Designing Core-selecting Payment Rules: A Computational Search Approach†

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Abstract

We study the design of core-selecting payment rules for combinatorial auctions (CAs), a challenging setting where no strategyproof rules exist. We show that the rule most commonly used in practice, the *Quadratic* rule, can be outperformed in terms of efficiency, incentives and revenue. In this paper, we present a new framework for an algorithmic search for good core-selecting rules. Within our framework, we use an algorithmic Bayes-Nash equilibrium solver to evaluate 366 rules across 30 settings to identify rules that are better than *Quadratic* in every dimension. We first study the well-known LLG domain and then show that our findings also generalize to the much larger and more complex LLLLGG domain. Our main finding is that our best-performing rules are *Large*-style rules, i.e., they provide bidders with large values with better incentives than *Quadratic*. Finally, we identify two particularly well-performing rules and suggest that they may be considered for practical implementation in place of *Quadratic*.

Keywords: Combinatorial Auctions, Payment Rules, Core

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1. Introduction

The spectrum auctions conducted by governments around the world over the last twenty years are a true success story for market design in general and auction design in particular. Sophisticated mechanisms have been used to sell resources worth billions of dollars, forming the basis for today's wireless industry (Cramton 2013). Recent versions of these markets have used a combinatorial auction (CA) mechanism. The advantage of CAs (e.g., in contrast to running multiple single-item auctions) is that buyers can express complex preferences over bundles of goods, which avoids the exposure problem and can increase efficiency.

There has been a large literature on the design of bidding languages, clearing algorithms, and activity rules for use in CAs (Cramton et al. 2006). However, finding optimal *payment rules* has remained elusive. In this work, we focus on the payments that are charged after the auction closes and the winners have been determined (i.e., we treat the CA as a one-shot game). This allows us to focus on *direct* payment rules which take as input the bidders' value reports and compute payments for the winning bidders. An important real-world application of such rules is the *Combinatorial Clock Auction (CCA)*, whose *supplementary round* is a sealed-bid package auction (Ausubel et al. 2006). The CCA has found wide-spread adoption in practice. For example, it has been used to conduct more than 15 spectrum auctions (Ausubel and Baranov 2017) and it has been used to auction off offshore wind rights (Ausubel et al. 2011).

Early proposals for CAs typically considered charging VCG prices (Varian and MacKie-Mason 1994). At first sight, VCG may seem like an appealing payment rule for a CA because it is strategyproof. Unfortunately, VCG is generally viewed as unsuitable in a CA domain when items can be complements because it often produces outcomes outside of the *core* (Ausubel and Milgrom 2006). Informally, this means that payments may be so low that a coalition of bidders may be willing to pay more in total than what the seller receives from the current winners. From a revenue perspective, VCG is also often undesirable, because CAs can produce very low or even zero revenue, despite high competition for the goods in the auction. For these reasons, recent auction designs have employed *core*-selecting payment rules that are guaranteed to charge payments in the (revealed) *core* (Ausubel and Milgrom 2002, Milgrom 2007, Day and Milgrom 2008).

1.1. The Quadratic Rule

Unfortunately, there exists no strategyproof core-selecting payment rule (Goeree and Lien 2016), and thus, designing an "optimal" core-selecting rule is a challenging market design problem. Parkes et al. (2001) were the first to introduce the idea of finding prices that minimize some distance metric to VCG, first for combinatorial exchanges, and later for CAs (Parkes 2002). Since then, a few rules have been proposed that also minimize some distance metric to VCG (e.g., (Day and Raghavan 2007)). Ultimately, Day and Cramton (2012) proposed the QUADRATIC rule, and this is also the rule most often used in practice. The QUADRATIC rule selects prices that are (1) enforced to be in the *core* (with respect to the submitted bids), within this (2) minimal in their total revenue to the seller, and then within this (3) minimal in the Euclidean distance to VCG prices.

The QUADRATIC rule is an integral part of the CCA, and it has now been used for more than 10 years by many governments around the world to allocate resources worth more than $20 billion (Ausubel and Baranov 2017). Nevertheless, we still have an incomplete understanding of this rule. Only recently has the research community started to grapple with the incentive properties of the Quadratic rule. For the so-called simple *LLG* domain, with 2 items and 3 bidders, Ausubel and Baranov (2019) as well as Goeree and Lien (2016) have independently derived the Bayes-Nash equilibrium of the Quadratic rule. It turns out that, even though the rule minimizes the Euclidean distance to VCG, the equilibrium strategies are far from truthful. This motivates the search for better minimum-revenue-core (MRC)-selecting payment rules in this paper.

1.2. Prior Work: A Manual Search for Better Payment Rules

The research community has already considered a number of alternative proposals for payment rules. Erdil and Klemperer (2010) argued that non-VCG reference points that are independent of the bidders' reports offer better incentives at the margin of truthful play. However, they do not offer a concrete payment rule, and they do not offer an argument about what happens when further deviations are necessary. Along the same lines, Day and Cramton (2012) studied the Quadratic rule with a ZERO reference point, instead of VCG. Using computational experiments (simulating truthful bidding), they found that, the use of ZERO tilts the payoff distribution in favor of winning bidders with higher values. However, they did not study this payment rule in Bayes-Nash equilibrium, nor did they analyze its effect on efficiency.

Ausubel and Baranov (2019) provided an analytical study of three core-selecting payment rules (Quadratic, Proxy, and Nearest-Bid), varying the distributional assumptions and the degree of risk-aversion. They found that core-selecting payment rules perform better in terms of efficiency and revenue when bidders' values are more correlated, while VCG performs worse. However, they did not identify a new payment rule with superior properties to Quadratic.

Parkes et al. (2001) already proposed three *weighted payment rules*. In particular, they suggested using the bidders' VCG payoffs to influence which core point to select. Ausubel and Baranov (2017) reported that weighted versions of the Quadratic rule have been used in recent CCAs conducted in Australia and Canada. However, they did not use VCG payoff but well-chosen reserve prices to power their weighted version of the Quadratic rule. The (intuitive) reason in favor of using reserve-price-weights is that those reserve prices are not manipulable while the VCG payoff is. However, no theoretical analysis of this reserve-price-weighted version of Quadratic exists.

We would like to highlight one common theme among this prior work: it always required an experienced auction designer to come up with a new payment rule. In some cases, we have limited theoretical results in favor or against the use of a particular rule. But in many cases, new rules were proposed using arguments that made intuitive sense, but without a comprehensive analysis. In this paper, we propose a radically different approach. Instead of relying on our own ingenuity of coming up with an even better payment rule, and having to analyze its properties in equilibrium by hand, we develop a *computational search approach* to automate this process.

1.3. A Computational Search for Better Payment Rules

The basic idea of our computational search approach is relatively simple: we construct a framework to parameterize the space of core-selecting payment rules, and we then use an algorithm to search this space and identify the best-performing rules. The details, however, are quite intricate.

First, the design space for core-selecting rules is infinitely large. To make it amenable to a computational search, we must thus choose a well-suited framework and parametrization. To this end, we introduce a *parameterized payment rule* we call FRACTIONAL*, with the following three parameters: a *reference point* R, a *weight* W, and an *amplification* A. In setting up this framework, we make sure that the parameters are general and human-understandable.

FRACTIONAL* minimizes the W-weighted Euclidean distance to the reference point R, whereby the weights can be strengthened or dampened by the amplification A. This provides a rich framework that captures all MRC-selecting payment rules. In this paper, we consider 6 different reference points, 11 different weights, and 6 different amplifications, yielding a total of 366 rules.

The second challenge relates to comparing the 366 rules in terms of their efficiency, incentives, and revenue. Because no core-selecting rule can be strategyproof, we must evaluate all of our rules *in equilibrium*. Given that many CAs like spectrum auctions are only conducted once, and bidders typically keep their valuations secret, the *Bayes-Nash equilibrium (BNE)* is the appropriate solution concept. As deriving 366 BNEs by hand (which typically involves solving a differential equation) is impractical, we use a recently developed *computational BNE solver* by Bosshard et al. (2017, 2020). Concretely, this is an algorithm that takes as input a payment rule and produces as output an ε-BNE. Because the algorithm only produces a numerical result, we obtain an ε-BNE with $\varepsilon > 0$, instead of a true BNE. However, the precision of the algorithm is extremely high (e.g., for LLG, we report ε-BNEs with $\varepsilon < 0.1\%$). For the BNE solver, a key question is the representation of the bidders' value space and its action/strategy space. For this work, we use an approach that lets us model the bidders' full, continuous value space (without any discretization). In terms of the action/strategy space, we employ a piece-wise linear approximation of the bidder's optimal strategy, which allows for a very good approximation of even non-linear functions.[1] Despite the use of this sophisticated BNE-solver, the final result of our computational search is "a set of (very) good core-selecting payment rules" that can easily be described via the three parameters.[2] This guarantees that our rules are broadly applicable and can directly be adopted in real-world CAs.

1.4. Towards Robust All-Rounder Rules

To evaluate our rules, we use two different domains. In a first step, we perform an extensive analysis in the stylized but well-know LLG domain. However, instead of just studying standard LLG (with uniform distributions and no correlation), we create 29 different variations of the LLG

[1] This is in contrast to some prior work that has employed computational BNE solvers, but with a restricted strategy space, for example using a simple multiplicative or additive shading strategy (Lubin and Parkes 2009, Lubin et al. 2015).

[2] This is in contrast to *automated mechanism design* (Sandholm 2003), where a mechanism is automatically created (by an algorithm) for each specific problem instance (at "run-time", so to say).

domain (varying, e.g., the marginal distribution as well as the correlation between bidders). Thus, with 366 rules evaluated on 29 domains, we study 10614 rule-domain combinations. This allows us to identify a set of 20 very good *all-rounder* rules, i.e., payment rules that outperform QUADRATIC (on average) across all 29 LLG domains in terms of efficiency, incentives and revenue.

In a second step, to check the robustness of our results, we then select seven of the best-performing rules from the first step and also evaluate them in the larger and more complex LL-LLGG domain. We find that that those rules also perform well in LLLLGG. Furthermore, we also select a rule which performs particularly badly in LLG and we find that it also performs worse than QUADRATIC in LLLLGG. This suggests that our results are robust, not only to changes in the distribution and correlation, but also to the structure and size of the domain.

1.5. Towards a Proposal for a new Rule

As a final step in our analysis, we aim to identify patterns to answer the question "what makes a rule a good rule?" One of the first findings that falls out of our analysis of the best 20 all-rounder rules in LLG is that using the *Shapley* value is extremely attractive for the design of our rules. Many of our best-performing rules use the Shapley value, either as a reference point or as weights.

A second interesting pattern is that our best-performing rules are *Large*-style rules in that they provide better incentives for bidders with larger values. It turns out that, for efficiency and revenue, it is most important to incentivize bidders with large values to bid truthful. By contrast Lubin and Parkes (2009) found *Small*-style rules performed well in the budget-balanced combinatorial exchanges (CE) they studied. This suggests that the structural differences between CEs and CAs may be important in the design of their payment rules.

Interestingly, there are different ways to achieve the effect of a *Large*-style rule, and our search for good all-rounders reveals the different combinations of reference points, weights, and amplifications that lead to this effect. Out of our best-performing rules, we highlight two rules that stand out in particular. The first rule is the Quadratic rule with a ZERO reference point and EQUAL weights. This rule is attractive because it is particulary simple and is still among our best-performing rules. It is also the same rule as previously studied by Day and Cramton (2012). The second rule also uses a ZERO reference point, but combines this with (dampened) VCG payoff weights. This is essentially a dampened version of the Quadratic rule with ZERO reference point. This rule is interesting because it is actually the best-performing rule according to our all-rounder analysis in LLG and it also generalized well to LLLLGG. Given these result, we suggest that both of these rules could be used in practice in place of Quadratic.

A last finding we want to emphasize is that there is not one best reference point, or one best set of weights. Instead, our results show that the *combination* of the three parameters matters. Our analysis leads to a set of well-performing rules, where for each rule, the reference point, the weights, and the amplification are perfectly tuned to complement each other.

Overall, our paper shows the power of a computational search approach in a mechanism design space. Furthermore, our results demonstrate that large improvements over Quadratic are possible

in terms of efficiency, incentives and revenue. We hope that some of our best-performing rules will spur new research and that we have also enlarged the space of payment rules that may be considered for implementation in practice.

2. Preliminaries

2.1. Formal Model

In a *combinatorial auction (CA)*, there is a set M of m distinct, indivisible items, and a set N of n bidders. Each bidder i has a *valuation function* v_i which, for every bundle of items $S \subseteq M$, defines bidder i's value $v_i(S) \in \mathbb{R}$, i.e., the maximum amount that bidder i would be willing to pay for S. To simplify notation, we assume that the seller has zero value for all items, although our setup generalizes to sellers with values (see Day and Cramton (2012) for specifying reserve prices).

We let $p = (p_1, ..., p_n)$ denote the payment vector, with p_i denoting bidder i's payment. We assume that bidders have quasi-linear utility functions of the form $u_i(S, p_i) = v_i(S) - p_i$. Bidders make reports about their values for bundles to the mechanism, denoted $\hat{v}_i(S)$, which may be non-truthful. We follow the majority of prior work (Ausubel and Baranov 2019, Goeree and Lien 2016, Bosshard et al. 2020) and assume that bidders only bid on bundles they are directly interested in (i.e., where removing any item from the bundle would strictly decrease the value).[3]

We define an *allocation* $X = (X_1, \ldots, X_n) \subseteq M^n$ as a vector of bundles, with $X_i \subseteq M$ being the bundle that i gets allocated. A mechanism's *allocation rule* maps the bidders' reports to an allocation. We only consider allocation rules that maximize reported *social welfare*, yielding an allocation $X^* = \arg\max_X \sum_{i \in N} \hat{v}_i(X_i)$, subject to X being feasible, i.e., $\bigcap X_i = \emptyset$. In addition to the allocation rule, a mechanism also specifies a *payment rule*, which determines the payment vector $p = (p_1, ..., p_n)$. Together, these define the *outcome* $O = \langle X^*, p \rangle$. An outcome O is called *individually rational (IR)* if, $\forall i$: $u_i(X_i^*, p_i) \geq 0$.

2.2. VCG Payments and VCG Payoff

Next, we define two auxiliary concepts based on the well-known VCG mechanism (Vickrey 1961, Clarke 1971, Groves 1973) which we later use as components in the design of our payment rules.

Definition 1 (VCG Payments) *Given an allocation X^*, and bidders' value reports $\hat{v}$, bidder i's* VCG payment *is defined as: $p_{VCG,i} = \sum_{j \neq i} \hat{v}_j(X^{-i}) - \sum_{j \neq i} \hat{v}_j(X^*)$, where X^{-i} is the welfare-maximizing allocation when all bidders except i are present.*

Note that VCG payments can be computed even if a different payment rule than VCG is used; in this case, the VCG payments are the payments the bidders *would have paid* if the VCG payment

[3] One noteworthy exception is the work by Ott and Beck (2013) who have studied a specific payment rule in the LLG domain while also considering bids on bundles a bidder is not directly interested in. In general, our framework can straightforwardly capture this as well. In future work, it would be interesting to extend our work in this direction.

rule *had been used*, but applied to the value reports submitted under the actual payment rule. We use those *VCG payments* in the definition of several of our more sophisticated payment rules.

Analogously, we also define a bidder's (reported) *VCG payoff*, which is the payoff the bidder *would have gotten*, given his reported value, if the VCG payment rule *had instead been* used in place of the actual payment rule.[4] Note that the payoff may be different from the bidder's utility which is always evaluated at the *true* values.

Definition 2 (VCG Payoff) *Given an allocation X^*, and bidders' value reports $\hat{v}$, bidder i's (reported)* VCG Payoff *is the difference between his reported value and his VCG payment: $\pi_{VCG,i} = \hat{v}(X_i^*) - p_{VCG,i}$.*

2.3. Bayes-Nash Equilibrium

In analyzing our payment rules, we assume each bidder i knows his own value function v_i, but only has distributional information about other bidder's value functions v_j. We assume each bidder's value v_i is sampled from a distribution function V_i, and that the distributions $V_1, \ldots V_n$ are common knowledge. Thus, from the bidder's perspective, the auction is a game of incomplete information which is why *Bayes-Nash equilibrium (BNE)* is the appropriate solution concept.

We let s_i denote bidder i's strategy, which is a mapping from his true value function v_i to a possibly non-truthful report $\hat{v}_i$. Given a value function and a strategy from each bidder, this determines the outcome of the auction. We let $u_i(s_1(v_1), s_2(v_2), \ldots, s_n(v_n))$ denote bidder i's utility for the outcome of the auction. We use v_{-i} to denote the value functions of all bidders except i, and analogously for the strategies s_{-i}. We say that a strategy profile s^* is an ε-BNE if no bidder has a profitable deviation from this strategy profile netting him more than ε in utility. Formally:

Definition 3 (ε-Bayes-Nash Equilibrium) *A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is an ε-*Bayes-Nash equilibrium *(ε-BNE) if, for all bidders $i \in N$, for all value functions v_i:*

$$\mathbb{E}_{v_{-i}}\left[u_i(s_i^*(v_i), s_{-i}^*(v_{-i}))\right] \geq \mathbb{E}_{v_{-i}}\left[u_i(\hat{v}_i, s_{-i}^*(v_{-i}))\right] - \varepsilon, \quad \textit{for all possible reports } \hat{v}_i, \tag{1}$$

where the expectation is taken w.r.t. the distribution over the other bidders' value functions.

In this work we use numerical algorithms with limited precision to find an equilibrium. We therefore adopt ε-BNEs as our solution concept, where ε is a suitably defined constant defined a priori for each setting we consider.

3. Computational Search Approach

In this section, we define the space of rules we investigate using our computational approach. These rules are all *core*-selecting, so we begin by introducing this concept.

[4] To distinguish the bidders from the auctioneer, we use "he/his" when referring to a bidder and "she/her" when referring to the auctioneer.

3.1. The *Core*

Informally, a payment rule is outside the *core* if a coalition of bidders is willing to pay more than what the seller receives in the mechanism. Payment rules that avoid such outcomes are said to be *core*-selecting (Day and Milgrom 2008). More formally:

Definition 4 (Core) *We let W denote the set of winners, X^* the welfare-maximizing allocation, $C \subseteq N$ denotes a coalition of bidders, and X^C is the allocation that would be chosen by the mechanism if only the bidders in the coalition C were be present. A price vector p is in the* core*, if, in addition to individual rationality, the following set of* core *constraints hold:*

$$\sum_{i \in W \setminus C} p_i \geq \sum_{i \in C} v_i(X^C) - \sum_{i \in C} v_i(X^*) \quad \forall C \subseteq N \tag{2}$$

Enforcing prices to be in the *core* puts lower bounds (constraints) on the payments of the winners, where each coalition of bidders leads to one *core* constraint.[5] Intuitively, the winners' payments must be *sufficiently large* such that there exists no coalition that is willing to pay more to the seller than the current winners' payments. In a CA with complements, VCG prices are often outside the core and, in the worst case, VCG may generate zero revenue despite high competition for the goods. Figure 1 illustrates the *core* in the so-called LLG domain with two goods and three players (see Section 5.1). Local bidder 1 bids 80 for good A; local bidder 2 bids 90 for good B, and the global bidder bids 100 for the bundle $\{A, B\}$. The figure also depicts several price vectors of particular interest (e.g., Shapley payoff), which we motivate and define formally in Section 3.3.

3.2. Design Framework for New Core-Selecting Payment Rules

Given a bid vector, a *core*-selecting payment rule always selects a price vector in the *core* (the shaded area in Figure 1). Day and Raghavan (2007) proposed to only select prices from the so-called *minimum-revenue*-core *(MRC)*, which is indicated as the diagonal line in Figure 1. The motivation for using this constraint is that this minimizes the total amount of deviation potential for all bidders. The QUADRATIC rule most commonly used in practice also employs the MRC constraint, and for this reason we also only consider payment rules which select prices from the minimum revenue *core*. Still, this leaves an infinite number of price vectors to choose from, and thus lots of room for the design of new payment rules, which all have different properties in BNE.

As depicted in Figure 1, the QUADRATIC rule selects a price vector in the MRC which minimizes the Euclidean distance to p_{VCG} (computed at the reported values of the bidders). Here, p_{VCG} serves as the *reference point* of the payment rule. Note that p_{VCG} (even though we call it "reference point") is not a fixed point but defined as a function of the bids. Bidders can, via their value reports, manipulate the reference point p_{VCG}. This motivates the use of alternative reference points, such

[5] Note that the core is defined in terms of bidders' true values. Goeree and Lien (2016) have shown that the outcome of a core-selecting CA can be outside the true core in BNE. Thus, core-selecting CAs only guarantee to produce outcomes in the *revealed* core, i.e., in the core with respect to *reported values*. Going forward, whenever we talk about the *core*, or *core*-selecting rules, we always mean the *revealed* core, unless we state it otherwise.

as ZERO $= \vec{0}$, which cannot be manipulated. We also consider weighted rules where a *weighting function* computes a weight per bidder based on the reported values. For QUADRATIC the weights are simply EQUAL $= \vec{1}$.

To define our space of rules, we generalize the Euclidean distance minimization in the QUADRATIC rule. Formally, our design framework is defined as follows:

Definition 5 (Algorithmic Framework for *Core*-Selecting Payment Rules) *Given a* reference point, weights, *and* amplification, *the unique* core-*selecting price vector is chosen to be:*

1. *Within the* core.
2. *Within this, in the* minimal revenue *core (MRC).*
3. *Within this, minimal in the weighted and amplified distance metric to the reference point.*

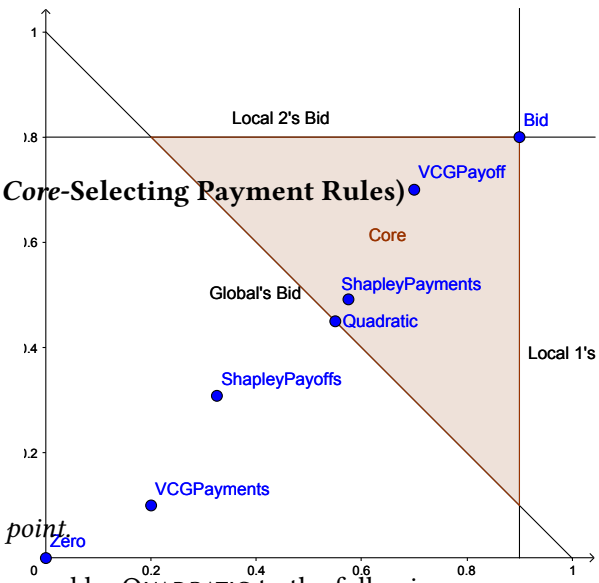


Figure 1: A graphical depiction of the *core*, the price vector corresponding to QUADRATIC and the reference points p_{VCG}, π_{VCG}, p_{SHAPLEY}, π_{SHAPLEY}, ZERO and BID.

In step (3) we generalize the Euclidean distance used by QUADRATIC to the following:

$$f_{w,a}(p, r^*) = \sqrt{\sum_{i=1}^{n} \frac{1}{w_i^a} |p_i - r_i^*|^2} \tag{3}$$

where p is the price vector, r^* is a *reference point*, w is a *weight* vector, and a is an *amplification*.

Proposition 1 *The algorithmic framework for core-selecting payment rules (Definition 5) is general and encompasses all possible MRC-selecting rules.*

Proof 1 *Given value reports $\hat{v}$, let $g^{MRC}(\cdot)$ be any payment rule that, for any vector of value reports $\hat{v}$ outputs a core payment vector $g^{MRC}(\hat{v})$ that is inside the MRC. Let $w = \vec{1}$ and define a reference point function $r^* = g^{MRC}(\hat{v})$. Clearly $p = r^*$ minimizes $f_{\vec{1},1}(p, g^{MRC}(\hat{v}))$ and $g^{MRC}(\hat{v})$ is in the MRC by definition.*

Of course, the proof of Proposition 1 hides the complexity in the definition of the reference point function itself. In particular, the computation of that reference point implicitly assumes knowledge of the MRC. Furthermore, even if we found a reference point in one CA domain (where $\hat{v}$ has a particular dimension) it would in general be unclear how to generalize that reference point to another (e.g., larger) CA domain. While our framework allows for arbitrarily complex reference points and weights, we restrict ourselves to reference points and weights that are defined independent of the domain and can be computed without prior knowledge of the core.

Working in a combinatorial exchange setting, Parkes et al. (2001) introduced a FRACTIONAL rule (sometimes called the PROPORTIONAL rule) with this proportional charging property restricted to $r^* = p_{\text{VCG}}$ and $w = \pi_{\text{VCG}}$. Because we use a generalized form of this distance function, we call our payment rules FRACTIONAL*, and use a wide array of reference points and weights, described in the next section. Additionally, we also introduce the amplification factor. When $a = 0$, the weights are ignored and the QUADRATIC rule is recovered. As $a \to \infty$, the weights will dominate over the quadratic term, and payments will be pushed as far as possible in the direction of the weights consistent with the *core* and MRC constraints.

Within this framework, we also consider taking the *inverse* of the weights, which effectively reverses the prioritization they construct. And lastly, because we include in our examinations reference points that may be within/above the core, we consider *mirroring* those points within/above the core across the nearest MRC facet so as to put the points outside and below the core. This has the effect of ensuring that the direction (and thus the sign) of the distance to the MRC line is always consistent, which it otherwise would not be.

3.3. Overview of all Core-Selecting Payment Rules

Consistent with the above framework, in this paper, we structure the design space of MRC-selecting payment rules to include the cross product of the following parameters, which results in 366 different MRC-selecting rules (with EQUAL weights, the amplification has no effect).[6]

1. **Reference Points:** ZERO, BID, BID^{M}, p_{VCG}, p_{SHAPLEY}, $p^{\text{M}}_{\text{SHAPLEY}}$,
2. **Weights:** EQUAL, BID, BID^{-1}, π_{VCG}, π^{-1}_{VCG}, p_{VCG}, p^{-1}_{VCG}, π_{SHAPLEY}, $\pi^{-1}_{\text{SHAPLEY}}$, p_{SHAPLEY}, p^{-1}_{SHAPLEY},
3. **Amplification:** We include $\{0.5, 1, 2, 3, 5, 10\}$

Recall that we use p for payments and π for payoffs. The M superscript denotes mirroring of reference points and the -1 superscript denotes inversion of weights. With BID we refer to the bid vector of the winning bidders as charged by the simple *First-Price* payment rule.

In the list of weights, π_{SHAPLEY} denotes the winner's *Shapley payoff* in the auction (which we define formally below), which is based on the well-known *Shapley value*. Like the core, the Shapley value is a concept from cooperative game theory (Shapley 1953); it measures the average contribution an agent provides to the grand coalition consisting of all agents. Unlike the core, though, the Shapley value, when applied to the coalition game including all bidders, can allocate surplus to losing bidders, which seems odd in an auction. One way to avoid this is to apply the Shapley value to a coalition game among those with winning bids, as was recently proposed by Lindsay (2018). However, in auctions, the payments defined by Lindsay (2018) are not guaranteed to be in the core. Therefore, to design core-selecting rules, we use the Shapley value among all bidders as a weight/reference point, but because we restrict payments to the core, no surplus is allocated to losing bidders. To keep our terminology consistent, we define the Shapley payoff as an analog of the Shapley value, but tailored to our auction domain. Note that in an auction, only coalitions

[6]This set includes all existing MRC-selecting rules we are aware of and many more.

that include the seller have positive value, which we consider in our definition. Additionally, since we consider non-cooperative (auction) domains, we compute the Shapley payoff based on the bidder's reports $\hat{v}$ not their values.

Definition 6 (Shapley Payoff) *Given the bidders' value reports $\hat{v}$, the* Shapley payoff *of player i is defined as:*

$$\pi_{SHAPLEY_i} = \sum_{C \subseteq N} \frac{|C|!(n-|C|)!}{(n+1)!} \sum_{j \in C \setminus \{i\}} \hat{v}_j(X^{C \cup \{i\}}) - \hat{v}(X^C) \tag{4}$$

where $X^{|C|}$ is the welfare-maximizing allocation when the bidders in C are present.[7]

The *Shapley payments* are the payments such that each bidder receives his Shapley payoff.

Definition 7 (Shapley Payments) *Given an allocation X^*, and bidders' value reports $\hat{v}$, bidder i's* Shapley payment *is defined as $p_{SHAPLEY_i} = \hat{v}_i(X_i^*) - \pi_{SHAPLEY_i}$*

3.4. Searching for Optimal Rules via a Computational BNE Solver

Given our framework for the design of core-selecting payment rules, we next consider how to search through the space of *candidate rules* to find one with high efficiency, good incentives and high revenue *in BNE*. To this end, we employ a BNE-finding algorithm that was recently introduced by Bosshard et al. (2017, 2020).[8] The algorithm represents the bidders' strategy space as a piecewise linear approximation with 80 control points. The search itself is obtained through iterated best response dynamics via a highly optimized fictitious play algorithm (Brown 1951).

One important sub-procedure of the search algorithm is the calculation of a best response to the other players' strategies, which requires solving a high-dimensional integral. When finding BNEs in the low-dimensional LLG domain (see Section 5.1), we obtain a speed-up by customizing the algorithm to use numerical integration for this step. In the more complex LLLLGG domain (see Section 6) we use Bosshard et al. (2020)'s carefully enhanced Monte-Carlo approach.

Because fictitious play algorithms are not guaranteed to converge in pure strategies (as required by the Bosshard et al. (2020) algorithm), it is possible that we may only find an ϵ-BNE for a reasonably large ϵ.[9] For LLG, we only report rules for which we find a 0.1%-BNE for all of our domains. In LLLLGG, we only report rules for which we find a 1.0%-BNE.

Equipped with the BNE algorithm, the next question is how to search through the design space for good payment rules. At first sight, some type of gradient decent in the parameter space may

[7] Note that our computation of the Shapley payoff involves all bidders (winning and losing bidders). We also experimented with an alternative approach where the Shapley payoff is computed based on the winning bidders only. However, we did not observe a significant difference between the two versions.

[8] https://github.com/marketdesignresearch/CA-BNE

[9] In theory, it is possible that multiple BNEs exist. Practically, when we have run the solver multiple times, we have never found two significantly different ε-BNEs for the same rule. Consequently, in our search, we deterministically seek a single BNE (the one closest to truth).

look promising. However, our results show that the rule space is highly non-convex over the different parameters. Moreover, the computational costs involved in computing each individual BNE (minutes to hours in LLG, days in LLLLGG) make gradient-based approaches impractical.

The approach we have chosen is an exhaustive search over our design framework in the discretized parameter space. We compute the BNEs of every rule defined in the previous section for every domain we consider. We then compare these rules according to multiple design goals, which we detail in Section 4, to find the optimal rule. In addition to being computationally feasible, another advantage of this approach is that all rules are easily interpretable (i.e., the reference points, weights and amplification factors are general and human-readable). Thus, the rules can be lifted from LLG and LLLLGG to be directly applied in any CA, regardless of size or structure.

4. Design Dimensions

In this section, we introduce formal metrics for the three goals we strive for when designing core-selecting payment rules: (1) high efficiency, (2) good incentives, and (3) high revenue.

4.1. High Efficiency

From a social planner's perspective (e.g., a government auctioning off spectrum) it is desirable to maximize the *social welfare* of the mechanism (i.e., the sum of the winners' values for their allocations). The *efficiency* of a mechanism is defined as the fraction of the social welfare that the mechanism achieves. As we evaluate the efficiency of our rules in BNE, we consider the *expected efficiency* in BNE. Thus, our measure for efficiency is the expected welfare of the mechanism divided by the expected welfare of the optimal allocation. Formally, given an auction instance I, let $SW_{OPT}(I)$ denote the social welfare obtained under the optimal allocation, evaluated at truth. Let $SW_M(I)$ denote the social welfare obtained by the mechanism M when all bidders play their BNE strategies, evaluated at truth. We define the efficiency of mechanism M as:

$$\text{Efficiency(M)} = \frac{\mathbb{E}_{\sim I}[SW_M(I)]}{\mathbb{E}_{\sim I}[SW_{OPT}(I)]} \tag{5}$$

where the expectation is taken over the auction instances in the domain being analyzed. This is the standard definition of efficiency used in prior work, e.g., by Goeree and Lien (2016).

4.2. Good Incentives

We next seek payment rules with "good incentives." Auction designers have argued in favor of the Quadratic rule because of its property to "induce truthful bidding" (Cramton 2013), or to "minimize the bidder's ability to benefit from strategic manipulation" (Day and Raghavan 2007), even though Quadratic is not strategyproof. One argument is that, if the rule is "approximately strategyproof", then finding a beneficial deviation from truthful bidding may be so hard that many bidders may just report truthfully (Day and Milgrom 2008). Of course, because there is no

strategyproof core-selecting CA, there will always remain some strategic opportunities for the participants; however, we would like these opportunities in BNE to be as small as possible.

Of course, the manipulability of a payment rule depends on a bidder's value; in particular, some rules may be more manipulable for bidders with small values while others may be more manipulable for bidders with large values. To capture the incentive properties of a rule in one number, we define an *aggregate incentives measure* (which we just call "incentives" going forward) as the total expected L2-distance between bidders' truthful value v and their bid $\hat{v}$ in BNE; we normalize this by the total expected value in the domain as this allows us to more fairly compare incentives across domains. Formally, we have:

$$\text{Incentives(M)} = \frac{\sum_{i=1}^{n} \sqrt{\int_{v_i} f(v_i) \cdot (v_i - \hat{v}_i)^2 \, dv_i}}{\sum_{i=1}^{n} \int_{v_i} f(v_i) \cdot v_i \quad dv_i} \tag{6}$$

where $f(v)$ is the probability density function (PDF) in the domain we are analyzing, and the bid $\hat{v}$ is the optimal bid in BNE for that domain.

4.3. High revenue

Another motivation for using core-selecting payment rules is to achieve high revenue, in particular higher revenue than VCG (Day and Milgrom 2008, Day and Raghavan 2007). Because the revenue achieved by a rule is domain dependent, we measure the *fraction of the VCG revenue* which a rule achieves in a particular domain, defined analogously to efficiency (except that, in contrast to efficiency, a rule can achieve more than 100% of the revenue achieved by VCG). Thus, our measure for the *revenue* achieved by a mechanisms M is defined as:

$$\text{Revenue(M)} = \frac{\mathbb{E}_{\sim I}[Revenue_M(I)]}{\mathbb{E}_{\sim I}[Revenue_{VCG}(I)]} \tag{7}$$

where the expectation is taken over the auction instances in the domain being analyzed.

5. Results for LLG

In this section, we study the Local-Local-Global (LLG) domain (defined below), and several novel variants. We first focus on LLG for several reasons: First, the existing theoretical results for simple rules provide a benchmark for our experiments. Second, solving for the BNE gets exponentially harder as the domain gets more complex. LLG is simple enough that we can solve for the BNE strategies for a large number of rules with high precision. That said, as we shall show, both the design space and the resulting BNE structure is surprisingly subtle and intricate. Thus, it is important to understand the BNEs of a small domain before moving to a larger one.

	Rule	Efficiency	Incentives	Revenue	
	QUADRATIC	98.03%	0.16	91.30%	
	Improvement Over QUADRATIC				**Avg.**
Best Efficiency	FRACTIONAL*(R=p_{SHAPLEY},W=π_{VCG},A=10)	0.29%	2.69%	1.43%	1.47%
Best Incentives	FRACTIONAL*(R=p_{SHAPLEY},W=π_{SHAPLEY},A=5)	0.26%	5.10%	1.24%	2.20%
Best Revenue	FRACTIONAL*(R=p_{SHAPLEY},W=π_{VCG},A=10)	0.29%	2.69%	1.43%	1.47%

Table 1: Results for LLG(MD=UNIFORM). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

5.1. LLG UNIFORM

In LLG, there are two items A and B, and three bidders. Two of the bidders are local bidders, each only interested in either item A or B, respectively. The third bidder is the global bidder who wants both items simultaneously. It remains to define the distribution from which the bidders' values are drawn. Existing work has focused on the case we refer to as UNIFORM where each player is drawn independently, with local bidders' values $\sim U[0, 1]$ and the global bidder's value $\sim U[0, 2]$.

BNEs of *core*-selecting payment rules are complex to study analytically, and existing theoretical results are only available for LLG. Prior work has shown that the BNE strategies of the local bidders require an *additive* shading in this setting:[10]

Proposition 2 (Goeree and Lien (2016), Ausubel and Baranov (2019)) *In UNIFORM, a Bayes-Nash equilibrium of the QUADRATIC rule is for the global bidder to be truthful, and for the local bidders to bid:* $\hat{v} = max(0, v - (3 - 2\sqrt{2})) \approx max(0, v - 0.17)$

The results for our computational investigation of this domain are provided in Table 1. The way to read this (and all following tables) is as follows. The QUADRATIC results are always provided in the first row (and here they correspond to the BNE in Proposition 2). The next three rows of the table then show the top rules by each dimension (efficiency, incentives, revenue). The cell entries for these rules now represent the *(multiplicative) improvement* of the respective metric/column relative to QUADRATIC. In this case, FRACTIONAL*(R=p_{SHAPLEY},W=π_{VCG},A=10) is best by both efficiency and revenue, providing evidence that the Shapley value can be useful in constructing these non-cooperative mechanisms. The rule is able to increase the efficiency (relative to QUADRATIC) by 0.29% and revenue by 1.43%. In this domain, the best rule by incentives is the FRACTIONAL*(R=p_{SHAPLEY},W=π_{SHAPLEY},A=5) rule, where the reference point is also Shapley, and where an amplified version of the Shapley payoff is used for weighting. By contrast, in terms of incentives, the non-MRC First-Price rule is worse than QUADRATIC by -412.6%, (see Appendix B).[11]

[10] See Ausubel and Baranov (2019) for a similar result where ZERO is used as the reference point.

[11] We include data on the First-Price rule for the Uniform domain; in most of our other domains it fails to converge.

	Rule	Efficiency	Incentives	Revenue	
	QUADRATIC	98.41%	0.11	95.19%	
	Improvement Over QUADRATIC				**Avg.**
Best Efficiency	FRACTIONAL*(R=ZERO,W=π_{VCG}, A=2)	0.24%	6.77%	2.65%	3.22%
Best Incentives	FRACTIONAL*(R=ZERO,W=p_{SHAPLEY}, A=0.5)	0.17%	8.46%	2.93%	3.85%
Best Revenue	FRACTIONAL*(R=ZERO,W=$\pi^{-1}_{\text{SHAPLEY}}$)	0.04%	-0.61%	4.00%	1.14%

Table 2: Results for LLG(MD = UNIFORM,Corr = SAME). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

5.2. LLG with Correlation

We now expand upon the basic LLG structure, by introducing correlation in the bidders' values: instead of drawing the values independently, we now draw their values from a joint distribution. We use *Copulae* to define these distributions, a method which lets us separate the specification of the *marginal* distributions through which each bidder views its distribution in isolation, from the *coupling* structure which describes the joint structure among these marginals. Formally, Sklar's Theorem (Sklar 1959) states that *all* multivariate cumulative distribution functions (CDFs) $F(x_1, \ldots, x_d) = \mathbb{P}(X_1 \leq x_1, \ldots, X_d \leq x_d)$ can be represented as $F(x_1, \ldots, x_d) = C(M_1(x_1), \ldots, M_d(x_d))$ where the M_i are the marginal CDFs in each dimension, e.g. $M_i(x) = \mathbb{P}(X_i \leq x)$, and C is a *copula* which is a joint CDF with uniform marginals. The theorem also provides that C will be unique if the F_i are continuous. The converse of the theorem lets us create multi-dimensional models by combining marginal distributions M_i with a copula C to create a joint distribution $C(M_1(x_1), \cdot, M_d(x_d))$. First, we consider several choices for C; in Section 5.3 we consider choices for M_i, and in Section 5.4 we then consider the cross product of these choices.

To model correlation, we adopt standard Gaussian copulae which use a multivariate normal distribution for the coupling function. We consider two correlation structures: (a) SAME which establishes correlation between both bidders interested in the *same* item and (b) CROSS which establishes correlation between the local players.[12] In both cases the correlation constant is 0.5.

Results for SAME side correlation are provided in Table 2. The third row of the table illustrates that a rule that is very good by one dimension may be worse on others. We will seek to address this in Section 5.5 by finding good all-rounders. Moreover, each dimension yields a distinct best rule in this domain. We also investigated cross-side correlation and varying the intensity of the correlation, results of which we include in Appendix D and E.

[12] Ausubel and Baranov (2019) consider a form of correlation where the local players are either exactly the same, or otherwise drawn independently. This approach is amenable to their form of theoretical analysis but is much less natural than approach taken here.

5.3. LLG with BETA Marginals (Uncorrelated)

We employ a BETA distribution for our marginals as it will adopt a diverse set of shapes that are similar to many familiar distributions with just two parameters.[13] We consider the application of several parameterizations of BETA distributions to the local players (see Figure 4 in Appendix A). We note that once one employs a skewed distribution, the relative bidder strength between the local and global bidders may no longer match that of the UNIFORM case. To address this, we linearly *calibrate* the distributions (unless explicitly mentioned) so as to ensure that the expected relative strength of the local and global bidders is identical to the UNIFORM case. We also experimented with uncalibrated domains, which we include in Appendix F, G, and H.

	Rule	Efficiency	Incentives	Revenue	
	QUADRATIC	97.79%	0.29	88.24%	
	Improvement Over QUADRATIC				**Avg.**
Best Efficiency	FRACTIONAL*(R=p_{VCG},W=π^{-1}_{VCG}, A=3)	1.19%	21.00%	14.68%	12.29%
Best Incentives	FRACTIONAL*(R=$p^{\text{M}}_{\text{SHAPLEY}}$,W=$p^{-1}_{\text{SHAPLEY}}$, A=3)	1.15%	25.74%	12.28%	13.06%
Best Revenue	FRACTIONAL*(R=BID,W=p_{SHAPLEY}, A=2)	0.99%	6.98%	15.80%	7.92%

Table 3: Results for LLG(MD = BETA(3, 1/3)). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

Table 3 provides results for LLG(MD=BETA(3, 1/3)), which is an example of the types of results we see in domains with skewed marginals. We again observe that a different rule is optimal for each dimension. However, all three rules perform quite well in all dimensions, relative to QUADRATIC. Results for the other distributions are provided in Appendix F.

5.4. Maximum Improvements Relative to Quadratic

	Rule	Efficiency	Incentives	Revenue	
	QUADRATIC	95.00%	0.21	143.59%	
	Improvement Over QUADRATIC				**Avg.**
Best Efficiency	FRACTIONAL*(R=BID,W=π_{SHAPLEY}, A=2)	3.99%	47.20%	10.02%	20.40%
Best Incentives	FRACTIONAL*(R=BID,W=π_{SHAPLEY}, A=2)	3.99%	47.20%	10.02%	20.40%
Best Revenue	FRACTIONAL*(R=BID,W=π_{SHAPLEY}, A=2)	3.99%	47.20%	10.02%	20.40%

Table 4: Results for LLG(MD = BETA(3, 1/3),CORR = CROSS, UNCALIBRATED). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

[13] Ausubel and Baranov (2019) consider a distribution similar to BETA(a,0), but they are limited to Pareto-like shapes.

As discussed in Section 5.2, modeling the joint distribution of value among the players using a copulae lets us mix and match between various marginal distributions and various types of correlation among these distributions. We have investigated the full cross product of correlations we discussed in Section 5.2 with the set of marginal distributions discussed in Section 5.3. The full set of results is presented in Appendix G and H.

We now briefly point towards those rules that achieve the *largest improvement* over QUADRATIC, in any single domain. In terms of efficiency, FRACTIONAL*(R = BID, W = π_{SHAPLEY}, A = 2.0) achieves a **3.99%** improvement over QUADRATIC in LLG(MD = BETA(3, 1/3), Corr = CROSS, UNCALIBRATED) (see Table 4). Considering the fact that many large-scale CAs allocate resources worth billions of dollars, an efficiency improvement of this magnitude is very significant. Note that the same rule, in the same domain, also achieves an incentive improvement of **47.20%**. This demonstrates that core-selecting rules exist for which, in some domains, their equilibrium strategies are significantly closer to truthful than QUADRATIC. This performance is even exceeded by FRACTIONAL*(R=ZERO, W=EQUAL) in the domain LLG(MD=UNIFORM, Corr=CROSSLARGE), where this rule achieves an incentive improvement of **48.25%** over QUADRATIC. In terms of revenue, FRACTIONAL*(R=BID^{M}, W=p^{-1}_{SHAPLEY}, A=2) achieves a **23.43%** improvement over QUADRATIC in LLG(MD=BETA(3, 1/3), Corr=CROSS).

5.5. Best All-Rounder Rules

In the previous section, we have evaluated our rules one domain at a time. When auctioneers have good information about their domain structure this enables the selection of very high-performing rules, even if these rules perform poorly elsewhere in the domain space. However, in practice, auctioneers may not know the exact structure of the domain in which they are operating. Accordingly, we now seek good "all-rounder" rules that are widely applicable.

Different auctioneers might place different emphasis on each of our evaluation dimensions. In the absence of such knowledge, we opt to take a simple average over all three dimensions and then rank our rules by this average. Table 5 shows the top-20. We see that the best rule achieves an 8.22% average improvement (across all three dimensions) over QUADRATIC, across all 29 domains.

Note that seven of the top-20 rules actually beat QUADRATIC in every dimension in every domain; those rules are highlighted in grey in Table 5. The other rules typically beat QUADRATIC in most (but not all) of the 29 domains (specifically: in 26, 27, or 28 domains). However, we do not consider it to be an exclusion criterion if a rule loses to QUADRATIC in one or multiple domains. In fact, we find that QUADRATIC actually performs quite well in some domains, where it is almost impossible to beat. It would not make sense to restrict our search for good all-rounder rules to only those that beat QUADRATIC everywhere.

Looking at Table 5 in more detail, we observe that Shapley-based rules are ubiquitous. Indeed, all seven rules that beat QUADRATIC everywhere use Shapley payments as a reference point. This is interesting, as Shapley-based reference points or weights had previously not been considered for the design of core-selecting payment rules. Looking into the calculation of the Shapley payment (Definition 7), it is intuitive that they capture a "fair" distribution of the surplus (and thus may naturally serve as a good reference point), while at the same time being somewhat robust against

		Efficiency	Incentives	Revenue	
	QUADRATIC	97.71%	0.17	63.72%	
	Best All-Rounder Rules	**Avg. Improvement over QUADRATIC**			**Avg.**
1	FRACTIONAL*(R=ZERO,W=π_{VCG},A=0.5)	0.92%	17.17%	6.57%	8.22%
2	FRACTIONAL*(R=ZERO,W=$\pi_{SHAPLEY}$,A=0.5)	0.86%	17.17%	5.90%	7.98%
3	FRACTIONAL*(R=ZERO,W=π_{VCG})	0.85%	16.92%	5.73%	7.83%
4	FRACTIONAL*(R=$p^{M}_{SHAPLEY}$,W=BID^{-1},A=3)	0.82%	16.74%	5.05%	7.54%
5	FRACTIONAL*(R=p_{VCG},W=$p^{-1}_{SHAPLEY}$)	0.82%	16.30%	5.34%	7.49%
6	FRACTIONAL*(R=$p^{M}_{SHAPLEY}$,W=$p^{-1}_{SHAPLEY}$,A=3)	0.81%	16.60%	5.00%	7.47%
7	FRACTIONAL*(R=ZERO,W=BID,A=0.5)	0.79%	16.23%	5.35%	7.46%
8	FRACTIONAL*(R=$p^{M}_{SHAPLEY}$,W=$\pi^{-1}_{SHAPLEY}$,A=3)	0.81%	16.59%	4.96%	7.46%
9	FRACTIONAL*(R=p_{VCG},W=BID^{-1})	0.80%	16.10%	5.16%	7.35%
10	FRACTIONAL*(R=p_{VCG},W=π^{-1}_{VCG},A=2)	0.79%	15.72%	4.94%	7.15%
11	FRACTIONAL*(R=ZERO,W=$p_{SHAPLEY}$,A=0.5)	0.75%	15.44%	5.01%	7.07%
12	FRACTIONAL*(R=ZERO,W=EQUAL)	0.87%	13.42%	6.82%	7.04%
13	FRACTIONAL*(R=p_{VCG},W=$\pi^{-1}_{SHAPLEY}$)	0.71%	14.56%	4.48%	6.58%
14	FRACTIONAL*(R=$p^{M}_{SHAPLEY}$,W=π^{-1}_{VCG},A=3)	0.66%	13.57%	3.89%	6.04%
15	FRACTIONAL*(R=$p_{SHAPLEY}$,W=π_{VCG},A=3)	0.65%	13.42%	3.84%	5.97%
16	FRACTIONAL*(R=$p^{M}_{SHAPLEY}$,W=BID^{-1},A=2)	0.58%	12.15%	3.56%	5.43%
17	FRACTIONAL*(R=$p^{M}_{SHAPLEY}$,W=$\pi^{-1}_{SHAPLEY}$,A=2)	0.57%	11.96%	3.49%	5.34%
18	FRACTIONAL*(R=$p^{M}_{SHAPLEY}$,W=$p^{-1}_{SHAPLEY}$,A=2)	0.56%	11.77%	3.46%	5.27%
19	FRACTIONAL*(R=p_{VCG},W=$p^{-1}_{SHAPLEY}$,A=0.5)	0.53%	11.09%	3.27%	4.96%
20	FRACTIONAL*(R=$p_{SHAPLEY}$,W=$\pi_{SHAPLEY}$,A=3)	0.55%	10.87%	3.05%	4.82%

Table 5: Results showing the top 20 all-rounder rules. The first row is the average performance of QUADRATIC over all 29 domains. The subsequent rows show the top rules by their average improvement over QUADRATIC. Rules that beat quadratic in every dimension in every domain are highlighted in grey.

manipulations. However, understanding the exact way in which the Shapley-based rules drive incentives (and then efficiency and revenue) requires a more detailed analysis of the individual rules, which is beyond the scope of this paper but which is an interesting subject for future work.

The Shapley-based rules are also particularly noteworthy from a computational point-of-view. The Shapley value is #P-hard for many standard games (Deng and Papadimitriou 1994), and we are not aware of polynomial-time algorithms for the auction game we consider. Thus, we currently only have algorithms that have run-time exponential in the number of bidders. For the LLG domain, this is not a problem, but in larger domains, this computational complexity becomes prohibitively expensive, and consequently we must omit the Shapley-based rules from our analysis

Rule	Efficiency	Incentives	Revenue
QUADRATIC	99.8%	0.266	110.5%

Table 6: Results for the Quadratic rule in LLLLGG(MD=UNIFORM).

of the larger LLLLGG domain in the next section.

6. Results for LLLLGG

In this section, we take the rules that worked well in the LLG domain and seek to find out if they also perform well in the larger LLLLGG domain introduced by Bosshard et al. (2017, 2020) as a generalization of LLG. This domain is significantly more complex, but numerical BNEs can just barely be computed for it using a powerful computer cluster.[14] Specifically, the LLLLGG domain has 8 goods and 6 bidders, each of whom is interested in two bundles. There are four local bidders, each interested in two distinct pairs of two goods. And there are two global bidders interested in two distinct sets of 4 goods. There are significant symmetries in the domain that reduce the complexity of the strategy space. The strategies of the local players can be represented as two 2D surfaces, and the strategy of the global player can be represented as a pair of symmetric 2D surfaces (which unifies their computation).

In LLLLGG(MD = UNIFORM), each local bidder draws his value for each bundle from $U[0, 1]$, while the global bidders draw their values from $U[0, 2]$. The results for the QUADRATIC rule in this domain are provided in Table 6 (see Appendix I for further results in LLLLGG(MD = UNIFORM)). Because the QUADRATIC rule already achieves an efficiency of 99.8% in this domain, there is not much room for improvement. We therefore also consider versions of LLLLGG modified in ways analogous to what we have done in LLG. However, because even LLLLGG(MD=UNIFORM) requires thousands of core-hours to solve for one high quality numerical BNE, we focus on a single modified domain. Specifically, we examine the domain analogous to the LLG one provided in Table 3, with BETA(3,1/3) marginals for the local players and uniform (between 0 and 2) values for the global bidders. All values are uncorrelated.

Because computing even a single BNE in LLLLGG(MD = BETA(3, 1/3)) is extremely costly, we cannot exhaustively test all of the rules in this domain. Instead, we select all of the (non-Shapley-based) top-20 rules we have previously identified from Table 5, leading to a set of seven rules to be tested in LLLLGG(MD = BETA(3, 1/3)). The results are shown in Table 7. We see that all of these rules also have a positive average improvement over QUADRATIC. Thus, the performance enhancements we observed for these rules in LLG generalize to this much larger and more complex setting. Furthermore, our top all-rounder rule from Table 5, FRACTIONAL*(R=ZERO,W=π_{VCG}, A=0.5), also performs extremely well in LLLLGG(MD=BETA(3, 1/3)), leading to an average improve-

[14] We computed 1.0%-BNEs using a cluster consisting of multiple Intel Xeon E5-650 v4 2.20GHz processors with 40 logical cores. The average runtime per rule in LLLLGG at this level of precision was 5 days.

	Rule	Efficiency	Incentives	Revenue	
	QUADRATIC	97.1%	0.896	138.8%	
	Improvement Over QUADRATIC				**Avg.**
1	FRACTIONAL*(R=ZERO,W=EQUAL)	2.1%	38.4%	3.5%	14.7%
2	FRACTIONAL*(R=ZERO,W=π_{VCG},A=0.5)	1.9%	34.1%	3.1%	13.0%
3	FRACTIONAL*(R=ZERO,W=π_{VCG})	1.7%	29.7%	3.4%	11.6%
4	FRACTIONAL*(R=ZERO,W=BID,A=0.5)	1.4%	22.6%	2.5%	8.8%
5	FRACTIONAL*(R=p_{VCG},W=BID^{-1})	0.9%	14.1%	1.5%	5.5%
6	FRACTIONAL*(R=p_{VCG},W=BID^{-1},A=0.5)	0.4%	6.2%	1.1%	2.6%
7	FRACTIONAL*(R=ZERO,W=BID)	0.2%	5.1%	0.7%	2.0%
8	FRACTIONAL*(R=p_{VCG},W=BID,A=5)	-2.8%	-38.1%	-8.4%	-16.5%
9	First-Price	-2.7%	-80.0%	-13.7%	-32.1%

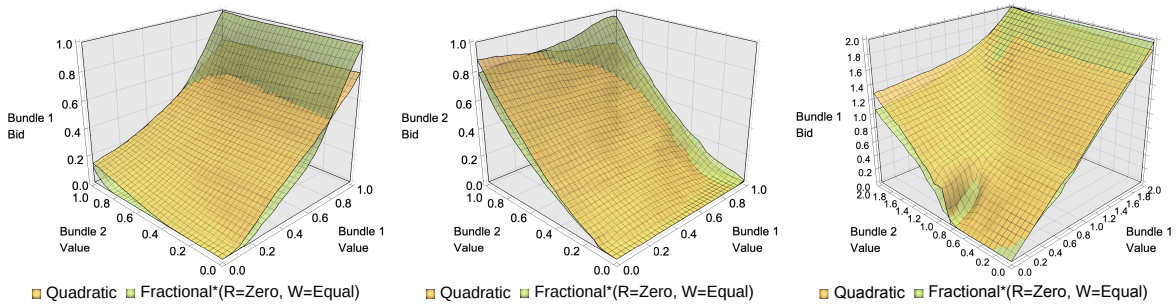
Table 7: Results for LLLLGG(MD=BETA(3, 1/3)). The first row shows the performance of QUADRATIC. In the subsequent rows we show seven of our best all-rounder rules from LLG, followed by one of the worst rules we identified in LLG. For comparison, we also include the First-Price rule at the bottom.

ment over QUADRATIC of 13.0%. Only one rule, FRACTIONAL*(R=ZERO,W=EQUAL), performs even better, achieving an average improvement over QUADRATIC of 14.7%.

For comparison, we also include the worst (non-Shapley-based) rule from LLG which converges in every domain. This rule, FRACTIONAL*(R=BID,W=p_{VCG},A=5), performs poorly in LLLLGG(MD=BETA(3, 1/3)) as well, with an average improvement of -16.5% over QUADRATIC, again confirming that our observations from the LLG domain generalize well to LLLLGG. Lastly, as a reference, we also include the First-Price rule, even though it is not an MRC-selecting rule.[15] We see that First-Price performs even worse than our worst-performing MRC-selecting rule.

To obtain an intuition for how our rules operate in LLLLGG(MD=BETA(3, 1/3)), we look at the BNE strategies. In Figure 2, we show the BNE strategies of QUADRATIC in yellow and we show the BNE strategies of one of our top rules, FRACTIONAL*(R=ZERO,W=EQUAL), in green. We observe that our rule induces more truthful bidding than QUADRATIC for local and global bidders with large values. Additionally, our rule also slightly reduces the global bidders' incentives to overbid when they have a small value. Figure 5 in Appendix J shows the BNE for the worst-performing rule. The main observation we see there is that the even -performing rule induces worse incentives (i.e., more shading) for the global bidders when their value is large, compared to QUADRATIC.

[15]Note that for the First-Price rule, we only obtain a 1.5% BNE.



(a) The local bidders' first strategy: their bid on bundle 1.

(b) The local bidders' second strategy: their bid on bundle 2.

(c) The global bidders' bid on bundle 1; their bid on bundle 2 is symmetric.

Figure 2: BNE strategies for QUADRATIC and FRACTIONAL*(R = ZERO,W = EQUAL) in LLLLGG(MD = BETA(3, 1/3))

7. Analysis and Discussion

So far, we have identified a set of rules that perform well in LLG and we have seen that their good performance generalizes well to LLLLGG. We now take a step back to evaluate whether we can identify any patterns, i.e., whether we can answer the question: "what makes a rule a good rule?"

We first take a closer look at the seven non-Shapley rules from Table 5 which we have analyzed in both LLG and LLLLGG. To gain some intuition for what incentives these rules provide, we plot the BNE strategies of all seven rules (in addition to the worst-performing rule) for LLG(MD = UNIFORM) in Figure 3. We observe a very clear pattern: all of our well-performing rules are *Large*-style rules, i.e., they provide much better incentives to bidders with large values (i.e., above 0.5) than QUADRATIC (by calling them "Large-style rules" we follow the terminology introduced by Parkes (2001)). In contrast, our worst-performing rule is a *Small*-style rule. The differences in incentives also directly translate into differences in utilities.[16] We have verified that this pattern, i.e., that our best-performing rules are *Large*-style rules, applies in the other LLG domains as well. Finally, in Figure 2, we have seen for LLLLGG(MD=BETA(3, 1/3)), one of our best-performing rules, FRACTIONAL*(R=ZERO,W=EQUAL), also provides better incentives to high-valued bidders than QUADRATIC. Thus, it also behaves like a *Large*-style rule in the LLLLGG domain.

Note that, a priori, it was not clear that *Large*-style rules would emerge as the best-performing rules. In fact, Lubin and Parkes (2009), who have previously studied payment rules in a *combinatorial exchange* domain, found that *Small*-style rules performed well in their setting. But in hindsight, it makes sense that in our framework, *Large*-style rules perform best. Recall that we restrict our search to MRC-selecting rules; thus, all rules charge the winning bidders the same total amount and they only differ in how they split up the payments between the bidders. Thus,

[16] We have computed the utilities obtained under the different rules for the different value quartiles. The bidders with values between 0.75 and 1 achieve 4.5% more utility under FRACTIONAL*(R=ZERO,W=EQUAL) than under QUADRATIC.

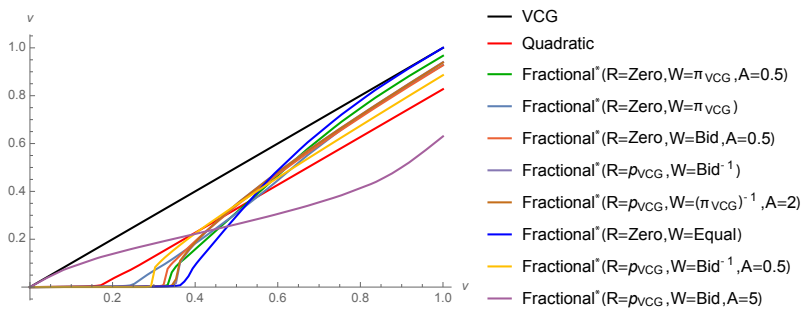


Figure 3: BNE strategies for VCG, QUADRATIC, seven of our best-performing rules, and one poorly-performing rule, in LLG(MD=UNIFORM)

any rule that wants to give preference to some bidders (compared to QUADRATIC) must disadvantage others. In a CA, the bidders with the largest values are most important in that making large-value bidders more truthful leads to higher efficiency and higher revenue. This explains why *Large*-style rules emerge as our best-performing rules.

In our framework, there are multiple ways to design a *Large*-style rule. One way is exemplified by the top rule in Table 5, FRACTIONAL*(R = ZERO,W = π_{VCG}, A = 0.5). This is an interesting rule as it uses a ZERO reference point, which heavily tilts the payments in favor of the large players (see Day and Cramton (2012)). Additionally, it uses π_{VCG} as weights (which tilts the payments back towards the small players) with a very small (0.5) amplification. Note that an amplification smaller than 1 de-emphasizes the weights, thus making the rule more like QUADRATIC, but not quite. Considering all parameters together, the rule is a dampened version of QUADRATIC with ZERO reference point. A rule that is very similar is FRACTIONAL*(R=ZERO,W=BID, A=0.5) (in row 7), except that it uses BID as the weight instead of π_{VCG}. Given that π_{VCG} = BID $- p_{\text{VCG}}$, it is intuitive that these two rules perform similarly.

An alternative way to construct a *Large*-style rule is to use a reference point with a more moderate impact/tilting, and instead create the effect via the weights. This is exemplified by the rule FRACTIONAL*(R=p_{VCG},W=BID^{-1}) in row 9 of Table 5. This rule uses the standard p_{VCG} reference point but combines it with BID^{-1} as weights. The inverted bid weighting has the effect of tilting the payments in favor of the large players, compared to the un-weighted version of QUADRATIC.[17]

We want to emphasize that there is no such thing as an "optimal reference point" or an "optimal weight." If we consider Table 5 again, we see that among the 20 rules, 4 of our 6 reference points show up and 9 of our 11 weights show up. No clear winner emerges. In fact, our results suggest that a search for an optimal reference point or an optimal weight seems to be misguided, because it

[17]That there are multiple ways to create *Large*-style rules in our framework makes clear that it does not create a *basis* for MRC-selecting rules. Recall that we built our framework from a carefully-chosen set of understandable components with the goal that our resulting rules would likewise be understandable, and also generalize well to different domains. If we were working with a basis then we would instead have a purely computational approach without these properties.

is really the *combination* of the reference point, the weights and the amplification that determine whether a payment rule performs well or not. This is well illustrated by the pair of rules in Table 5 in rows 7 and 9. The rule in row 7, FRACTIONAL*(R = ZERO,W = BID, A = 0.5), uses BID as weights, while the rule in row 9, FRACTIONAL*(R=p_{VCG},W=BID^{-1}), uses BID^{-1} as weights. Thus, whether the BID should be inverted depends on the reference point. In fact, if we combine the VCG payment reference point with the non-inverted BID weight, we obtain a very badly-performing rule, FRACTIONAL*(R = p_{VCG},W = BID), whose average improvement over QUADRATIC is -8.88%. If we additionally add an amplification of 5, we obtain our worst-performing rule, FRACTIONAL*(R= p_{VCG},W=BID, A=5), whose average improvement over QUADRATIC is -29.24%. This highlights the importance of choosing the right *combination* of reference point, weights, and amplification.

Based on our results in LLG and LLLLGG, we have identified seven very good rules (i.e., rules 1-7 in Table 7) that systemically outperform QUADRATIC. We want to highlight two of those rules. First, FRACTIONAL*(R=ZERO,W=π_{VCG}, A=0.5) stands out as it was the top rule according to our all-rounder analysis in Section 5 and because it also performed very well in LLLLGG(MD=BETA(3, 1/3)). Second, FRACTIONAL*(R=ZERO,W=EQUAL) stands out because it is particularly simple and it was the top rule in LLLLGG(MD=BETA(3, 1/3)). Furthermore, this rule had previously been studied by Day and Cramton (2012), albeit only at truth and not in BNE.

One limitation of our overall approach in this paper is the objective we have adopted in our search for the best all-rounder rules (i.e., the average improvement over all dimensions across all domains). Of course, other objectives are conceivable, including 1) maximizing the minimum average improvement across all domains, 2) maximizing the minimum improvement across all three dimensions, 3) maximizing the three dimensions in lexicographic order, etc. One might also add new dimensions into consideration, like the *fairness* of a rule. Our *Large*-style rules shift the benefit to large-valued players which may be considered highly unfair. However, balancing between efficiency and fairness raises challenging new questions, e.g., regarding how to measure fairness. Lubin et al. (2015) have already made some progress in this direction, but more work is still needed. Our computational search approach is agnostic to the particular objective of the search. In this paper, we do not argue in favor of any one objective; instead, we simply adopted the most straightforward one possible. Future work should explore other objectives, ideally informed by conversations with governments or other entities that are interested in the CA outcome.

8. Conclusion

In this paper, we have presented a computational search approach for finding good minimum-revenue-core-selecting payment rules. Using a suitable parametrization of the design space, we have been able to systematically search through this space and identify rules that outperform the commonly-used *Quadratic* rule on each dimension (efficiency, incentives, and revenue).

We have followed a two-step approach. First, we have studied the well-known but stylized LLG domain, which is amenable to an extensive search through our rule space. Within LLG, we have identified 20 very good all-rounder rules which beat QUADRATIC by a significant margin across

all domains (on average). Out of those, we have selected seven rules for evaluation in the larger LLLLGG domain. We have shown that those rules also perform better than QUADRATIC in this more complex domain, thus demonstrating the robustness of the new rules we have identified.

Importantly, we found that our best-performing rules are *Large*-style rules, i.e., they provide particularly good incentives to bidders with large values. While this is true for all of our best rules, we highlight two high performers in particular: FRACTIONAL*(R=ZERO,W=π_{VCG},A=0.5) and FRACTIONAL*(R=ZERO,W=EQUAL). Our results suggest that both of these rules are good candidates to be used in practice as alternatives for QUADRATIC. At the same time, we also suggest continued study of these rules, both analytically and via lab experiments.

Our results suggest that previous approaches that have tried to optimize core-selecting payment rules by modifying only the reference point or the weights may have been misguided. We have shown that it is the *combination* of reference point, weights, and amplification, that determines whether a rule is a *Large*-style rule or a *Small*-style rule, and thus if it is good or bad.

In terms of our computational search approach, we believe that our work illustrates the power of an automated search for a good mechanism, rather than designing it by hand. The design of core-selecting rules lends itself to this approach, because the design space can be nicely parameterized, and then searched through. However, computational complexity is a serious concern, and future work should thus explore smart ways to scale this approach to even larger settings.

Interestingly, computational complexity was also a concern for the design of the payment rules themselves. Some of our most promising rules were based on the Shapley value which is too computationally expensive to compute exactly in large domains. There exists some work on Shapley value approximation algorithms. Future work could explore the design of suitable approximations for our domain and evaluate the impact of using an approximate instead of the exact Shapley value. Relatedly, the success of our Shapley-based rules also motivate theoretical questions. Future work could seek an analytical explanation for the observed advantage of Shapley payments over VCG payments.

Going forward, we encourage other researchers to also consider using a computational search approach for similar auction design questions. Orthogonally, we hope that some of the best rules we have identified may spur new theoretical research, new computational research, or that they may be considered for implementation in practice.

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Appendices

A. Beta Distributions Used for Marginals

We illustrate the various Beta distributions we use for marginals in the following figure:

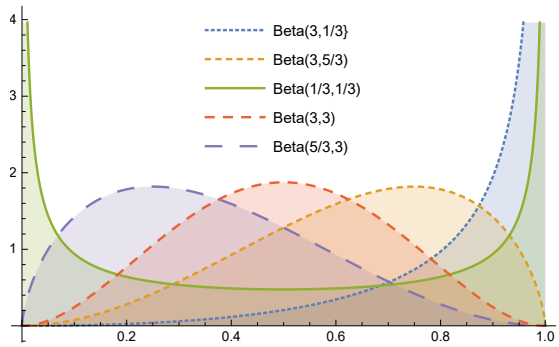


Figure 4: The various BETA marginal distributions we consider

B. The First-Price Rule in LLG(MD=UNIFORM)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	98.03%	0.16	91.30%
Improvement Over QUADRATIC			
First Price	0.7%	-412.6%	2.6%

Table 8: Results for the First Price rule in LLG(MD=UNIFORM).

Table 8 provides the results for the First-Price rule in LLG(MD = UNIFORM). The First-Price rule increases the efficiency relative to QUADRATIC by 0.7%, despite strategies that involve significantly more shading than under QUADRATIC. This efficiency gain, however, is dependent on the bidders being able to coordinate to find these extreme strategies in equilibrium. Because every marginal change in the bid creates a marginal change in utility under the First-Price rule, finding such a stable strategy profile is not easy. This coordination is possible in simple domains such as LLG, but does not generalize: when we look at more complex settings like LLLLGG (see Table 7) we see that First-Price has much lower efficiency than QUADRATIC (or our best rules).

Results For All Distributions

In Section 5, we introduce several variants of the LLG domain in which we analyze our payment rules. In the following appendices we present the best rules for each of these domains.

Note that the number of domains in this set, 29, is slightly smaller than the full cross product of all the domain features we have considered. There are several reasons for this:

- We excluded all variations of LLG(MD=BETA(1/3, 3)), which is a strongly left-skewed distribution. This is the reason why we only have five instead of six Beta distributions in Figure 4. The reason for excluding this domain is that it leads to pathological effects when evaluating core-selecting payment rules. In the uncalibrated setting, VCG is almost always in the core, which makes this an uninteresting case. In the calibrated setting, when a local bidder gets lucky and has a high value, then the other local bidder still almost certainly has a very low value. This implies that the high-valued bidder essentially has no ability to manipulate the core payment. In fact, shading his bid might risk losing to the global bidder. Thus, the distribution of this domain completely determines the incentives of any core-selecting rule (which are all very close to truthful), which renders comparing individual rules uninteresting.
- For one domain, (LLG(MD=BETA(3, 1/3),CORR=SAME, UNCALIBRATED)) the BNE algorithm failed to converge to our threshold of 0.1% on sufficiently many rules that we felt we could not effectively generalize from the results. Specifically, for this domain only 5% of the rules converged. For all other domains at least 182 out of the 366 converged. We note this one omitted domaind with a † below.
- For one domain (LLG(MD=BETA(1/3, 1/3),CORR=SAME)) the QUADRATIC rule failed to converge, so we could not compare the other rules to our chosen benchmark. We note this case with a ‡ below.

C. UNIFORM Marginals and No Correlation

C.1. LLG(MD=UNIFORM)

	Rule	Efficiency	Incentives	Revenue	
	QUADRATIC	98.03%	0.16	91.30%	
	Improvement Over QUADRATIC				**Avg.**
Best Efficiency	FRACTIONAL*(R=p_{SHAPLEY},W=π_{VCG}, A=10)	0.29%	2.69%	1.43%	1.47%
Best Incentives	FRACTIONAL*(R=p_{SHAPLEY},W=π_{SHAPLEY}, A=5)	0.26%	5.10%	1.24%	2.20%
Best Revenue	FRACTIONAL*(R=p_{SHAPLEY},W=π_{VCG}, A=10)	0.29%	2.69%	1.43%	1.47%

Table 9: Results for LLG(MD=UNIFORM). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

D. UNIFORM Marginals and Same-Side Correlation

We investigated using both smaller, 0.25, and larger, 0.75, correlation constants in our copulae. Results on individual domains are provided below:

D.1. LLG(MD=UNIFORM,Corr=SAMESMALL)

	Rule	Efficiency	Incentives	Revenue	
	QUADRATIC	98.13%	0.14	93.14%	
	Improvement Over QUADRATIC				**Avg.**
Best Efficiency	FRACTIONAL*(R=ZERO,W=π_{VCG})	0.32%	7.43%	2.82%	3.52%
Best Incentives	FRACTIONAL*(R=ZERO,W=p_{SHAPLEY}, A=0.5)	0.29%	7.89%	2.60%	3.59%
Best Revenue	FRACTIONAL*(R=ZERO,W=EQUAL)	0.24%	2.00%	3.29%	1.84%

Table 10: Results for LLG(MD=UNIFORM,Corr=SAMESMALL). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

D.2. LLG(MD=UNIFORM,Corr=SAME)

	Rule	Efficiency	Incentives	Revenue	
	QUADRATIC	98.41%	0.11	95.19%	
	Improvement Over QUADRATIC				**Avg.**
Best Efficiency	FRACTIONAL*(R=ZERO,W=π_{VCG}, A=2)	0.24%	6.77%	2.65%	3.22%
Best Incentives	FRACTIONAL*(R=ZERO,W=p_{SHAPLEY}, A=0.5)	0.17%	8.46%	2.93%	3.85%
Best Revenue	FRACTIONAL*(R=ZERO,W=$\pi^{-1}_{\text{SHAPLEY}}$)	0.04%	-0.61%	4.00%	1.14%

Table 11: Results for LLG(MD = UNIFORM,Corr = SAME). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

D.3. LLG(MD=UNIFORM,Corr=SAMELARGE)

	Rule	Efficiency	Incentives	Revenue	
	QUADRATIC	98.90%	0.06	98.37%	
	Improvement Over QUADRATIC				**Avg.**
Best Efficiency	FRACTIONAL*(R=p_{SHAPLEY},W=p^{-1}_{VCG}, A=10)	0.17%	8.72%	1.74%	3.54%
Best Incentives	FRACTIONAL*(R=ZERO,W=π_{VCG}, A=2)	0.13%	10.05%	2.40%	4.20%
Best Revenue	FRACTIONAL*(R=ZERO,W=p^{-1}_{VCG}, A=10)	-0.09%	4.14%	2.54%	2.20%

Table 12: Results for LLG(MD=UNIFORM,Corr=SAMELARGE). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

E. UNIFORM Marginals and Cross-Side Correlation

E.1. LLG(MD=UNIFORM,Corr=CROSSSMALL)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	97.87%	0.17	92.93%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=p^{M}_{SHAPLEY},W=$\pi^{-1}_{\text{SHAPLEY}}$, A=5)	0.57%	10.90%	2.09%	4.52%
Best Incentives	FRACTIONAL*(R=p^{M}_{SHAPLEY},W=π^{-1}_{VCG}, A=5)	0.55%	12.08%	2.13%	4.92%
Best Revenue	FRACTIONAL*(R=p^{M}_{SHAPLEY},W=π^{-1}_{VCG}, A=5)	0.55%	12.08%	2.13%	4.92%

Table 13: Results for LLG(MD=UNIFORM,Corr=CROSSSMALL). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

E.2. LLG(MD=UNIFORM,Corr=CROSS)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	97.69%	0.18	95.27%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=ZERO,W=π_{VCG}, A=0.5)	1.04%	23.77%	4.20%	9.67%
Best Incentives	FRACTIONAL*(R=ZERO,W=π_{VCG}, A=0.5)	1.04%	23.77%	4.20%	9.67%
Best Revenue	FRACTIONAL*(R=ZERO,W=π_{VCG}, A=0.5)	1.04%	23.77%	4.20%	9.67%

Table 14: Results for LLG(MD=UNIFORM,Corr=CROSS). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

E.3. LLG(MD=UNIFORM,Corr=CROSSLARGE)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	97.35%	0.19	97.51%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=ZERO,W=EQUAL)	2.02%	48.25%	11.88%	20.72%
Best Incentives	FRACTIONAL*(R=ZERO,W=EQUAL)	2.02%	48.25%	11.88%	20.72%
Best Revenue	FRACTIONAL*(R=ZERO,W=EQUAL)	2.02%	48.25%	11.88%	20.72%

Table 15: Results for LLG(MD=UNIFORM,Corr=CROSSLARGE). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

F. BETA Marginals and No Correlation

F.1. LLG(MD=BETA(5/3, 3))

Rule	Efficiency	Incentives	Revenue
QUADRATIC	97.95%	0.12	89.93%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=p_{VCG},W=π^{-1}_{VCG})	0.19%	3.19%	0.56%	1.32%
Best Incentives	FRACTIONAL*(R=p_{VCG},W=π^{-1}_{VCG})	0.19%	3.19%	0.56%	1.32%
Best Revenue	FRACTIONAL*(R=p_{VCG},W=π^{-1}_{VCG})	0.19%	3.19%	0.56%	1.32%

Table 16: Results for LLG(MD=BETA(5/3, 3)). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

F.2. LLG(MD=BETA(5/3, 3), UNCALIBRATED)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	98.83%	0.14	80.99%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=p_{VCG},W=π^{-1}_{VCG})	0.10%	2.78%	1.19%	1.36%
Best Incentives	FRACTIONAL*(R=p_{VCG},W=π^{-1}_{VCG})	0.10%	2.78%	1.19%	1.36%
Best Revenue	FRACTIONAL*(R=p_{VCG},W=π^{-1}_{VCG}, A=2)	0.09%	-0.69%	1.28%	0.23%

Table 17: Results for LLG(MD=BETA(5/3, 3), UNCALIBRATED). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

F.3. LLG(MD=BETA(1/3, 1/3), CALIB = UNCALIB)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	98.39%	0.15	97.26%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=$p_{SHAPLEY}$,W=π_{VCG}, A=5)	0.43%	9.95%	2.55%	4.31%
Best Incentives	FRACTIONAL*(R=$p_{SHAPLEY}$,W=$\pi_{SHAPLEY}$, A=5)	0.41%	10.46%	2.36%	4.41%
Best Revenue	FRACTIONAL*(R=$p^{M}_{SHAPLEY}$,W=$\pi^{-1}_{SHAPLEY}$, A=5)	0.43%	5.96%	2.82%	3.07%

Table 18: Results for LLG(MD=BETA(1/3, 1/3), CALIB = UNCALIB). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

F.4. LLG(MD=BETA(3, 3), CALIB = UNCALIB)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	97.80%	0.17	88.67%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=p_{VCG},W=$\pi^{-1}_{\text{SHAPLEY}}$)	0.44%	6.69%	3.57%	3.57%
Best Incentives	FRACTIONAL*(R=p_{VCG},W=π^{-1}_{VCG})	0.39%	7.58%	2.97%	3.65%
Best Revenue	FRACTIONAL*(R=p_{VCG},W=$\pi^{-1}_{\text{SHAPLEY}}$)	0.44%	6.69%	3.57%	3.57%

Table 19: Results for LLG(MD=BETA(3, 3), UNCALIBRATED). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

F.5. LLG(MD=BETA(3, 5/3))

Rule	Efficiency	Incentives	Revenue
QUADRATIC	97.78%	0.22	88.63%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=p_{VCG},W=BID^{-1})	0.62%	9.71%	5.86%	5.40%
Best Incentives	FRACTIONAL*(R=$p^{\text{M}}_{\text{SHAPLEY}}$,W=$p^{-1}_{\text{SHAPLEY}}$, A=3)	0.58%	10.89%	5.07%	5.51%
Best Revenue	FRACTIONAL*(R=p_{VCG},W=π^{-1}_{VCG}, A=3)	0.55%	3.64%	6.02%	3.40%

Table 20: Results for LLG(MD=BETA(3, 5/3)). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

F.6. LLG(MD=BETA(3, 5/3), UNCALIBRATED)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	96.77%	0.19	101.20%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=p_{VCG},W=BID^{-1})	0.90%	9.59%	4.22%	4.90%
Best Incentives	FRACTIONAL*(R=ZERO,W=p_{SHAPLEY}, A=0.5)	0.87%	10.89%	3.97%	5.24%
Best Revenue	FRACTIONAL*(R=p_{VCG},W=BID^{-1})	0.90%	9.59%	4.22%	4.90%

Table 21: Results for LLG(MD=BETA(3, 5/3), UNCALIBRATED). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

F.7. LLG(MD=BETA(3, 1/3))

Rule	Efficiency	Incentives	Revenue
QUADRATIC	97.79%	0.29	88.24%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=p_{VCG},W=π^{-1}_{VCG}, A=3)	1.19%	21.00%	14.68%	12.29%
Best Incentives	FRACTIONAL*(R=$p^{\text{M}}_{\text{SHAPLEY}}$,W=$p^{-1}_{\text{SHAPLEY}}$, A=3)	1.15%	25.74%	12.28%	13.06%
Best Revenue	FRACTIONAL*(R=BID,W=p_{SHAPLEY}, A=2)	0.99%	6.98%	15.80%	7.92%

Table 22: Results for LLG(MD=BETA(3, 1/3)). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

F.8. LLG(MD=BETA(3, 1/3), UNCALIBRATED)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	95.08%	0.21	141.89%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=ZERO,W=π_{VCG}, A=0.5)	2.69%	20.42%	6.41%	9.84%
Best Incentives	FRACTIONAL*(R=$p^{\text{M}}_{\text{SHAPLEY}}$,W=$p^{-1}_{\text{SHAPLEY}}$, A=3)	2.58%	24.97%	6.33%	11.30%
Best Revenue	FRACTIONAL*(R=p_{VCG},W=π^{-1}_{VCG}, A=3)	2.68%	20.38%	6.48%	9.85%

Table 23: Results for LLG(MD=BETA(3, 1/3), UNCALIBRATED). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

G. BETA Marginals and Same-Side Correlation Results

G.1. LLG(MD=BETA(5/3, 3),CORR=SAME)

	Rule	Efficiency	Incentives	Revenue	
	QUADRATIC	98.29%	0.08	95.06%	
	Improvement Over QUADRATIC				**Avg.**
Best Efficiency	FRACTIONAL*(R=p^{M}_{SHAPLEY},W=p^{-1}_{SHAPLEY}, A=10)	0.17%	7.04%	2.25%	3.16%
Best Incentives	FRACTIONAL*(R=BID,W=π_{SHAPLEY}, A=2)	0.14%	7.95%	2.57%	3.55%
Best Revenue	FRACTIONAL*(R=ZERO,W=p^{-1}_{SHAPLEY}, A=10)	-0.02%	-1.24%	3.78%	0.84%

Table 24: Results for LLG(MD=BETA(5/3, 3),CORR=SAME). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

G.2. LLG(MD=BETA(5/3, 3),CORR=SAME, UNCALIBRATED)

	Rule	Efficiency	Incentives	Revenue	
	QUADRATIC	99.31%	0.06	88.33%	
	Improvement Over QUADRATIC				**Avg.**
Best Efficiency	FRACTIONAL*(R=ZERO,W=π_{VCG}, A=2)	0.05%	6.78%	2.70%	3.18%
Best Incentives	FRACTIONAL*(R=ZERO,W=p_{SHAPLEY}, A=0.5)	0.01%	10.07%	4.39%	4.82%
Best Revenue	FRACTIONAL*(R=ZERO,W=BID^{-1}, A=10)	-0.12%	-3.73%	6.93%	1.03%

Table 25: Results for LLG(MD = BETA(5/3, 3),CORR = SAME, UNCALIBRATED). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

G.3. LLG(MD=BETA(1/3, 1/3),CORR=SAME, CALIB = UNCALIB)‡

Table omitted because the QUADRATIC benchmark fails to converge, as described above.

G.4. LLG(MD=BETA(3,3),CORR=SAME, CALIB = UNCALIB)

	Rule	Efficiency	Incentives	Revenue	
	QUADRATIC	98.04%	0.13	94.21%	
	Improvement Over QUADRATIC				**Avg.**
Best Efficiency	FRACTIONAL*(R=ZERO,W=BID, A=0.5)	0.37%	12.94%	6.38%	6.56%
Best Incentives	FRACTIONAL*(R=BID^M,W=p^{-1}_{VCG}, A=0.5)	0.37%	13.29%	7.01%	6.89%
Best Revenue	FRACTIONAL*(R=ZERO,W=p^{-1}_{SHAPLEY}, A=2)	0.11%	-0.34%	10.56%	3.44%

Table 26: Results for LLG(MD = BETA(3,3),CORR = SAME, UNCALIBRATED). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

G.5. LLG(MD=BETA(3, 5/3),CORR=SAME)

	Rule	Efficiency	Incentives	Revenue	
	QUADRATIC	97.96%	0.17	94.23%	
	Improvement Over QUADRATIC				**Avg.**
Best Efficiency	FRACTIONAL*(R=BID^M,W=p^{-1}_{VCG})	0.55%	15.05%	10.24%	8.61%
Best Incentives	FRACTIONAL*(R=BID,W=p_{VCG}, A=0.5)	0.52%	15.86%	8.41%	8.26%
Best Revenue	FRACTIONAL*(R=ZERO,W=$\pi^{-1}_{\text{SHAPLEY}}$, A=3)	0.01%	-7.38%	13.31%	1.98%

Table 27: Results for LLG(MD=BETA(3, 5/3),CORR=SAME). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

G.6. LLG(MD=BETA(3, 5/3),CORR=SAME, UNCALIBRATED)

	Rule	Efficiency	Incentives	Revenue	
	QUADRATIC	96.67%	0.16	107.77%	
	Improvement Over QUADRATIC				**Avg.**
Best Efficiency	FRACTIONAL*(R=ZERO,W=π_{VCG}, A=0.5)	0.97%	10.98%	6.97%	6.30%
Best Incentives	FRACTIONAL*(R=BID^M,W=BID^{-1}, A=2)	0.91%	12.88%	6.17%	6.65%
Best Revenue	FRACTIONAL*(R=ZERO,W=BID^{-1})	0.69%	1.55%	8.96%	3.73%

Table 28: Results for LLG(MD = BETA(3, 5/3),CORR = SAME, UNCALIBRATED). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

G.7. LLG(MD=BETA(3, 1/3),Corr=SAME)

	Rule	Efficiency	Incentives	Revenue	
	QUADRATIC	97.74%	0.26	94.85%	
	Improvement Over QUADRATIC				**Avg.**
Best Efficiency	FRACTIONAL*(R=ZERO,W=EQUAL)	1.02%	19.03%	16.97%	12.34%
Best Incentives	FRACTIONAL*(R=BID,W=π_{SHAPLEY},A=2)	0.97%	22.90%	11.74%	11.87%
Best Revenue	FRACTIONAL*(R=ZERO,W=BID^{-1},A=0.5)	0.61%	0.07%	20.67%	7.12%

Table 29: Results for LLG(MD=BETA(3, 1/3),Corr=SAME). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

G.8. LLG(MD=BETA(3, 1/3),Corr=SAME, UNCALIBRATED)†

Table omitted because enough rules failed to converge to meet our criterion, as described above.

H. BETA Marginals and Cross-Side Correlation Results

H.1. LLG(MD=BETA(5/3, 3),CORR=CROSS)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	97.68%	0.13	94.26%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=ZERO,W=π_{VCG}, A=0.5)	1.01%	24.18%	4.03%	9.74%
Best Incentives	FRACTIONAL*(R=ZERO,W=π_{VCG}, A=0.5)	1.01%	24.18%	4.03%	9.74%
Best Revenue	FRACTIONAL*(R=ZERO,W=$\pi_{SHAPLEY}$, A=0.5)	0.97%	23.27%	4.33%	9.52%

Table 30: Results for LLG(MD=BETA(5/3, 3),CORR=CROSS). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

H.2. LLG(MD=BETA(5/3, 3),CORR=CROSS, UNCALIBRATED)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	98.65%	0.15	82.40%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=p_{VCG},W=π^{-1}_{VCG}, A=3)	0.62%	25.24%	7.46%	11.10%
Best Incentives	FRACTIONAL*(R=p_{VCG},W=π^{-1}_{VCG}, A=3)	0.62%	25.24%	7.46%	11.10%
Best Revenue	FRACTIONAL*(R=ZERO,W=EQUAL)	0.62%	23.84%	7.62%	10.69%

Table 31: Results for LLG(MD=BETA(5/3, 3),CORR=CROSS, UNCALIBRATED). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

H.3. LLG(MD=BETA(1/3, 1/3),CORR=CROSS, CALIB = UNCALIB)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	97.67%	0.18	102.48%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=ZERO,W=π_{VCG}, A=0.5)	1.20%	25.45%	3.56%	10.07%
Best Incentives	FRACTIONAL*(R=$p^{M}_{SHAPLEY}$,W=BID^{-1}, A=3)	1.11%	25.63%	3.71%	10.15%
Best Revenue	FRACTIONAL*(R=$p^{M}_{SHAPLEY}$,W=$\pi^{-1}_{SHAPLEY}$, A=3)	1.11%	25.62%	3.71%	10.15%

Table 32: Results for LLG(MD=BETA(1/3, 1/3),CORR=CROSS, UNCALIBRATED). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

H.4. LLG(MD=BETA(3,3),Corr=CROSS, CALIB = UNCALIB)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	97.64%	0.17	90.39%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=ZERO,W=EQUAL)	1.45%	34.24%	12.36%	16.02%
Best Incentives	FRACTIONAL*(R=BID^M,W=$\pi^{-1}_{\text{SHAPLEY}}$, A=2)	1.42%	34.98%	11.66%	16.02%
Best Revenue	FRACTIONAL*(R=ZERO,W=EQUAL)	1.45%	34.24%	12.36%	16.02%

Table 33: Results for LLG(MD = BETA(3,3),Corr = CROSS, UNCALIBRATED). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

H.5. LLG(MD=BETA(3, 5/3),Corr=CROSS)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	97.65%	0.22	89.60%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=ZERO,W=EQUAL)	1.55%	36.92%	15.28%	17.92%
Best Incentives	FRACTIONAL*(R=BID,W=π_{SHAPLEY}, A=2)	1.53%	38.14%	14.17%	17.95%
Best Revenue	FRACTIONAL*(R=BID,W=p_{SHAPLEY}, A=2)	1.53%	35.81%	15.42%	17.59%

Table 34: Results for LLG(MD=BETA(3, 5/3),Corr=CROSS). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

H.6. LLG(MD=BETA(3, 5/3),Corr=CROSS, UNCALIBRATED)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	96.64%	0.19	103.92%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=ZERO,W=EQUAL)	2.20%	35.84%	10.85%	16.29%
Best Incentives	FRACTIONAL*(R=BID,W=π_{SHAPLEY}, A=2)	2.18%	37.44%	10.58%	16.73%
Best Revenue	FRACTIONAL*(R=ZERO,W=EQUAL)	2.20%	35.84%	10.85%	16.29%

Table 35: Results for LLG(MD = BETA(3, 5/3),Corr = CROSS, UNCALIBRATED). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

H.7. LLG(MD=BETA(3, 1/3),Corr=CROSS)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	97.76%	0.30	88.39%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=BID,W=BID, A=2)	1.75%	45.11%	23.28%	23.38%
Best Incentives	FRACTIONAL*(R=BID^M,W=$\pi^{-1}_{\text{SHAPLEY}}$, A=2)	1.73%	46.97%	20.75%	23.15%
Best Revenue	FRACTIONAL*(R=BID^M,W=p^{-1}_{SHAPLEY}, A=2)	1.70%	41.82%	23.43%	22.32%

Table 36: Results for LLG(MD=BETA(3, 1/3),Corr=CROSS). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

H.8. LLG(MD=BETA(3, 1/3),Corr=CROSS, UNCALIBRATED)

Rule	Efficiency	Incentives	Revenue
QUADRATIC	95.00%	0.21	143.59%

	Improvement Over QUADRATIC				Avg.
Best Efficiency	FRACTIONAL*(R=BID,W=π_{SHAPLEY}, A=2)	3.99%	47.20%	10.02%	20.40%
Best Incentives	FRACTIONAL*(R=BID,W=π_{SHAPLEY}, A=2)	3.99%	47.20%	10.02%	20.40%
Best Revenue	FRACTIONAL*(R=BID,W=π_{SHAPLEY}, A=2)	3.99%	47.20%	10.02%	20.40%

Table 37: Results for LLG(MD=BETA(3, 1/3),Corr=CROSS, UNCALIBRATED). The first row shows the performance of QUADRATIC. The subsequent rows show the top rules for each dimension.

I. Results for LLLLGG(MD=UNIFORM)

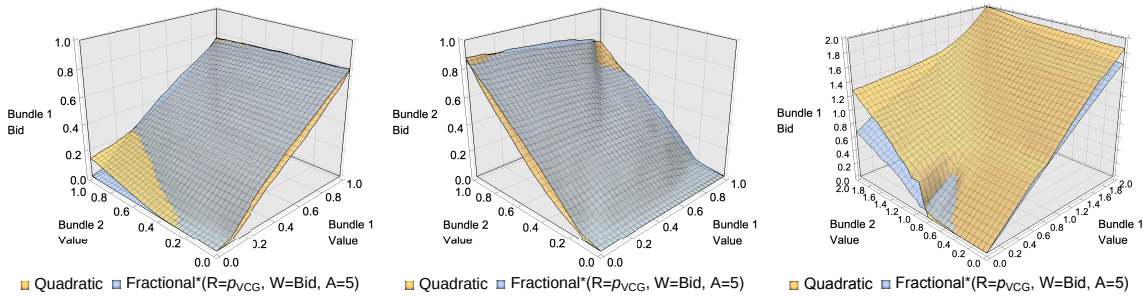
Below we present the result for LLLLGG(MD=UNIFORM), where the QUADRATIC rule already has very high efficiency in BNE such that there is essentially no room left for improvement.

	Rule	Efficiency	Incentives	Revenue	
	QUADRATIC	99.8%	0.266	110.5%	
	Improvement Over QUADRATIC				**Avg.**
1	FRACTIONAL*(R=p_{VCG},W=BID^{-1})	0.0%	23.7%	2.4%	8.7%
2	FRACTIONAL*(R=p_{VCG},W=BID^{-1}, A=0.5)	0.1%	21.1%	1.9%	7.7%
3	FRACTIONAL*(R=ZERO,W=BID)	0.0%	16.4%	1.6%	6.0%
4	FRACTIONAL*(R=ZERO,W=EQUAL)	-0.1%	3.8%	0.8%	1.5%
5	FRACTIONAL*(R=ZERO,W=BID, A=0.5)	-0.1%	2.2%	0.9%	1.0%
6	FRACTIONAL*(R=ZERO,W=π_{VCG}, A=0.5)	-0.1%	2.2%	0.3%	0.8%
7	FRACTIONAL*(R=ZERO,W=π_{VCG})	-0.1%	0.5%	0.5%	0.3%
8	FRACTIONAL*(R=p_{VCG},W=BID, A=5)	-.4%	-299.8%	-51.6%	-117.28%
9	First-Price	-3.5%	-371.9%	-11.4%	-128.9%

Table 38: Results for LLLLGG(MD=UNIFORM). The first row shows the performance of QUADRATIC. In the subsequent rows we show seven of our best all-rounder rules from LLG, followed by one of the worst rules we identified in LLG. For comparison, we also include the First-Price rule at the bottom.

J. BNE Strategies of a Badly-Performing Rule in LLLLGG

In Figure 5, we show the BNE strategies for QUADRATIC and for one of the worst MRC rules we found in LLG, FRACTIONAL*(R=p_{VCG},W=BID, A=5), in LLLLGG(MD=BETA(3, 1/3)). We observe that the FRACTIONAL*(R=p_{VCG},W=BID, A=5) induces worse incentives than QUADRATIC for global bidders with large values. It also provides worse incentives to the local bidders when they have a large value for both bundles. Note that due to the skewedness of the marginal distributions, these cases (with high values) are particularly likely in this domain, and thus matter even more than usual for efficiency and revenue. This explains why FRACTIONAL*(R=p_{VCG},W=BID, A=5) achieves worse efficiency and revenue than QUADRATIC. Thus, again, our findings from LLG for a badly-performing rule also translate well to LLLLGG in this case.



(a) First strategy of the local player. (b) Second strategy of the local player. (c) First strategy of the global player; her second strategy is symmetric.

Figure 5: BNE strategies for QUADRATIC and FRACTIONAL*(R=p_{VCG},W=BID,A=5) in LLLLGG(MD=BETA(3, 1/3))