) \cdot P(O|T, L) \cdot P(S) \cdot P(L|S) \cdot P(B|S) \\
&= \sum_{O,L,T,X} P(A) \cdot P(D|O, B) \cdot P(X|O) \cdot P(T|A) \cdot P(O|T, L) \cdot \underbrace{\sum_{S} P(S) \cdot P(L|S) \cdot P(B|S)}_{\phi_1(L,B)} \\
&= \sum_{O,T,X} P(A) \cdot P(D|O, B) \cdot P(X|O) \cdot P(T|A) \cdot \underbrace{\sum_{L} P(O|T, L) \cdot \phi_1(L, B)}_{\phi_2(T,O,B)} \\
&= \sum_{O,X} P(A) \cdot P(D|O, B) \cdot P(X|O) \cdot \underbrace{\sum_{T} P(T|A) \cdot \phi_2(T, O, B)}_{\phi_3(O,A,B)} \\
&= \sum_{O} P(A) \cdot P(D|O, B) \cdot \underbrace{\sum_{X} P(X|O)}_{\phi_4(O)} \cdot \phi_3(O, A, B)
\end{aligned}
$$

Computing the Inference Query $P(A, B, D)$

$$
\begin{aligned}
P(A,B,D) &= \sum_{O,L,S,T,X} P(A,T,S,L,B,O,X,D) \\
&= \sum_{O,L,S,T,X} P(A) \cdot P(D|O,B) \cdot P(X|O) \cdot P(T|A) \cdot P(O|T,L) \cdot P(S) \cdot P(L|S) \cdot P(B|S) \\
&= \sum_{O,L,T,X} P(A) \cdot P(D|O,B) \cdot P(X|O) \cdot P(T|A) \cdot P(O|T,L) \cdot \underbrace{\sum_{S} P(S) \cdot P(L|S) \cdot P(B|S)}_{\phi_1(L,B)} \\
&= \sum_{O,T,X} P(A) \cdot P(D|O,B) \cdot P(X|O) \cdot P(T|A) \cdot \underbrace{\sum_{L} P(O|T,L) \cdot \phi_1(L,B)}_{\phi_2(T,O,B)} \\
&= \sum_{O,X} P(A) \cdot P(D|O,B) \cdot P(X|O) \cdot \underbrace{\sum_{T} P(T|A) \cdot \phi_2(T,O,B)}_{\phi_3(O,A,B)} \\
&= \sum_{O} P(A) \cdot P(D|O,B) \cdot \underbrace{\sum_{X} P(X|O)}_{\phi_4(O)} \cdot \phi_3(O,A,B) \\
&= P(A) \cdot \underbrace{\sum_{O} P(D|O,B) \cdot \phi_4(O) \cdot \phi_3(O,A,B)}_{\phi_5(A,B,D)} = P(A) \cdot \phi_5(A,B,D)
\end{aligned}
$$

Is the Rewritten Expression for the Inference Query $P(A, B, D)$ Better?

- Started with 8 (conditional probability) tables and joint pdf with 2^8 rows
- Pushed the summation (marginalisation) past the product
- Created intermediate results:
  - $\phi_1(L, B)$ has 2^2 rows
  - $\phi_2(T, O, B)$ has 2^3 rows
  - $\phi_3(O, A, B)$ has 2^3 rows
  - $\phi_4(O)$ has 2^1 rows
  - $\phi_5(A, B, D)$ has 2^3 rows

Probability of Most Likely Configuration

How to find the probability for the mode of the joint probability distribution?

Probability of Most Likely Configuration

How to find the probability for the mode of the joint probability distribution?

Change the semiring: From sum-product to max-product

- Sum-product semiring: $P(A, B, D) = \sum_{O,L,S,T,X} P(A, T, S, L, B, O, X, D)$

- Max-product semiring: $P(A, B, D) = \max_{O,L,S,T,X} P(A, T, S, L, B, O, X, D)$

Our previous optimisation remains the same!

Side Note: Working with Conditional Probability Tables

Task: Compute $\phi_1(S, L, B) = P(S) \cdot P(L|S) \cdot P(B|S)$

Recall the conditional probability tables (rows for $S = \texttt{F}, B = \texttt{F}, L = \texttt{F}$ not shown):

S	$P(S)$
T	.4

S	B	$P(B\|S)$
T	T	.6
F	T	.3

S	L	$P(L\|S)$
T	T	.1
F	T	.01

$P(S) \cdot P(L|S) \cdot P(B|S)$ is computed by the natural join (on S) of these tables:

S	L	B	$P(S)$	$P(L\|S)$	$P(B\|S)$	$P(S, L, B)$
T	T	T	.4	.1	.6	.024
		F	.4	.1	.4	.016
	F	T	.4	.9	.6	.216
		F	.4	.9	.4	.144
F	T	T	.6	.01	.3	.0018
		F	.6	.01	.7	.0042
	F	T	.6	.99	.3	.1782
		F	.6	.99	.7	.4158

Side Note: Working with Conditional Probability Tables

Task: Compute $\phi_1(L, B) = \bigoplus_S \phi_2(S, L, B)$

We marginalise out variable S according to semiring operation $\bigoplus$.

S	L	B	$P(S, L, B)$
T	T	T	.024
		F	.016
	F	T	.216
		F	.144
F	T	T	.0018
		F	.0042
	F	T	.1782
		F	.4158

Side Note: Working with Conditional Probability Tables

Task: Compute $\phi_1(L, B) = \bigoplus_S \phi_2(S, L, B)$

We marginalise out variable S according to semiring operation $\bigoplus$.

S	L	B	$P(S, L, B)$
T	T	T	.024
		F	.016
	F	T	.216
		F	.144
F	T	T	.0018
		F	.0042
	F	T	.1782
		F	.4158

$\overset{\sum_S}{\Rightarrow}$

L	B	$P(L, B)$
T	T	.024 + .0018
	F	.016 + .0042
F	T	.216 + .1782
	F	.144 + .4158

Side Note: Working with Conditional Probability Tables

Task: Compute $\phi_1(L, B) = \bigoplus_S \phi_2(S, L, B)$

We marginalise out variable S according to semiring operation $\bigoplus$.

L	B	$P(L, B)$
T	T	.024
T	F	.016
F	T	.216
F	F	.4158

$\overset{\max_S}{\Longleftarrow}$

S	L	B	$P(S, L, B)$
T	T	T	.024
		F	.016
	F	T	.216
		F	.144
F	T	T	.0018
		F	.0042
	F	T	.1782
		F	.4158

$\overset{\sum_S}{\Longrightarrow}$

L	B	$P(L, B)$
T	T	.024 + .0018
	F	.016 + .0042
F	T	.216 + .1782
	F	.144 + .4158

TL;DR: The Unusual Power of Semirings

Why are Semirings Relevant in Computer Science?

- They enable generic problem solving
  - by changing the semiring
  - the algorithm remains the same
- They reduce problem complexity
  - thanks to the distributivity law

Different semirings give different semantics of

- the same problem
- the same algorithm
- the same complexity
- the same implementation