Exercise 5: Graphs and Networks, Dynamical Systems and Fractals

Formal Methods II, Fall Semester 2013

Solution Sheet

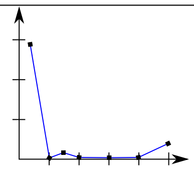
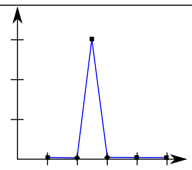
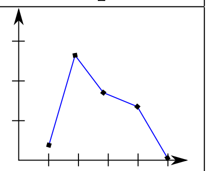
Graphs and Networks

1. (2 points)

$$\bar{D} = 6 \quad (1\text{pt})$$

$$\bar{C} = \frac{\frac{8}{28} + \frac{4}{6}}{2} = \frac{0.286 + 0.667}{2} = 0.476 \quad (1\text{pt})$$

2. (4 points)

	Graph a	**Graph b**	**Graph c**
degree distribution			
avg. degree	≈ 2	5	≈ 2.67
avg. path length	rel. low (≈ 3.1)	low (1)	rel. high
clustering coefficient	very low (0)	very high (1)	rel. low
betweenness	1:high, 2:low	1,2: low	1:low, 2:high

3. (a) (2 points) By definition, each node has $2k$ neighbors. The number of possible connections between these $2k$ neighbors is

$$N_{possible} = \frac{1}{2} \cdot 2k \cdot (2k-1) = k \cdot (2k-1)$$

since each of the $2k$ neighbors can be connected to its $2k-1$ other neighbors. The factor $\frac{1}{2}$ is added since the connections are not directed.

To count the number of actual connections, we consider each neighbor from left to right, counting only once each connection - i.e. counting only connections going to the right - see Figure 1. On the left side, each one of the k neighbors has $k-1$ connections. For the right side, the first neighbor has $k-1$ connections, the second $k-2$ connections, and so on.

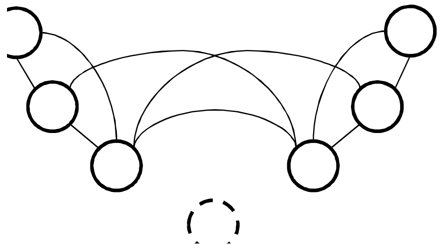


Figure 1: Neighborhood of a node for $k = 3$.

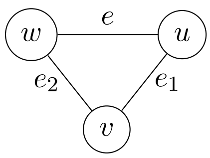
Using the equivalence $1 + 2 + ... + n = \sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ we therefore have:

$$\begin{aligned} N_{actual} &= \underbrace{(k-1) + (k-1) + ... + (k-1)}_{for\ the\ k\ left\ neighbors} + \underbrace{(k-1) + (k-2) + ... + 1}_{for\ the\ k\ right\ neighbors} \\ &= k \cdot (k-1) + \frac{1}{2}(k-1)k \\ &= \frac{3}{2}k^2 - \frac{3}{2}k = \frac{1}{2}k \cdot (3k-3). \end{aligned}$$

The clustering coefficient is therefore:

$$C = \frac{N_{actual}}{N_{possible}} = \frac{\frac{1}{2}k \cdot (3k-3)}{k \cdot (2k-1)} = \frac{3k-3}{4k-2} \qquad \square$$

(b) (3 points) Let us assume a node v in the regular graph, where two of the neighbors – u and w – are connected by an edge e. A total of three edges are involved in the case, as can be seen in the figure below:



During the rewiring, any of these edges can be rewired with probability p, and therefore, are not rewired with probability $1 - p$. Edge e will only be a connection between neighbor nodes if none of the three original edges are rewired. This probability is expressed as $(1-p)^3$. Thus, $C(p)$ can expressed as:

$$C(p) = C \cdot (1-p)^3 = \frac{3k-3}{4k-2}(1-p)^3$$

which leads to:

$$\frac{C(p)}{C(0)} = \frac{C \cdot (1-p)^3}{C} = (1-p)^3$$

Dynamical Systems – Iterative Maps

4. (2 points)

	Map a	**Map b**	**Map c**	**Map d**
Attractor	point	strange	point	periodic (2-point)
Value(s)	$x = 0.459$	-	$x = 0.636$	$x = 0.558/x = 0.765$

The solutions are considered correct if they are within $x \pm 0.05$.

5. (2 points + 1 bonus)

(a) Strange (or chaotic) attractor.

(b)

$$x_{t+1} = r \sin(\pi\, x_t) \rightarrow r = \frac{x_{t+1}}{\sin(\pi\, x_t)}$$

$$r = \frac{0.6}{\sin(\pi\, 0.6)} = 0.63088$$

Bonus Any value $r = 0.72 \pm 0.05$ for the lower bound and $r = 0.83 \pm 0.05$ for the upper bound is accepted.

Fractals

6. (2 points + 1 bonus)

(a) This fractal – called *hexaflake* – can be constructed as follows: Start out with a hexagon, then in every iteration exchange each hexagon with a flake of 7 hexagons, each of them having a side length of $\frac{1}{3}$ of the former iteration.

(b) The Hausdorff dimension of this fractal is:

$$D = \frac{\log(7)}{\log(3)} \approx 1.7712$$

Bonus The hexaflake contains an infinite number of *Koch snowflakes*.

7. (3 points) The first two applications of the rule lead to the following strings:

0 F (Axiom)

1 FF[+F][−F]F

2 FF[+F][−F]FFF[+F][−F]F[+FF[+F][−F]F][−FF[+F][−F]F]FF[+F][−F]F

The corresponding geometric realization of the axiom and the first two applications of the rule are shown below: