Computation and Economics – Spring 2012
Section: Math-Refresher

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1. Notation

- A set is a collection of objects

  $A = \{a, b, c, \dots\}$, a is an **element** of A

  $a \in A$

- Let A be a set : $B \subseteq A$ **subset**

  $\{1, 2, 3\} = A$, $B = \{1, 2\} \Rightarrow B \subseteq A$

- $A \cup B$ **union** , $A \cap B$ **intersection** $\quad A = \{1, 2\}$, $B = \{2, 3\}$

  $A \cup B = \{1, 2, 3\}$, $A \cap B = \{2\}$

- $\emptyset$ **empty set** $\quad \emptyset = \{\}$

- $\mathcal{P}(A)$ **power set** : $A = \{\{\}, \{1\}, \{2\}, \{1, 2\}\}$

1. Notation

- $x = (x_1, x_2, \dots)$ ordered list

  x_1 component

  $(1,2) \neq (2,1)$

  $\{1,2\} = \{2,1\}$

- Productspace: A, B sets : $A \times B$

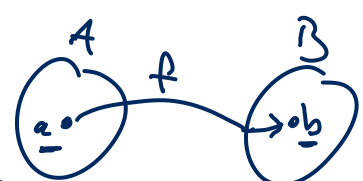
  $A = \{1,2\}$, $B = \{3,4\}$

  $\Rightarrow A \times B = \{(x_1, x_2) \mid x_1 \in A, x_2 \in B\} = \{(1,3),(1,4),(2,3),(2,4)\}$

- Mappings

  $f : A \longrightarrow B$

  $f(a) = b$ OR $a \overset{f}{\longmapsto} b$



$u : \mathbb{R} \longrightarrow \mathbb{R}$

$u(x) = 2x^2 - x^3$

$u(0) = 0$

1. <u>projection</u>

$$\pi_2 : A_1 \times A_2 \longrightarrow \underline{A_2}$$

$$(a_1, a_2) \longmapsto a_2$$

$$\pi_1 : A_1 \times A_2 \to A_1$$

$$(a_1, a_2) \mapsto a_1$$

$$\pi_2 : \underset{\{1,2\}}{A_1} \times \underset{\{3,4,5\}}{A_2} \to \{3,4,5\}$$

$$(1,3) \in A_1 \times A_2$$

$$(1,3) \longmapsto 3$$

2. Mathematical Proofs

2.1 direct implication :

A precondition

B consequence

$\Rightarrow A_1 \Rightarrow A_2 \Rightarrow \ldots \Rightarrow$

Z: $\underline{a, b \in \mathbb{N}}$, $\underline{\text{one of them is even}}$, then $\underline{a \cdot b \text{ is even}}$

(Def. $n \in \mathbb{N}$ is $\underline{\text{even}}$ if $n = 2 \cdot n'$, $n' \in \mathbb{N}$)

$\Rightarrow \quad a = 2a' \quad$ or $\quad b = 2b'$

$\Rightarrow \quad a \cdot b = 2\underbrace{a'b}_{\in \mathbb{N}} \quad$ or $\quad a \cdot b = a\,2\,b' = 2\underbrace{b'a}_{\in \mathbb{N}}$

$\Rightarrow$ $a \cdot b$ must be even

IB

2.2 proof by complete enumeration

Z: Agent in second-price sealed-bid auction is best off bidding his value

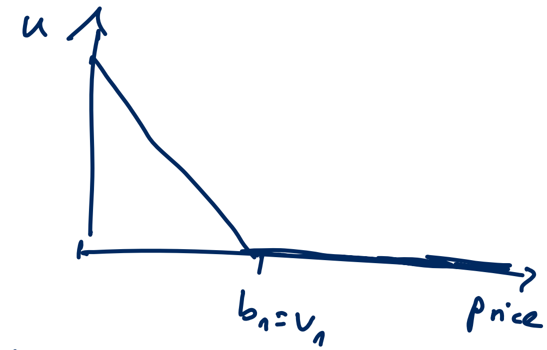
- n agents, each with a private value v_i for item
- $u_i = \begin{cases} 0 & \text{if he does NOT win the auction} \\ v_i - \text{price} & \text{if he wins and pays price "price"} \end{cases}$
- agents submit bids b_i, unknown
- Say i is highest bidder b_i, then the price is

$$\max \{ b_j \mid j \neq i \} = \text{price}$$

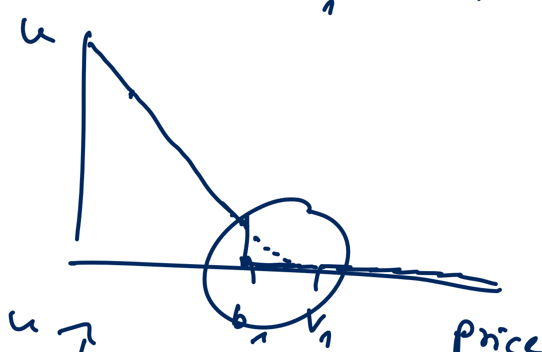
$$b_{-1} = \max \{ b_j \mid j \neq 1 \}$$

2.2. 1) $b_1 = v_1$ 2) $b_1 < v_1$ 3) $b_1 > v_1$

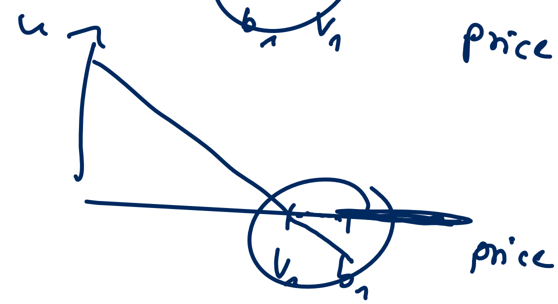
1)
$$u_1^{=} = \begin{cases} 0, & b_1 \le b_{-1} \\ v_1 - b_{-1}, & b_{-1} < b_1 \end{cases}$$



2)
$$u_1^{<} = \begin{cases} 0 & b_1 < v_1 \le b_{-1} \\ 0 & b_1 < b_{-1} \le v_1 \\ v_1 - b_{-1} & b_{-1} \le b_1 < v_1 \end{cases}$$



3)
$$u_1^{>} = \begin{cases} 0, & v_1 < b_1 \le b_{-1} \\ v_1 - b_{-1}, & v_1 < b_{-1} \le b_1 \\ v_1 - b_{-1}, & b_{-1} < v_1 < b_1 \end{cases}$$



$\Rightarrow$ optimal to bid my value

dominant strategy

$\square$

2.3 Contradiction : prove that the opposite is impossible

ξ: There are ∞ many primes

/A: $p_1, \ldots, p_n$ is list of all primes

$d := \underline{p_1 \cdot \ldots \cdot p_n} + 1 \Rightarrow$ no prime can divide every

d is prime $\Rightarrow$ ↯

$\Rightarrow d' \mid d$ $\quad d'$ divisor of d

d' is prime

d' not prime $\Rightarrow d'' \mid d'$

$\Rightarrow d'''''''$ is prime and $\underset{\shortparallel}{d''''''} \mid d$ ↯

$\Rightarrow$ /A was wrong $\Rightarrow \infty$ many primes $\square$

2.4. proof by induction

- show A for $n=0$ (base clause)
- $\boxed{\text{if } A \text{ holds for } N}$ then A holds for $N+1$

$n=0 \Rightarrow n=1 \Rightarrow n=2$ ----

$$\text{z: } \sum_{k=0}^{n} \delta^k = \frac{1-\delta^{n+1}}{1-\delta}$$

$n=0$:

$$\sum_{k=0}^{0} \delta^k = \delta^0 = 1 = \frac{1-\delta^{(0+1)}}{1-\delta} \quad \checkmark$$

2.4

$N \to N+1$: $\sum_{k=0}^{N+1} \delta^k = \overset{!}{=} \frac{1-\delta^{(N+1)+1}}{1-\delta}$

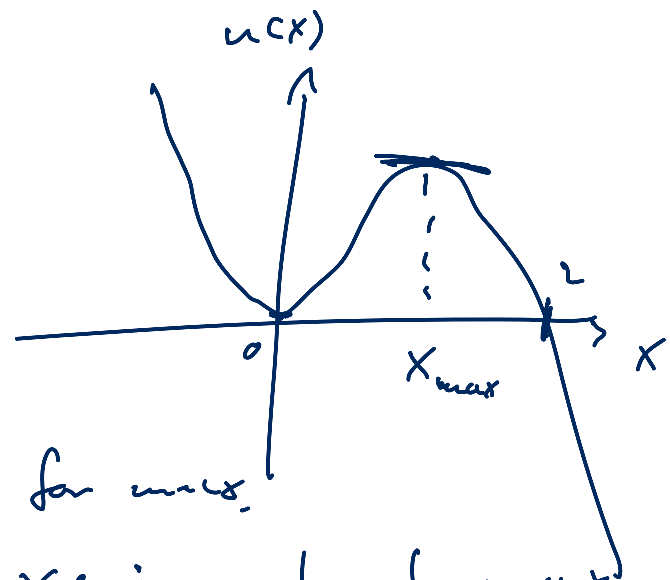
$$= \boxed{\sum_{k=0}^{N} \delta^k} + \delta^{N+1} = \boxed{\frac{1-\delta^{N+1}}{1-\delta}} + \delta^{N+1} \frac{1-\delta}{1-\delta}$$

induction hypothesis

$$= \frac{1-\delta^{N+1} + (1-\delta)\,\delta^{N+1}}{1-\delta} = \frac{1-\delta^{(N+1)+1}}{1-\delta}$$

3.2

$$u(x) = 2x^2 - x^3$$



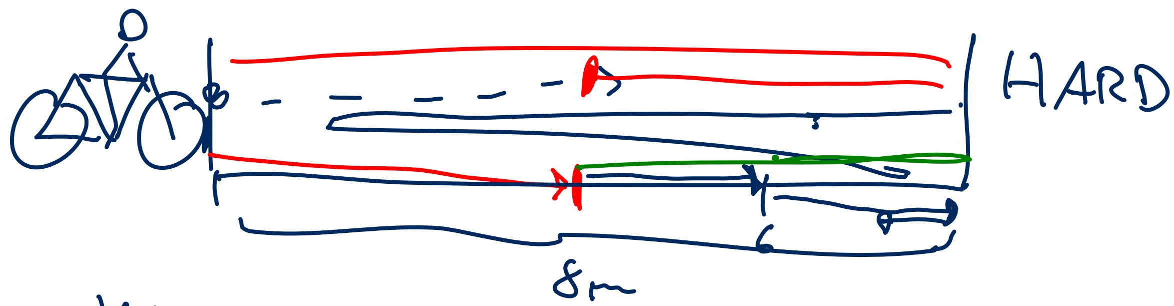
- $u'(x) = 0$

  $\Rightarrow x_1, x_2, \ldots$ candidates for max.

- if $u''(x_i) < 0 \Rightarrow x_i$ is a local max

  if $u''(x_i) > 0 \Rightarrow x_i$ is a local min.

3.3 Series



$v_{student} = 8\,m/s$ $\qquad v_{fly} = 24\,m/s$

~~1. Sec.~~ $1/8$ second : both meet at 4m

$1/4$ second.

$$\text{fly flies} = 12 + 6 + 3 + 1{,}5 + 0{,}75 + \ldots \overset{!}{=} 24$$

- after 1s student hits wall.
  after 1s fly has gone 24

$$d_{fly} = 12 + 6 + 3 + 1{,}5 + \dots$$

$$= 12 \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

$$|\delta| < 1$$

$$\sum_{k=0}^{\infty} \delta^k = \frac{1}{1-\delta} \qquad \left(\sum_{k=0}^{N} \delta^k = \frac{1 - \underbrace{\delta^{N+1}}_{\to 0}}{1-\delta} \right)$$

geometric series

$1 \qquad \frac{1}{2}$

- $\sum_{k=0}^{\infty} \frac{1}{k} = \infty$ $\quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} = \frac{0 - 1\cdot(-1)}{(1-\delta)^2}$

- $f(\delta) = \sum_{k=0}^{\infty} \delta^k \quad \Rightarrow \quad f'(\delta) \text{ "="} \sum_{k=0}^{\infty} k\delta^{k-1}$

$(x^2)' = 2x \qquad \left(\frac{1}{1-\delta}\right)' = \frac{1}{(1-\delta)^2}$

$$= \frac{1}{\delta}\sum_{k=0}^{\infty} k\delta^k = \frac{1}{(1-\delta)^2} \Rightarrow \sum_{k=0}^{\infty} k\delta^k = \frac{\delta}{(1-\delta)^2}$$

$|\delta| < 1$

$$\left((1-\delta)^{-1} \right)' = -1\,((1-\delta))^{-2} \cdot \underline{(1-\delta)'}$$

$$(u(v(x)))' = u'(v(x)) \cdot v'(x)$$ -1

4 $(\Omega, \mathcal{A}, P)$ probability space

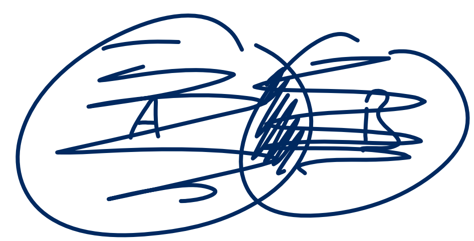
- Ω any set (sample space), $\omega \in \Omega$ elementary event
- $\mathcal{A}$ is a set of subsets of Ω (events)
- $P : \mathcal{A} \to [0,1]$ probability distribution

$\Omega = \{H, T\}$ $\mathcal{A} = \{\{\}, \{H\}, \{T\}, \{H,T\}\}$

$P(\{H\}) = \frac{1}{2}$, $P(\{T\}) = \frac{1}{2}$, $P(\phi) = 0$, $P(\Omega) = 1$

$P(\phi) = 0$, $P(\Omega) = 1$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$$\Omega = \{1, \dots, 6\}$$

$$P(\{1\}) = \frac{1}{6}$$

el. events

$$P(\underbrace{\{\text{even}\}}_{\text{Event}}) = P(\{2\}) + P(\{4\}) + P(\{6\})$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

Random variable : $(\Omega, \mathcal{A}, P)$

$$X : \Omega \longrightarrow I \qquad (I = \mathbb{R})$$

$$X(1) = -1$$

$$X(2) = 1$$

$$E(X) := \sum_{x \in I} x \, P[X = x]$$

expectation

$\Omega = \{1, \ldots, 6\}$ $\quad W = \{2, 4, 6\}$

$B = \{4, 5, 6\}$

$$P(W) = \frac{1}{2}$$

$$\underline{P(W \mid B)} := \frac{P(W \cap B)}{P(B)}$$

given B occurs, what is the prob. of W also occuring?

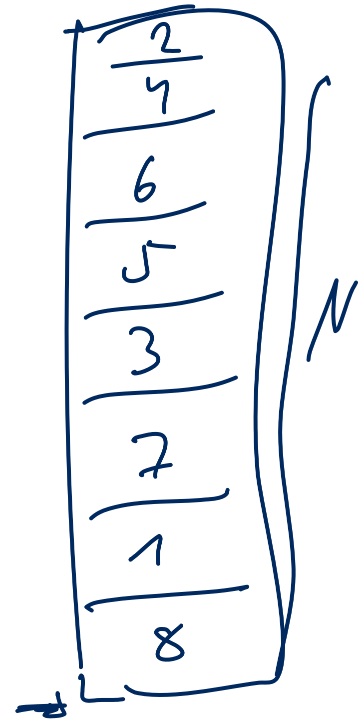
Bayes' Rule!

5

$0, 1, 2, \ldots, n-2$

```
bubbleSort(Array A)
for (n=A.size; n>1; n=n+1)
    for (i=0; i<n-1; i=i+1)
        if (A[i] < A[i+1])
            A.swap(i; i+1)
```

→ how many comparisons do we have to make?



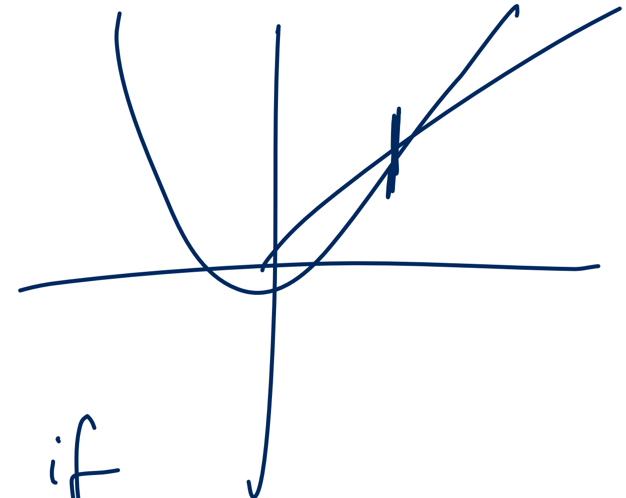
outer loop: $N-1$

inner loop: $n-1$

$$\sum_{n=2}^{N} \sum_{i=0}^{n-1} 1 = \sum_{n=2}^{N} (n-1) = \sum_{n=1}^{N-1} n = \frac{N(N-1)}{2} = \frac{1}{2}N^2 - \frac{1}{2}N$$

O-notation

$$f(n) = \frac{1}{2}n^2 - \frac{1}{2}n$$



$f \in O(g)$ $\quad g: \mathbb{N} \to \mathbb{N}$ $\quad$ if

$$\exists a, c \in \mathbb{N}: \; f \le a \cdot g + c$$

$$f \in O(x^2) \;\Leftarrow\; a = 1, \; c = 0$$

$f \notin O(x)$ by contradiction

$$x = (2a + 1) + \frac{c}{2a+1}$$