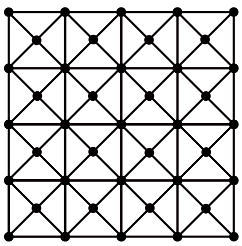
Exercise 5: Graphs and Networks, Dynamical Systems and Fractals

Formal Methods II, Fall Semester 2013

Distributed: 29.11.2013
Due Date: 13.12.2013
Send your solutions to: tobias.klauser@uzh.ch or deliver them in class.

Graphs and Networks

1. (2 points) Consider a network whose nodes are arranged in a planar regular lattice as shown here. The network is considered to be infinite.



What is the average node degree $\bar{D}$ and the mean clustering coefficient $\bar{C}$[1]?

2. (4 points) For each of the three graphs a, b and c shown in Figure 1 on the back of this sheet, estimate the *average path length*[2], *average degree*, *mean clustering coefficient* as well as the *betweenness* of nodes 1 and 2. Additionally, sketch the degree distributions of the three graphs in a histogram (the degree on the x-axis and the number of nodes having degree x on the y-axis). You don't have to calculate the numbers exactly, it's enough to *estimate* them and to state whether the values are rather high or low for the respective graph.

3. (a) (2 points) Prove that for an undirected regular graph (such as the leftmost graph depicted in Figure 2) where each node has a neighborhood range of k, the clustering coefficient (as defined in Section 9.4.4 of the script) is

$$C = \frac{3k-3}{4k-2}$$

for any vertex.

[1] A formal definition of $\bar{C}$ is given in task 3.

[2] The average path length is the average of the shortest path lengths between any two distinct nodes of the graph.

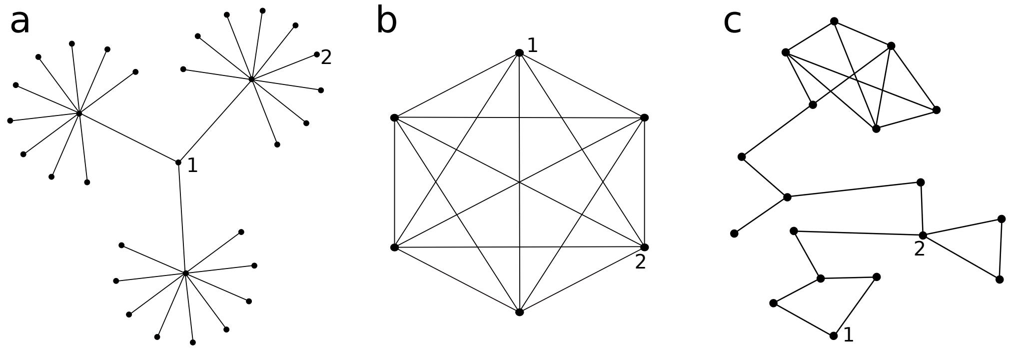


Figure 1: Graphs for task 2

(b) (3 points) In Section 9.4.2 of the script, a random rewiring procedure (from Watts and Strogatz, 1998) is introduced to interpolate between a regular ring lattice and a random network (see Figure 2). Starting with an undirected regular graph where each node has a neighborhood range of k, the rewiring procedure involves going through each edge in turn and, with probability p, moving one end of that edge to a new location chosen uniformly at random from the lattice (double edges and self-edges are forbidden).

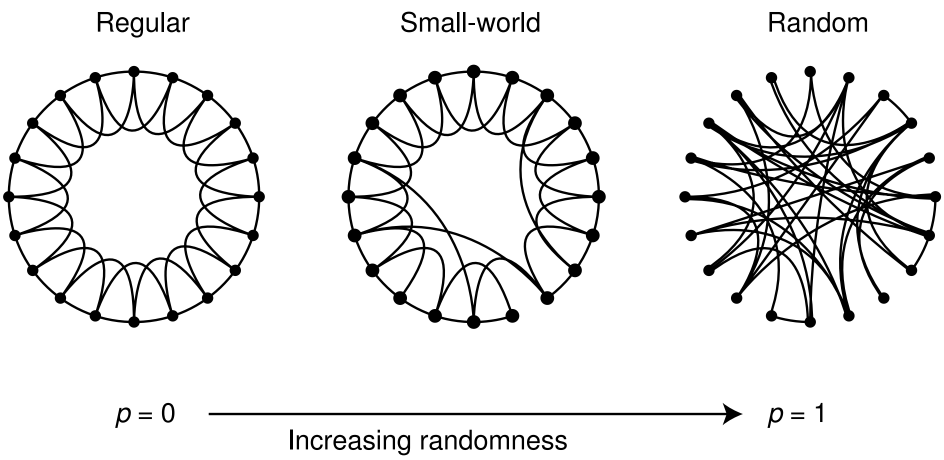


Figure 2: Random rewiring procedure (from Watts and Strogatz, 1998)

We define the mean clustering coefficient $\bar{C}(p)$ as:

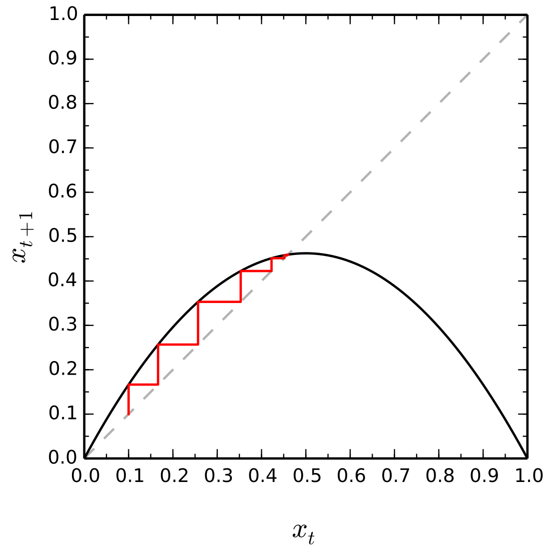
$$\bar{C}(p) = \frac{1}{n} \sum_{i=1}^{n} \underbrace{\frac{\text{number of } \textit{actual} \text{ links between the neighbors of vertex } v_i}{\text{number of } \textit{possible} \text{ links between the neighbors of vertex } v_i}}_{\text{Clustering coefficient of vertex } v_i}$$

Give a formula which describes $\frac{\bar{C}(p)}{\bar{C}(0)}$ by using k and p (note that edges rewired back to each other can be ignored).[3]

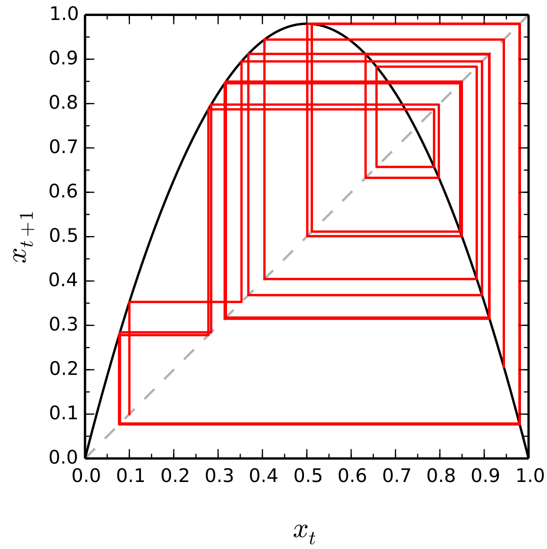
[3]Hint: You can check your solution by comparing it with Figure 9.4 of the script; mind the log-log-plot, though.

Dynamical Systems – Iterative Maps

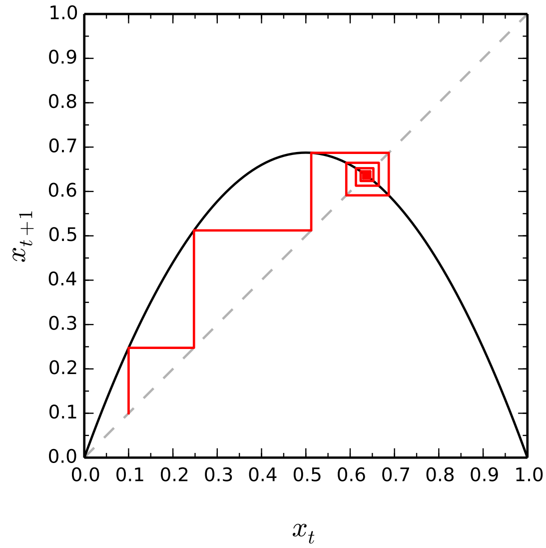
4. (2 points) As seen in the lecture, the phase plot (or *cobweb* diagram) of the logistic map $x_{t+1} = rx_t(1 - x_t)$ can be used to analyze its evolution over time. For each of the diagrams given below (all with initial value $x_0 = 0.1$ and run for 25 iterations), indicate the type of attractor: point, periodic or strange. In case of a point or a periodic attractor also give the approximate value(s) it converges to or oscillates between respectively.



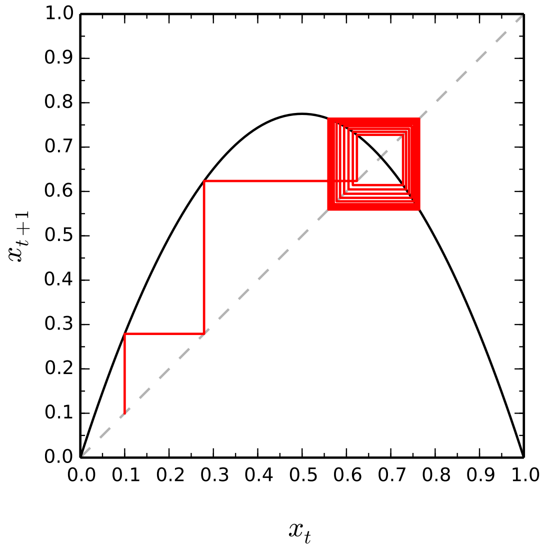
(a)



(b)



(c)



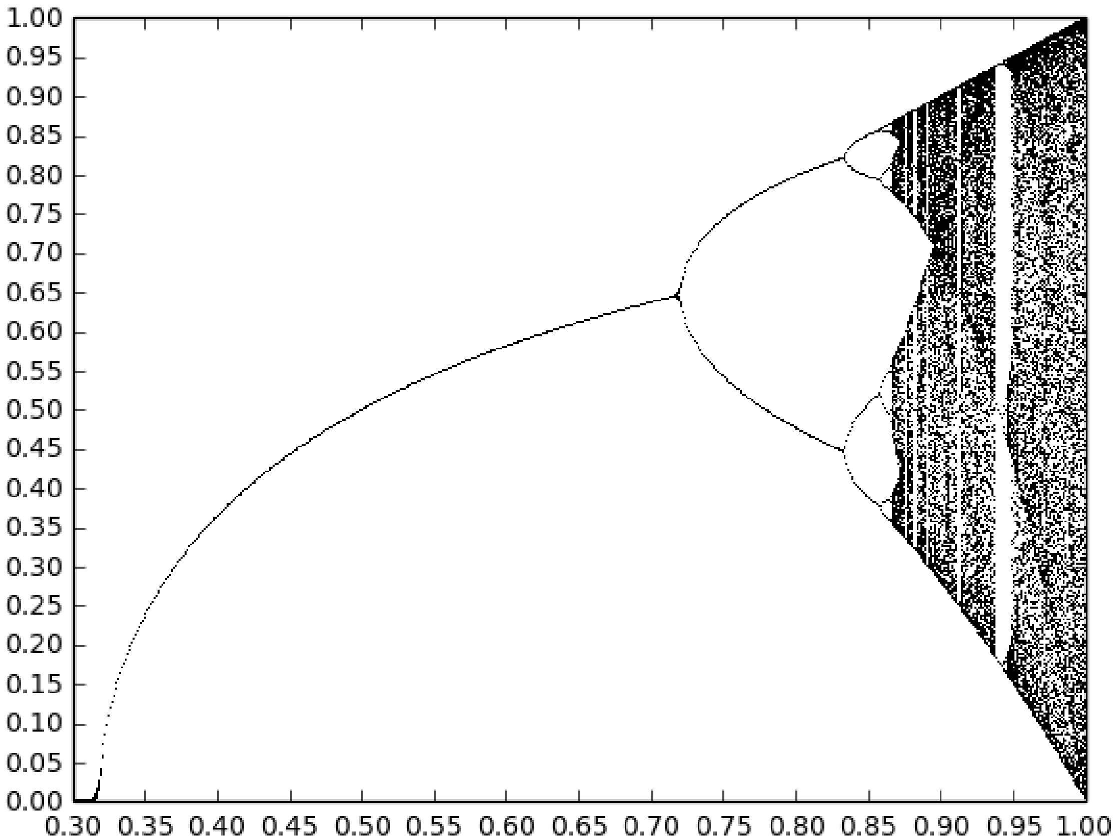
(d)

5. The sine map $x_{t+1} = r \cdot sin(\pi x_t)$ is another type of iterative map and has characteristics similar to those of the logistic map. Consider the bifurcation diagram of the sine map for $0.3 \leq r \leq 1.0$ as shown below.

   (a) (1 point) What type of attractor will the system evolve to for $r = 0.9$?

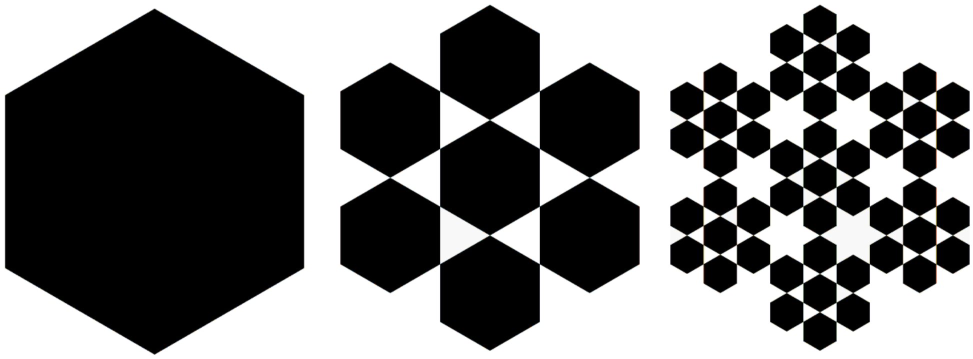
   (b) (1 point) At approximately which value of r will the value of the sine map converge towards $x_t = 0.6$?

Bonus (1 point) In which range of values for r (approximately) will the sine map oscillate between two different values (2-point periodic attractor)?



Fractals

6. Consider the fractal depicted below:



   (a) (1 point) Give a description of the procedure to generate this fractal.

   (b) (1 point) Calculate its fractal dimension D, using formula 8.4 given in the script. Briefly explain how you end up with the particular numbers.

Bonus (1 point) Which other, well-known fractal is contained within the one shown above?

7. (3 points) Given is the following L-system:

   - Axiom: F
   - Production rule: F $\rightarrow$ FF[+F][−F]F
   - Angle: 45°

   Write down the strings resulting from the first two applications of the rule and draw their respective geometric realizations.