Portfolio Compression in Financial Networks: Incentives and Systemic Risk

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Abstract

We study portfolio compression, a post-trade mechanism that eliminates cycles in a financial network. We study the incentives for banks to engage in compression and its systemic effects in terms of all banks’ equities. We show that, contrary to conventional wisdom, compression may be socially and individually detrimental and incentives may be misaligned with the social good. We show that these effects depends on the parameters of the financial system and the compression in a complex and non-monotonic way. We then present sufficient conditions under which compression is incentivized for participating banks or a Pareto improvement for all banks. More in detail, compression is universally beneficial when interbank payments are subject to high default costs, when recovery rates are high, or when the participating banks’ balance sheets are sufficiently homogeneous. Furthermore, we show that banks only have an incentive to reject a compression if there are feedback paths of links that are not compressed. Our results contribute to a better understanding of the implications of recent regulatory policy.

1 Introduction

The 2008 financial crisis is widely regarded as the result of a shock on a complex, opaque, unregulated network of dependencies among financial institutions.¹ In normal times, connections to other institutions provide financial actors with means to secure funding and hedge risks. Since potential losses are distributed to many other institutions, diversification stabilizes the system as a whole. In the 2008 crisis, however, it quickly became clear that this network could just as well amplify and spread losses between markets and institutions. Like a disease, financial distress traveled through the network and “infected” institutions — an idea known as financial contagion.

Financial regulators realized that, in the process of creating the financial network, financial institutions (just “banks” from now on) had not just taken on individual

¹For this paragraph, see the well-known speeches by then Executive Director of Financial Stability at the Bank of England Andrew Haldane (2009) and then Vice Chair of the Federal Reserve Janet Yellen (2013).
risk, but also created *systemic risk*, which endangered the financial system as a whole. After the crisis, different regulatory policies were put into place to reduce the “excessive systemic risk arising from the complexity and interconnectedness that characterize our financial system” (Yellen, 2013). Many of these policies were aimed directly at reducing interconnectedness. After all, reducing interconnectedness should also limit the “channels” through which financial contagion could spread. For some financial products, this was achieved using *central clearing counterparties (CCPs)*: central nodes that act as middlemen to all trades and serve as buffers in case one of the participants in the trade is unable to pay its obligations.

For other products, *portfolio compression* was made mandatory. The idea is simple: banks’ daily business activities have the side effect of creating *cycles* of obligations in the network, where (say) a bank A has an obligation to a bank B, which has an obligation to a third bank C, which has an obligation back to bank A — all for the same financial product with the same parameters. Removing these cycles reduces interconnectedness without affecting any bank’s net position with respect to any product. Portfolio compression is the process of doing just that. This paper studies the effects of portfolio compression on systemic risk and the incentives for banks to engage in it.

Portfolio compression (just *compression* from now on) originated in the private sector and has only later been endorsed by regulators. Compression has several immediate benefits for the participating banks: it reduces their exposure to other banks, it reduces their operational costs to keep track of their claims and obligations, and, perhaps most importantly, it reduces the sizes of their balance sheets and thus the amount of capital they need to hold available.

In practice, several financial service providers offer compression services. We explain the process at the example of TriOptima’s service triReduce. First, participating institutions submit the trades they would like to compress. Second, the service provider combines the information submitted by all participants to construct the network and calculates an *unwind proposal*, i.e., a collection of suggested contract

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2 Reducing interconnectedness has another important benefit, namely improving transparency. — A less dense network is easier to analyze for regulatory bodies when the next crisis strikes. In fact, many of the reforms after the crisis were geared towards only making the network of interconnections *visible* to regulatory bodies. See Financial Stability Board (2017) for an overview of the new regulatory measures for the OTC derivatives market, which is the main financial market relevant for the present piece of work.

3 EMIR regulations include an “obligation to have procedures to analyse the possibility to conduct the exercise” of portfolio compression when counterparties have more than 500 contracts with each other (European Securities and Markets Authority, 2017).

4 Basel III regulations require banks to hold liquid assets proportional to their risk-weighted assets (Basel Committee on Banking Supervision, 2011). This creates an incentive to, other things equal, keep assets and liabilities as low as possible.

5 The most important ones are: TriOptima, LMRKTS, Markit, Catalyst, and SwapClear (D’Errico and Roukny, 2019).

modifications and terminations that together constitute the compression. The unwind proposal is sent to the banks. Crucially, the unwind proposal itself carries no legal force. Instead, all involved banks need to agree to the proposal in a third step. Once this has happened, the compression is implemented. In practice, banks generally seem to agree to unwind proposals.7

Compression is used in markets for OTC derivatives where insufficient standardization prohibits the use of a CCP, specifically: interest rate swaps, currency swaps, and credit default swaps. According to TriOptima, their service has removed over USD 1.5 quadrillion in notional since its inception in 2003. Aldasoro and Ehlers (2018) cite compression as one of the main reasons for the drop in size of the credit default swap market after the 2008 financial crisis.

At first sight, one might think that compression is always (weakly) beneficial for all banks: compression reduces the liabilities of involved banks at least as much as their assets (at fair value) and thus benefits involved banks, and it has no effect on the balance sheets of the other banks. However, in this paper, we show that, in a networked system, this simple local view of compression is incorrect, as it does not capture systemic effects.

We study compression in a static setting ex-post to a shock, considering the following time line of events. Initially, all banks were solvent. Then compression was performed. At some point, a shock hit the banks, which caused some of the banks to default on their obligations and triggered a contagion process. We compare the final outcome of this process (modeled by the Rogers and Veraart (2013) clearing model) to the outcome had compression not been performed. In our analysis, we focus on two questions: first, was compression economically efficient in retrospect in terms of banks’ equities? We consider both a social welfare as well as a Pareto perspective. Second, had the involved banks known the shock upfront, should they still have agreed to compression, i.e., was compression incentivized in retrospect?

By incentivized we mean that all banks involved in the compression would agree to it in step three of the aforementioned compression process. We assume that banks derive a utility from any given (compressed or uncompressed) network equal to their equity and decide based on that. Note that in this paper, our goal is not to perform an equilibrium analysis. We do not consider compressions actions that banks perform.

Further complications, which we not model in this paper, apply. The main reason for this is that compression is also performed among products whose characteristics (such as maturity, coupons, start and end dates) do not match exactly, but only approximately. This increases the amount of compression that can be done, but it may necessitate compensation payments. These payments are determined using a special market mechanism called a compression auction (Duffie, 2018). In this paper, we abstract away from these complications by assuming that all contracts in the network have the same characteristics. We further do not consider any exotic variants of compression, where one may be allowed to both add and remove liabilities. That is, we only consider conservative compression (D’Errico and Roukny, 2019) in this paper.

Over-the-counter, i.e., traded directly between banks, rather than through an exchange. In this work, we only consider OTC derivatives markets.
Rather, we assume that compressions are suggested by a central financial service provider that is not a bank.

For a given network, shocks, and compression, it is easy to answer the above questions regarding economic efficiency and incentives algorithmically: simply evaluate the clearing model on the uncompressed and the compressed network and compare the two outcomes. However, we want to answer these questions at a more general level: what determines whether or not compression will be efficient or incentivized? When are incentives misaligned with the social good? How do these effects depend on the structure of the financial system?

Portfolio compression is systemically important, as evidenced by regulators’ attention to it. However, while there is an extensive literature on netting in the context of CCPs (for example, Duffie and Zhu, 2011; Duffie, Scheicher and Vuilleme, 2015; Amini, Filipović and Minca, 2015; Cui et al., 2018), there has only been a surprisingly small amount of work on compression (without a CCP). Where compression was studied, authors mostly focused on algorithmic questions to achieve the optimal compressed amount subject to risk tolerances (O’Kane, 2017; D’Errico and Roukny, 2019). The impact of compression was therefore measured in terms of eliminated notional, rather than the consequences of a shock. Many researchers have studied the effect of different aspects of network structure in general on systemic risk (e.g., Elliott, Golub and Jackson, 2014; Glasserman and Young, 2015; Acemoglu, Ozdaglar and Tahbaz-Salehi, 2015; Demange, 2016). However, these results can at best provide a very broad estimate of what the specific changes to the network structure due to compression may imply. We discuss this in Appendix A. Feinstein et al. (2017) studied the sensitivity of the outcome of a crisis to changes in relative interbank liabilities, where the total liabilities of each bank remain the same. Compression, however, explicitly reduces the liabilities of each bank involved, which is why their theory does not apply.

In a recent research note, we were, to the best of our knowledge, the first to present an example where portfolio compression could be detrimental in terms of social welfare (Schuldenzucker, Seuken and Battiston, 2018). The only other piece of work we are aware of that deals with the effects of compression on systemic risk specifically is Veraart (2019). The author studied compression in much the same way as we do and obtained sufficient conditions under which compression is beneficial for all banks. The author also showed that compression can have negative effects both on the banks involved in it and on other banks and this can depend on assets external to the financial network. These results can be viewed as special cases of some of the results obtained in the present paper.

In the present paper we show that, in contrast to conventional wisdom, compression is not in general socially beneficial in terms of banks’ equities. There are cases where compression is socially detrimental or even hurts every bank in the system in a Pareto
sense. We further show that whether or not compression is socially beneficial depends on the parameters of the financial system and on the compression in a complex and non-monotonic way. The same applies to banks’ incentives to agree to compression. Their incentives may further be misaligned with the social good (Section 3).

We then derive sufficient conditions under which these complications do not apply, i.e., where compression is always a Pareto improvement with respect to banks’ equities. These conditions are local in the sense that they only depend on properties of individual banks and banks’ assets in the uncompressed system, but do not require any global network computation that depends on the particular compression in question. We find that compression is beneficial when banks are well capitalized or when all banks are involved in the compression to a certain degree. These conditions scale with the default costs in the system: our results become stronger when defaulting banks lose a higher share of their incoming payments due to frictions such as legal costs or early settlement fees (Section 4).

From these “local” conditions, we then turn to the network structure itself. We find that whether or not the involved banks have an incentive to agree to compression crucially depends on the presence of feedback paths, i.e., paths of liabilities that are not involved in the compression and that lead from an involved bank to another involved bank. If feedback paths do not exist and a normality condition is met, compression is always incentivized. This provides a possible explanation why banks virtually always agree to compression in practice while our results in Section 3 imply that this is a complex strategic decision: banks may not be taking the possibility of feedback paths into account (Section 5).

Finally, we turn to a high-level property of the financial network structure, namely homogeneity of the involved banks. We show that if a collection of asset and liability measures and the compressed amount of liabilities are equal across all involved banks, then compression is beneficial for all (involved and non-involved) banks. A study of an example network suggests a quantified version: the lower the default costs in the system, the more homogeneity is required to make compression beneficial for all banks (Section 6).

With the present paper, we are among the first to conduct a principled theoretical study of the effects of portfolio compression on systemic risk. We consider this paper a first step at the beginning of a larger program of research. Given that compression has already found its way into regulatory policy, we believe that a thorough understanding of compression is urgently needed. We discuss possible next steps in this direction in Section 7.
2 Preliminaries

We assume that we are given a financial network ex-post to a shock. We then perform network clearing in the compressed financial network and in the network had compression not been done and we compare the two outcomes by their efficiency and incentives. We now formally define the three elements of this approach: our model of (post-shock) financial systems and clearing payments, our formalization of portfolio compression, and our measure of utilities, efficiency, and incentives in a financial system.

2.1 Basic Notation

Throughout this paper, we employ the following notation. Matrices and vectors always range over arbitrary finite sets of indices. That is, we do not implicitly assume that vectors range over sets of form \( \{1, \ldots, n\} \). Ordering of matrices and vectors is always point-wise. That is, if \( N \) and \( M \) are sets and \( p, q \in \mathbb{R}^{N \times M} \) are matrices, we write \( p \leq q \) iff \( p_{ij} \leq q_{ij} \) \( \forall i \in N, j \in M \). We write \([0, q]\) for the set of \( p \) such that \( 0 \leq p \leq q \), i.e., \( 0 \leq p_{ij} \leq q_{ij} \) \( \forall i, j \). Vectors are treated analogously. If \( A \subseteq N \times M \), we write \( p_A \) for the restriction of \( p \) to pairs of indices in \( A \), i.e., \( p_A = (p_{ij})_{(i,j) \in A} \). Vectors are treated analogously. We use \( \cup \) to denote concatenation of vectors and matrices, respectively, with disjoint index sets. Thus, if \( A, B \subseteq N \times M \), \( A \cap B = \emptyset \), \( p \subseteq \mathbb{R}^A \), \( q \subseteq \mathbb{R}^B \), we write \( p \cup q \) for the matrix with indices \( A \cup B \) where \( (p \cup q)_{ij} = p_{ij} \) if \( (i, j) \in A \) and \( (p \cup q)_{ij} = q_{ij} \) if \( (i, j) \in B \).

2.2 Financial Systems and Clearing Payments

We use the clearing model from Rogers and Veraart (2013). In this sub-section, we provide a brief description of this model.

Financial System. Let \( N \) be a set of banks. We assume that each bank \( i \in N \) holds an amount of external assets \( e_i \geq 0 \). For any two banks \( i, j \in N \), let \( l_{ij} \geq 0 \) denote the amount that the writer \( i \) owes to the holder \( j \) of the contract (i.e., the liability). We also call this number the notional of the contract, where a notional of zero indicates the absence of a contract. If \( l_{ij} > 0 \), we also call \( j \) a creditor of \( i \) and \( i \) a debtor of \( j \). Assume that \( l_{ii} = 0 \) \( \forall i \), i.e., no bank has a contract with itself. We do not exclude the possibility that two banks are creditors of each other, i.e., \( l_{ij}, l_{ji} > 0 \). Note that the collection of all notional \( l = (l_{ij})_{i,j \in N} \) can be viewed as the adjacency matrix of a weighted graph with nodes \( N \) and \( e = (e_i)_{i \in N} \) can be viewed as a vector of node weights.

Following Rogers and Veraart (2013), we assume that we are further given two default cost parameters \( \alpha, \beta \in [0, 1] \). If a bank is in default, it is only able to pay to its creditors a share of \( \alpha \) of its external assets and a share of \( \beta \) of its incoming
payments from other banks. That is, a share of $1 - \alpha$ and $1 - \beta$, respectively, is lost.\footnote{Default costs may originate from various sources, corresponding to different combinations of the $\alpha$ and $\beta$ parameters. For example, we can model a setting where the external assets are illiquid, which leads to a price discount when they need to be sold quickly in case it defaults, by choosing $\alpha < 1$. Smaller $\alpha$ values model lower liquidity of the external assets. If the external assets describe perfectly liquid “cash” holdings, we set $\alpha = 1$. Similarly, interbank payments may be lossless ($\beta = 1$) or they may be subject to time delays, legal costs, or early settlement costs ($\beta < 1$). It is easy to extend the model to allow for per-bank $\alpha$ and $\beta$ values or several asset classes with different default cost levels. Our results easily generalize to these extensions with minor adjustments.}

If we set $\alpha = \beta = 1$ (no default costs), we receive the clearing model in Eisenberg and Noe (2001).

A tuple $X = (N, e, l, \alpha, \beta)$ is called a financial system.

**Assets and Liabilities.** Given such a financial system $X$, the total liabilities of a bank $i$ are

$$l_i := \sum_{j \in N} l_{ij}$$

and the relative liability of $i$ to $j$ is

$$\pi_{ij} := \begin{cases} \frac{l_{ij}}{l_i} & \text{if } l_i > 0 \\ 0 & \text{if } l_i = 0. \end{cases}$$

Note that $\sum_j \pi_{ij} = 1$ unless $l_i = 0$.

The matrix of liabilities defines how much money every bank is supposed to pay to its creditors. The amount of money that said bank will actually be able to pay will be lower in case it defaults. We capture these amounts in a matrix of payments $p_{ij} \in [0, l_{ij}]$. We are ultimately looking for a payment matrix that is clearing in a sense that will be defined shortly.

Given $p$, the total assets $a_i(p)$ of a bank $i$ at $p$ consist of its external assets and the incoming payments from other banks:

$$a_i(p) := e_i + \sum_{j \in N} p_{ji}$$

If a bank’s assets are insufficient to cover its liabilities, it is called in default. A defaulting bank $i$ has its assets reduced according to the factors $\alpha$ and $\beta$. That is, its assets after default costs are:

$$a_i'(p) := \alpha e_i + \beta \sum_{j \in N} p_{ji}$$

**Clearing Payment Matrix.** We call $p$ clearing if it follows the following fundamental principles of bankruptcy law:

1. Banks that are not in default pay their liabilities in full.
2. Banks that are in default pay out all their assets to creditors, after default costs have been subtracted.

3. By the principle of proportionality, the assets of defaulting banks are split up among creditors in proportion to the respective liability.

We thus call \( p \) clearing if it is the fixed point of the following function:

\[
\Psi : [0, l] \rightarrow [0, l]
\]

\[
\Psi_{ij}(p) := \begin{cases} l_{ij} & \text{if } a_i(p) \geq l_i \\ \pi_{ij}a'_i(p) & \text{if } a_i(p) < l_i \end{cases}
\] \hspace{1cm} (1)

In other words, \( p \) is clearing iff \( \Psi(p) = p \). The following theorem shows that a clearing matrix of payments always exists is essentially unique (unique up to a point-wise decrease in payments).\(^{10}\)

**Theorem** (Rogers and Veraart (2013, Theorem 3.1)). For any financial system \( X = (N, e, l, \alpha, \beta) \) there is a matrix \( p \) of payments such that i) \( p \in [0, l] \), ii) \( p = \Psi(p) \), and iii) if \( p' \) is another matrix with these properties, then \( p' \leq p \) point-wise.

We call \( p \) like in the theorem the maximal clearing payment matrix. While there could be several clearing \( p \), the point-wise maximal \( p \) is a canonical choice. Note in particular that maximizing \( p \) also maximizes the assets of each individual bank. Therefore, in this paper, we only consider the maximal clearing matrix of payments.

**Remark 1** (Algorithms for Computing Clearing Payments). Two well-known algorithms can be used to compute the maximal clearing payment matrix. The simplest one is to consider the iteration sequence defined by \( p^0 := l \) and \( p^{n+1} := \Psi(p^n) \). Since \( \Psi \) is monotonic with respect to the point-wise ordering, this sequence converges from above to the maximal clearing payment matrix.\(^{11}\) However, as soon as there is a cycle of defaulting banks, the sequence does not converge in finite time. An improved algorithm, called the fictitious default algorithm by Eisenberg and Noe (2001) and the greatest clearing vector algorithm by Rogers and Veraart (2013), skips over linear

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\(^{10}\) Note that this existence result is specific to financial networks where the liabilities between banks are fixed numbers. This model is appropriate when banks only enter into debt contracts, i.e., loans from one bank to another, or into financial derivatives that only depend on variables external to the financial system. In the latter case, liabilities can be assumed to be fixed or the purpose of clearing. In contrast, clearing payments need not exist (and need not be essentially unique) when banks can also enter into credit default swaps, i.e., default insurance, on other banks in the network (Schuldenzucker, Seuken and Battiston, 2019).

\(^{11}\) This is a consequence of the following theorem. If \( L \) is a complete lattice with maximal element \( \top \) and \( F : L \rightarrow L \) is monotonic and continuous from above, then the iteration sequence defined by \( x^0 := \top \) and \( x^{n+1} = F(x^n) \) is decreasing and converges to a fixed point \( x^* \) of \( F \) that dominates every other fixed point of \( F \). This theorem is a special case of the Kleene and Tarski-Kantorovich fixed-point theorems (see Granas and Dugundji (2003)) and has become a standard tool in the theory of financial networks, for example Rogers and Veraart (2013); Barucca et al. (2016). We provide a simple proof in Schuldenzucker, Seuken and Battiston (2019, Lemma 3). The statement for clearing payment matrices follows via \( L := [0, l] \) with the point-wise ordering and \( F := \Psi \).
stretches of the iteration sequence by solving a linear equation system and terminates with an exact solution after polynomially many steps.

Remark 2 (Individual payments, total payments, and recovery rates). Most pieces of prior work do not consider the matrix of individual payments \( p_{ij} \) as the fundamental object of clearing, but the vector of total payments \( p_i \) or the vector of recovery rates \( r_i \).

\[
    r_i := \begin{cases} 
        1 & \text{if } a_i(p) \geq l_i \\
        \frac{a_i(p)}{l_i} & \text{if } a_i(p) < l_i.
    \end{cases}
\]

It is easy to see that the definition (1) of \( \Psi \) implies that \( p_{ij} = \pi_{ij} p_i = l_{ij} r_i \), so that these three objects all define each other and being “clearing” can be defined in terms of either of them. In this paper, we find that operating on the individual payments is most convenient to establish relationships between the uncompressed and the compressed network.

2.3 Portfolio Compression

Portfolio compression is the process of netting liabilities between any number of banks while preserving each bank’s net position \( \sum_j l_{ji} - \sum_j l_{ij} \). We only consider conservative compression (D’Errico and Roukny, 2019) in this paper, i.e., point-wise reductions in liabilities. Again following D’Errico and Roukny (2019), we thus define a compression as any way how a given financial system can be compressed subject to these two constraints, i.e., as follows:

Definition 1 (Compression). Let \( X = (N, e, l, \alpha, \beta) \) be a financial system. A compression for \( X \) is a circulation in the weighted graph associated to \( l \) in the sense of network flow theory. That is, a compression is a matrix \( c \in [0, l] \) such that \( c_i := \sum_j c_{ij} = \sum_j c_{ji} \). If \( c \) is a compression for \( X \), the financial system \( X \) compressed by \( c \) is \( X^c := (N, e, l - c, \alpha, \beta) \). We also use a superscript \( \cdot^c \) for the liabilities, assets, etc., evaluated in \( X^c \). In particular, we write \( p^c \) for the maximal clearing payments in \( X^c \). Let \( N(c) = \{ i \in N \mid c_i > 0 \} \) be the set of banks involved in the compression.

Remark 3 (Cycles and Compressions). Veraart (2019) considered compression by cycles (i.e., closed directed paths in the liability graph) rather than circulations. A cycle can be viewed as a special case of a compression: if \( C \) is a cycle and \( 0 \leq \mu \leq \min_{(i,j) \in C} l_{ij} \), then \( c \) defined by

\[
    c_{ij} = \begin{cases} 
        \mu & \text{if } (i,j) \in C \\
        0 & \text{otherwise}
    \end{cases}
\]

is obviously a compression for \( X \). In this case, \( c_i = \mu \) if \( i \in C \) and \( c_i = 0 \) otherwise. By slight abuse of notation, we write \( c = (C, \mu) \) for this kind of compression. By the flow decomposition theorem (see, e.g., Korte and Vygen (2012, Chapter 8)), for any
There exists a finite (not generally unique) set $\mathcal{E}$ of pairs $(C, \mu)$ like above such that:

$$c_{ij} = \sum_{(C, \mu) \in \mathcal{E}} \mu$$

s.t. $(i,j) \in C$

Our most important example in the following considerations will be compressions of form $(C, \min_{(i,j) \in C} l_{ij})$, where $C$ is a cycle. However, we will show at the end of Section 4 that, to correctly capture the systemic effects of more complex compressions, it is not sufficient to apply flow decomposition and then consider individual cycles. Thus, our definition of compressions as circulations offers additional generality.

A compression of maximal value $\sum_i c_i$, where the largest possible amount of notional is eliminated, can be computed efficiently via linear programming (D’Errico and Roukny, 2019) or via combinatorial algorithms (e.g., Korte and Vygen (2012, Chapter 8)). Like for most flow problems, the naive greedy algorithm, where cycles are successively compressed until none are left, does not in general lead to a compression of maximal value.

### 2.4 Utilities, Efficiency, Welfare

Given a matrix of payments $p$, define the balance $B_i(p)$ of bank $i$ as the difference between assets and liabilities:

$$B_i(p) := a_i(p) - l_i$$

Note that $B_i(p) < 0$ if and only if $i$ defaults under $p$. Define the equity $E_i(p)$ as the balance if this is non-negative (i.e., if $i$ is not in default):

$$E_i(p) := \max (0, B_i(p)) = \max (0, a_i(p) - l_i)$$

If there is no risk of confusion, we leave out the $p$ argument and simply write $a_i$, $B_i$, and $E_i$ for the respective values under the maximal clearing payment matrix. We likewise write $a_i^c$, $B_i^c$, and $E_i^c$ for these values in the compressed financial system $X^c$ under the maximal payment matrix $p^c$.

The equity is the profit that the owners (i.e., the shareholders) of a bank get to keep after clearing. Just like how a stock price can never be negative, the equity value of a bank can never be negative, too. This is a consequence of limited liability of equity holders. By the principle of absolute priority of debt, equity is zero in case of default.\(^{12}\)

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\(^{12}\)There does not seem to be a standard for the definition of the term “equity”. While, for example, Barucca et al. (2016) and Veraart (2019) use the term “equity” to refer to what we call the balance, Eisenberg and Noe (2001) use the term in the same manner as we do. In this paper, it is important for us to differentiate between the two concepts of equity and balance.
We assume that banks derive a utility from a financial system that is equal to their equity under the maximal clearing matrix of payments. Recall that a compression is implemented if and only if each of the involved banks agrees to it, i.e., if it weakly increases the equity of each involved bank. We can now define Pareto efficiency, social welfare, and incentives in the usual way.

**Definition 2.** The social welfare in a financial system $X$ is the total equity

$$E_\Sigma := \sum_{i \in N} E_i.$$

If $c$ is a compression for $X$, then $c$ is called a (weak) Pareto improvement if $c$ weakly increases the equity of each bank in the system, i.e., if $E_i^c \geq E_i \forall i \in N$. The Pareto improvement is strict if the inequality is strict for at least one $i$. We call $c$ (weakly) incentivized if it weakly increases the equity of each bank involved in the compression (but it may reduce the equity of other banks). That is, we call $c$ incentivized if $E_i^c \geq E_i \forall i \in N(c)$. We call $c$ strictly incentivized if the inequality is strict for at least one $i$. Obviously, every Pareto improvement is incentivized and weakly increases social welfare.

### 2.5 Discussion of Our Formal Model

If $\alpha = \beta = 1$ (no default costs), then we know from Eisenberg and Noe (2001) that $E_\Sigma = e_\Sigma = E_\Sigma^c$, where $e_\Sigma := \sum_i e_i$. Thus, in this case, compression cannot be welfare-improving and there can in particular be no strict Pareto improvements. However, strictly incentivized compressions can exist and they can reallocate equity from banks outside the compression to those inside. Prior work that studied systemic risk in the model without default costs (Glasserman and Young, 2015; Demange, 2016) considered the aggregate payments (equivalently, the total balance) as a welfare measure. This was necessary because total equity is a trivial measure if there are no default costs. For the present paper, where we do assume default costs, it is easy to see that $E_\Sigma < e_\Sigma$ unless no bank defaults or $\alpha = \beta = 1$, so that total equity is not a trivial measure. We consider total equity a measure of social welfare that is better motivated and better in line with banks’ incentives than total payments or total balance.$^{13}$

Recall that for we do not assume that banks can actually make choices about compression after as shock has struck and clearing has already taken place. Banks that have already defaulted will be subject to severe limits on their capacity to contract.

$^{13}$For example, it is not clear what immediate value society would derive from payments being high *per se*. In contrast, equity has a clear interpretation as the amount of money that “leaves” the financial system after all obligations have been settled. Note that the interests of society or the real economy can be represented explicitly in this framework by adding them as synthetic nodes to the network. These nodes will likely be leaves in the network and thus cannot be involved in any compression.
Instead, we take an ex-post perspective. A Pareto improvement is thus a compression where everyone would agree in retrospect that they should have compressed. An incentivized compression is one where the involved banks would agree on this.

Recall further that we do not perform an equilibrium analysis in this paper. A compression is not an action that banks perform as part of some game. Rather, we assume that a certain compression is provided exogenously by a central financial service provider (that is not a bank) and we are interested in efficiency and incentives for this single given compression. Our results in the present paper may form the basis for a future study of “stable” networks where no further incentivized compressions exist. It may further serve as a first step towards a study of the mechanism design problem where banks strategically report the liabilities to compress and the financial service provider chooses the compression according to a known set of rules.

To the best of our knowledge, the following are the only known general results on the effect of compression on efficiency in prior work:

**Theorem** (Essentially Veraart (2019)). Let $c$ be a compression for $X = (N, e, c, \alpha, \beta)$.

1. If $\alpha = \beta = 0$, then $c$ is a Pareto improvement.

2. None of the banks in $N(c)$ default in $X$ iff none of the banks in $N(c)$ default in $X^c$. If any of these two equivalent conditions holds, then $E = E^c$ point-wise, i.e., all banks are indifferent regarding compression.

**Proof.** Veraart (2019) proved these statements when $c$ is a cycle. It is easy to see that her arguments generalize to arbitrary compressions.

Beyond these results, we note that compression affects various measures of network structure and prior work has studied the effect of such measures on systemic risk quite extensively. By connecting these two effects, we can receive a first indication how compression may affect systemic risk. The respective predictions vary across different models. We provide a discussion in Appendix A.

An immediate implication of part 2 of this theorem is that compression cannot set all banks involved in it strictly better off: any incentivized compression leaves some of the involved banks indifferent. Likewise, no compression can set all involved banks strictly worse off.

**Corollary 1.** If $c$ is a compression for $X$, then there exist $i, j \in N(c)$ such that $E_i \leq E_i^c$ and $E_j \geq E_j^c$.

**Proof.** If no $i$ like in the statement exists, then $E_i > E_i^c \ \forall i \in N(c)$. In particular, $E_i > 0 \ \forall i \in N(c)$ and thus none of the banks in $N(c)$ default in $X$. By the above theorem, part 2, this implies $E = E^c$. Contradiction. Likewise, if $E_j^c > E_j \ \forall j \in N(c)$, then none of those banks default in $X^c$ and again, $E = E^c$. 

12
3 Detrimental Effects of Compression

In this section, we illustrate using a series of examples that compression can have beneficial as well as detrimental effects on efficiency, both in terms of social welfare and in a Pareto sense, and that this effect depends on the parameters of the financial system in a complex and non-monotonic way. Incentives to compress may further be misaligned with the social good.

3.1 Compression May Reduce Social Welfare

The basis for our examples in this section is a variant of a financial system first introduced in our prior work (Schuldenzucker, Seuken and Battiston, 2018); see Figure 1a. There are five banks A–E. Liabilities are depicted as blue arrows with notionals next to them. External assets are depicted inside boxes on top of the banks. There is a single cycle, A–B–C. We are interested in the effect of compressing that cycle, i.e., we consider the compression $c = (A–B–C, 2)$. The result of the compression is depicted in Figure 1b. Let $\alpha = \beta = 0.5$. The way we would typically think about this example is that all banks were initially solvent and may or may not have
performed compression; then A was hit by a shock.\textsuperscript{14}

Compression reduces social welfare. To see this, observe that in both the compressed and uncompressed case, banks B and C have external assets so high that they cannot default and bank A will always default. The key to our construction is that the recovery rate of A, and equivalently the payment $p_{AD}$, depends on whether or not compression is performed. Since there is no cycle of defaulting banks, clearing payments can be computed easily in topological order. In the uncompressed case, we have $a'_{A} = \alpha \cdot 0.5 + \beta \cdot 2 = 1.25$ and thus $p_{AD} = p_{AB} = 1/2 \cdot a'_{A} = 0.625$. In effect, $a_{D} = 4.125 > 4 = l_{D}$ and $D$ does not default. In the compressed case, we have $a'_{A} = \alpha \cdot 0.5 = 0.25$ and thus $p'_{AD} = a'_{D} = 0.25$ and $a_{D} = 3.75 < 4 = l_{D}$. Thus, $D$ defaults. As the external assets of $D$ are relatively large, the default costs due to $\alpha$ lead to a significant drop in social welfare compared to the uncompressed case. This dominates social welfare. More in detail, the equities in both cases are:

\begin{align*}
E_{A} &= 0 & E_{A}^{c} &= 0 \\
E_{B} &= 0.625 & E_{B}^{c} &= 2 \\
E_{C} &= 2 & E_{C}^{c} &= 2 \\
E_{D} &= 0.125 & E_{D}^{c} &= 0 \\
E_{E} &= 4 & E_{E}^{c} &= 1.75 \\
E_{\Sigma} &= 6.75 & E_{\Sigma}^{c} &= 5.75
\end{align*}

The total share of value lost due to default costs is $E_{\Sigma} - E_{\Sigma}^{c} = \frac{1.25}{8} \approx 0.15$ in the uncompressed and $E_{\Sigma}^{c} - E_{\Sigma} \approx 0.28$ in the compressed case. This spread can be made arbitrarily close to $1 - \alpha$ by increasing $e_{C}$ and $l_{DE}$ by the same amount.\textsuperscript{15}

### 3.2 Dependence on External Assets and Default Costs

One may wonder how our result depends on the choice of parameters, specifically how the difference in total equity $\Delta E_{\Sigma} := E_{\Sigma}^{c} - E_{\Sigma}$ depends on the default cost parameters and the level of the external assets $e_{A}$ of A. We can determine two boundary cases analytically: if $\alpha = \beta = 0$, then Veraart (2019) has shown that compression constitutes a (weak) Pareto improvement so $\Delta E_{\Sigma} \geq 0$. In our case, $\Delta E_{\Sigma} = E_{B} = 2$ in case $\alpha = \beta = 0$. If $\alpha = \beta = 1$, we know from Eisenberg and Noe (2001) that $\Delta E_{\Sigma} = e_{\Sigma} - e_{\Sigma} = 0$. The existence of our counterexample (where $\Delta E_{\Sigma} < 1$) now implies that $\Delta E_{\Sigma}$ must be non-monotonic in the default cost parameters at least for $e_{A} = 0.5$.

\textsuperscript{14}If $e_{A} \geq 2$ and one of the other banks is exposed to shock instead, then compression is always a Pareto improvement: for $B$ and $C$, it is clear that it is better to isolate these banks from the others as long as $A$ can pay. For $D$ and $E$, compression makes no difference.

\textsuperscript{15}It can in fact be made arbitrarily close to 1 by replacing the bank $E$ by a cycle of intermediation.
Figure 2 Difference in total equity $\Delta E_{\Sigma}$ between the compressed and uncompressed network under variation in $e_A$ and $\alpha = \beta$ in Figure 1. The blue plane is zero. Resolution: 0.01 steps in both parameters.

Figure 2 depicts the graph of $\Delta E_{\Sigma}$ for all relevant values of $\alpha = \beta$ and $e_A$. The blue transparent plane is zero. It is easy to see analytically that D does not default in $X$ iff $\alpha e_A + \beta \cdot 2 = \alpha(e_A + 2) \geq 1$ and D defaults in $X^c$ iff $\alpha e_A < 0.5$. The intersection of these two regions is the blue “canyon,” where compression is socially detrimental. Note how this set has a complex three-dimensional shape that depends on both parameters. Note in particular that if we fix a value of $e_A > 1.0$, the set $\{\alpha \mid \text{compression is socially detrimental for } e_A \text{ and } \alpha = \beta\}$ is non-contiguous. Analogously, for $\alpha = \beta \approx 0.6$, this is the case as we vary the $e_A$ values.

### 3.3 Individual Values for Default Cost Parameters

In the discussion so far, we have assumed that $\alpha = \beta$. One may wonder to which extent this is necessary for our result.

For the $\alpha$ parameter, it is easy to construct an equivalent example that does not depend on the value of $\alpha$ at all. To do this, add a source bank $s$ and replace the external assets of all banks by an equivalent liability from $s$. Then give $s$ sufficiently high external assets so that it cannot default. There is now no bank that holds any external assets and can possibly default, so the value of $\alpha$ is irrelevant. The resulting financial system will behave like our original example for $\alpha = \beta$. Thus, for any value of $\alpha$ there exists a financial system and a compression that decreases social welfare. Note that this technique is general and can transform any financial system where $\alpha = \beta$ into an equivalent one where $\alpha$ is arbitrary.

---

15Note that only values $e_A \in [0, 2)$ are relevant since no bank defaults if $e_A \geq 2$ and thus, $\Delta E_{\Sigma} = 0$ trivially.
The remaining two boundary cases to explore are thus $\beta = 0$ and $\beta = 1$. For $\beta = 0$, we will show in Section 4 below that this already implies that compression is a (weak) Pareto improvement, and is in particular (weakly) socially beneficial. For $\beta = 1$, compression is (weakly) socially detrimental in our example for any value of $\alpha$ and $e_A$. This is because then $E_\Sigma = e_\Sigma - (1 - \alpha) \sum_{i \in D} e_i$, where $D$ is the set of defaulting banks, (Rogers and Veraart, 2013, Lemma 4.8) and, by the above discussion, only the default of $D$ depends on any payments from others. It follows from the analytical expressions for the default of $D$ in Section 3.2 that if $D$ defaults in $X$, then also in $X^c$. Thus, $\Delta E_\Sigma \leq 0$ and $\Delta E_\Sigma = 0$ iff $\alpha = 1$ or $\alpha e_A \geq 0.5$. Figure 3 shows the corresponding plot.

3.4 Detrimental Effects for Involved Banks and Pareto worsening

Note that the banks A, B, and C that are part of the compressed cycle in our running example benefit from compression (some weakly, some strongly). Thus, this compression is incentivized. We will show in Section 5 that this is always the case for this kind of network topology. For the parameter values discussed above, compression also reduces social welfare, so that incentives are misaligned with the social good.

This property is not universal. If we identify banks C and E, we receive the financial system in Figure 4. In this financial system, bank C suffers from compression. To see this note that C cannot default, so the outcome of clearing is the same as in Section 3.1, except for that C’s equity is increased by what was E’s equity before. We thus have $E_C^c = 3.75 < 6 = E_C$, i.e., compression would harm C and C would therefore not be willing to agree to it. Thus, $c$ is not incentivized. Notice how C’s
veto to compression is at the expense of B, who would benefit from compression. Keeping the cycle does not cause an immediate cost for C because B does not default and thus allows C to support A “for free”. This eventually benefits C. We will study such phenomena in greater detail in Section 6 when we discuss homogeneity.

We can take the idea from the previous example one step further to receive a financial system where the equity of every bank decreases (strictly for some banks), i.e., compression is a Pareto worsening. To do this, consider Figure 1 again and notice that C is actually indifferent between compression and non-compression, B gains 1.375 and E loses 2.25 from it. By eliminating E and splitting up the liability from D to E into two separate liabilities to B and C, we can simultaneously split up the loss and make compression strictly harmful for both B and C. See Figure 5. We now have:

\[
\begin{align*}
E_A &= 0 \\
E_B &= 3.625 \\
E_C &= 3 \\
E_D &= 0.125 \\
E_\Sigma &= 6.75
\end{align*}
\]

\[
\begin{align*}
E'_A &= 0 \\
E'_B &= 3.3125 \\
E'_C &= 2.4375 \\
E'_D &= 0 \\
E'_\Sigma &= 5.75
\end{align*}
\]

Of course, the effect can be made arbitrarily much stronger by increasing \(e_D\), \(l_{DB}\), and \(l_{DC}\) like before.

Note that, by Corollary 1, it is not possible that all banks strictly suffer from compression.
Figure 6: Individual equities and total equity dependent on the compressed amount $\zeta$ in Figure 1. Resolution: 0.01

3.5 Partial Compression

One can imagine a situation where only part of a feasible compression is executed. This might be, for example, because in reality, opportunities for compression come up over time and are executed in an on-line fashion. One might assume that partial compression has a monotonic effect on social welfare: if compression is beneficial, performing part of the compression should be less beneficial, but still beneficial and vice versa. However, this is not the case even when we consider a single cycle. Figure 6 shows the total equity and individual banks’ equities under the compression $c = (A-B-C, \zeta)$ in our original example from Figure 1a. $\zeta = 0$ corresponds to no compression and $\zeta = 2$ corresponds to full compression of the cycle. We also allow negative values of $\zeta$, which we interpret as increasing all liabilities in the cycle by $-\zeta$.

Observe that, as we proceed from 0 to 2 in the graph, the total equity first increases, then experiences a discontinuous drop as D defaults (and thus $E_E$ drops sharply) and then the total equity increases again, but not to its previous level. Observe further that each of the individual equities is monotonic, only their sum $E_\Sigma$ is not. Specifically, the banks A, B, and C involved in the compression monotonically profit from more compression while the other banks D and E are monotonically hurt by it. This is a consequence of the network topology in this example, which we will discuss in greater detail in Section 5.

3.6 Pareto Effects Without Additional Defaults

The effects presented in the previous subsections are driven by the fact that the default of bank D depends on whether or not compression is done. However, if $\beta < 1$, then compression can also Pareto-decrease equities while the set of defaulting banks
stays the same, by changing the paths that money takes. To see this, let $\alpha = \beta = 0.5$ and consider Figure 7. We see immediately that $C$ defaults and that this implies that all banks except $H$ default. If we compress the only cycle $(A-B-C, 1)$, the same banks default. Thus, $H$ is the only bank with positive equity and $E_H = p_{AH} + p_{FH}$ and $E^c_H = p^c_{AH} + p^c_{FH}$. To compute the payments in the uncompressed case, we first resolve the cycle. By $p = \Psi(p)$ and since all banks default, we have:

$$
\begin{align*}
    p_{AB} &= 1/2 \cdot 1/2 \cdot p_{CA} = 1/4 \cdot p_{CA} \\
    p_{BC} &= 1/2 \cdot p_{AB} \\
    p_{CA} &= 1/2 \cdot 1/2 \cdot (0.5 + p_{BC})
\end{align*}
$$

Solving this equation system yields $p_{AH} = p_{AB} = 1/31$, $p_{BC} = 1/62$, and $p_{CD} = p_{CA} = 4/31$. At each step of the path $D-E-F-G-H$, a factor of $1 - \beta = 0.5$ is lost due to default costs. We thus have $p_{FH} = (1/2)^3 \cdot 4/31 = 1/8 \cdot 1/31$. Overall, $E_H = (1 + 1/8) \cdot 1/31$. In the compressed network, all the money must take the less efficient path $C-D-E-F$, so $p^c_{AH} = 0$ and $E^c_H = p^c_{FH} = (1/2)^3 \cdot 1/2 = 1/64 < (1 + 1/8) \cdot 1/31 = E_H$. Thus, compression leads to a Pareto worsening without any new defaults. Of course, one can make the effect arbitrarily much stronger by increasing the length of the longer path.

If we give $A$ positive external assets rather than $C$, setting $e_C = 0$ and $e_A = 0.5$, we can observe the opposite effect: compression leads to a Pareto improvement because it prevents money from flowing via the inefficient cycle and the longer path, and instead routes it directly to $H$.

### 4 Local Sufficient Conditions for a Pareto Improvement

In the previous section, we have seen that compression need not constitute a Pareto improvement, may in fact be a Pareto worsening, and that incentives for the involved
banks may be misaligned with the social good. In this section, we present conditions under which this cannot happen, i.e., where compression is always a Pareto improvement. Our conditions will be local, i.e., they will not require any network calculations beyond, in some cases, a lower bound for banks’ assets under the maximal clearing values.

We begin by defining two formal tools that will be useful in the rest of our analytical examination.

Definition 3. Let $c$ be a compression for $X = (N, e, l, \alpha, \beta)$, the change in relative liability due to $c$ is $\Delta \pi_{ij} := \pi_{ij}^c - \pi_{ij} \in [-1, 1)$. The adjusted compressed payments are the matrix $p' := p^c + c$

Remark 4. The following facts are easy to verify. The proof is omitted.

- If $c_i = 0$, then $\Delta \pi_{ij} = 0 \forall j$. That is, only relative liabilities of nodes involved in the compression change.

- $\Delta \pi_{ij} > 0$ iff $c_{ij} < \pi_{ij} c_i$. Likewise, $\Delta \pi_{ij} \leq 0$ iff $c_{ij} \geq \pi_{ij} c_i$. That is, the “inside” nodes of a compression are exactly those where the compressed amount is at least in proportion to the relative liabilities.

- $\sum_j \Delta \pi_{ij} = 0$, except for when $c_i = l_i > 0$, in which case $\Delta \pi_{ij} = -\pi_{ij} \forall j$ and thus $\sum_j \Delta \pi_{ij} = -1$.

- If $c = (C, \mu)$ is a cycle and $\mu < l_i$, then $\Delta \pi_{ij} = -(1 - \pi_{ij}) \frac{\mu}{l_i - \mu} < 0$ if $(i, j) \in C$ and $\Delta \pi_{ij} = \pi_{ij} \frac{\mu}{l_i - \mu} > 0$ if $(i, j) \notin C$. If $\mu = l_i$, then $\Delta \pi_{ij} = -\pi_{ij} \forall j$

- $p' \in [0, l]$. $p' = l$ iff no bank defaults in $X^c$.

- $B_i(p') = B_i^c(p^c) = B_i^c$. In particular, $E_i(p') = E_i^c$ and if $p' \geq p$, then $c$ is a Pareto improvement.

The change in relative liabilities allows us to distinguish between banks “inside” and “outside” the compression in a mathematically meaningful way. Intuitively, if $\Delta \pi_{ij} \leq 0$, we consider the edge $(i, j)$ “rather inside” $c$ and if $\Delta \pi_{ij} > 0$ we consider $(i, j)$ “rather outside” $c$. Note that simply testing whether $c_{ij} > 0$ is not a good way to do this for general compressions. For example, for any financial system and compression, we could simply increase both $l_{ij}$ and $c_{ij}$ by a sufficiently small positive number. Then we would suddenly have $c_{ij} > 0$ for all $(i, j)$ and $N(c) = N$ even though the mathematical properties of the compression have not changed.

The last property in Remark 4 means that we can “transfer” the clearing payments from the compressed financial system $X^c$ to the original system $X$ by considering the adjusted compressed payments instead. The payment matrix $p'$ is not usually clearing because i) proportionality is not generally satisfied and ii) the $c$ part of $p'$ is not subjected to default costs due to $\beta$. Instead, we can interpret $p'$ as payments in $X$ that have a priority structure: payments in $c$ are privileged before other payments.
and also avoid default costs. The following lemma provides a technical sufficient condition under which this compression is beneficial along all dimensions. The lemma will be fundamental for all other, more conceptual sufficient conditions presented in this section.

**Lemma 1.** Let $c$ be a compression for a financial system $X = (N, e, l, \alpha, \beta)$. Let $p$ be the maximal clearing payment matrix in $X$ and let $p \lor c$ be the point-wise maximum of $p$ and $c$, i.e., $(p \lor c)_{ij} = \max(p_{ij}, c_{ij})$. Assume that for all $i, j \in N$ where $\Delta\pi_{ij} > 0$ and $i$ defaults under $p \lor c$ in $X$, we have:

$$\Delta\pi_{ij}a'_i(p \lor c) + c_{ij} \geq \beta \pi_{ij}c_i$$

Then $p' \geq p$. In particular, compression by $c$ weakly increases the balance of each bank and is a Pareto improvement.

The proof of the lemma can be found in Appendix B. Recall that $\Delta\pi_{ij} = 0$ if $i \notin N(c)$. Thus, to verify the precondition of the theorem, we only need to look at banks $i$ that are involved in the compression. The precondition of the lemma is not completely local because it depends on the maximal clearing payments $p$ in $X$. However, it only depends on $p$: once $p$ is known, $a'_i(p \lor c) = \alpha e_i + \beta \sum_j \max(p_{ji}, c_{ji})$ can be easily computed. Often, we can bound $p$ from below, which may already be enough to apply Lemma 1. We demonstrate some of these applications in the following.

We call $c$ immediately beneficial at $(i, j)$ for $X$ if the preconditions of the lemma for the pair $(i, j)$ are satisfied. Note that this alone does not imply that $p'_{ij} \geq p_{ij}$ or that bank $j$ benefits from compression. That is because compression may be detrimental to other banks, which may negatively affect $i$ and $j$.\footnote{We explore the idea of these feedback effects further in Section 6.} — The precondition needs to hold for all pairs $(i, j)$ for the lemma to have any implications. We will now describe a few simpler and more interpretable conditions that imply immediate benefits for some pair. By definition, if for each pair $(i, j)$ where $\Delta\pi_{ij} > 0$ and $i$ defaults (under $p$ or $p \lor c$), one of these conditions hold, then $c$ is a Pareto improvement.

An immediate implication of the theorem is that compression is always a Pareto improvement if $\beta = 0$. This generalizes Veraart (2019, Theorem 3.7), where it was shown that compression is always a Pareto improvement if $\alpha = \beta = 0$. We thus learn that the condition $\alpha = 0$ was not actually necessary.

**Corollary 2.** For any financial system $X = (N, e, l, \alpha, \beta = 0)$, any compression is immediately beneficial for all pairs. In particular, $p' \geq p$ and $c$ is a Pareto improvement.

**Proof.** If $\beta = 0$, the right-hand side in the precondition of Lemma 1 is 0 and the left-hand side is always non-negative. Thus, the precondition holds trivially. \(\square\)
Compression is immediately beneficial when the recovery rate of the contract writer is not too low.

**Theorem 1.** Let \( X = (N, c, l, \alpha, \beta) \) be a financial system and assume that

\[
r_i \geq \beta.
\]

Then any compression is immediately beneficial for any pair \((i, j)\), for \( j \in N \).

**Proof.** Let \( i, j \) be such that \( \Delta \pi_{ij} > 0 \) and \( i \) defaults in \( X \) under \( p \lor c \). We have \( r_i = a'_i(p)/l_i \geq \beta \) by assumption and thus \( a'_i(p \lor c) \geq a'_i(p) \geq \beta l_i \). We now receive:

\[
\Delta \pi_{ij} a'_i(p \lor c) + c_{ij} \geq \beta \Delta \pi_{ij} l_i + c_{ij} = \beta \pi_{ij} c_i - \beta c_{ij} + c_{ij} \geq \beta \pi_{ij} c_i
\]
as required, where the second line is by the identity \( c_{ij} = \pi_{ij} c_i - \Delta \pi_{ij} l_i \).

The previous theorem is only meaningful when \( \beta < \alpha \). This is because, if \( \beta \geq \alpha \), it is easy to see that if we have \( r_i \geq \beta = \max(\alpha, \beta) \), then already \( r_i = 1 \), i.e., \( i \) does not default under \( p \) and in particular not under \( p \lor c \).\(^{18}\) A default cost regime where \( \beta < \alpha \) is one in which external assets are relatively liquid, but interbank payments are subject to high costs such as delays or legal uncertainty (see Section 2.2). Under this regime, the theorem states that the creditors of a defaulting bank will still benefit from compression as long as said bank is not too deep in default. Note how this is independent of the compression applied.

Theorem 1 does still not provide a fully local condition because it depends on the recovery rates. We could derive a fully local condition from the theorem using the estimate \( r_i \geq \alpha e_i/l_i \). However, the following theorem shows that a weaker version of this condition is still sufficient.

**Theorem 2.** Let \( c \) be a compression for \( X = (N, c, l, \alpha, \beta) \) and assume that \( \alpha > 0 \) and

\[
l_i = c_i \quad \text{or} \quad \frac{c_i}{l_i - c_i} \geq \frac{\beta}{\alpha}.
\]

Then \( c \) is immediately beneficial for any pair \((i, j)\) for \( j \in N \).

**Proof.** The condition implies that \( \alpha e_i \geq \beta(l_i - c_i) \) and thus \( a'_i(p \lor c) \geq a'_i(c) = \alpha e_i + \beta c_i \geq \beta l_i \). The statement now follows like in the proof of Theorem 1.

The term \( \frac{e_i}{l_i - c_i} = \frac{l_i}{c_i} \) is the reciprocal of the loan-to-value ratio \( \frac{l_i}{c_i} \), which is a common measure used to gauge the riskiness of giving a loan of \( l_i \) to a would-be

\(^{18}\)This holds for any bank under any clearing vector. We provide a proof of this “gap lemma” in Schuldenzucker, Seuken and Battiston (2019, Lemma 1). There, we consider an extension of the model from the present paper to other contract types, but the argument is the same.
debtor that has own capital of value $e_i$. This is used especially in the context of mortgages.

The precondition of the theorem implies that $\alpha e_i \geq \beta l_i^c$ and thus $r_i^c \geq \beta$. It can thus be seen as a counterpart to Theorem 1 in $X^c$. If $\beta \geq \alpha$, then like in Theorem 1, the precondition implies that $i$ does not default in $X^c$. It might, however, default in $X$.

Our final sufficient condition enables us to relax the condition that we need to consider all pairs $(i,j)$ for which $\Delta \pi_{ij} > 0$. Recall from Remark 4 that

$$\Delta \pi_{ij} \leq 0 \iff e_{ij} \geq \pi_{ij} e_i \iff \frac{c_{ij}}{l_{ij}} \geq \frac{c_i}{l_i} \geq 1.$$ 

That is, compression is immediately beneficial if the relative decrease in the individual liability $l_{ij}$ is at least as big as the relative decrease in the total liabilities $l_i$ of $i$. The following theorem allows us to relax this condition from “at least as big” to “not too much smaller.”

**Theorem 3.** Let $c$ be a compression for a financial system $X = (N, e, l, \alpha, \beta)$. Let

$$\eta : [0, 1) \to [0, \infty)$$

$$\eta(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{x}{1-x} = \frac{1}{\frac{1}{x}-1} & \text{otherwise.} \end{cases}$$

Assume that

$$\frac{\eta(c_{ij}/l_{ij})}{\eta(c_i/l_i)} \geq \beta.$$ 

Then $c$ is immediately beneficial for the pair $(i,j)$.

**Proof.** Whenever $\Delta \pi_{ij} > 0$, we must have $c_{ij} < l_{ij}$ and in particular $c_i < l_i$, otherwise $\pi_{ij}^c = 0$ and thus $\Delta \pi_{ij} \leq 0$. We must further have $c_i > 0$, otherwise $\Delta \pi_{ij} = 0$. Hence, all fractions above are well-defined. We now have:

$$\frac{\eta(c_{ij}/l_{ij})}{\eta(c_i/l_i)} = \frac{c_{ij}}{l_{ij}} \geq \frac{c_{ij}}{l_i} = \frac{c_{ij}}{\pi_{ij}^c}.$$

Thus, $c_{ij} \geq \beta \pi_{ij}^c c_i$. This implies the precondition of Lemma 1. \qed

The function $\eta$ is a continuous, monotonic, non-linear, and convex. We have $\eta(0) = 0$ and $\lim_{x \to \infty} \eta(x) = \infty$. Figure 8 provides an illustration. It is easy to see that $x \geq y \Rightarrow \eta(x)/\eta(y) \geq 1$. Thus, the theorem provides a true relaxation over the condition $\Delta \pi_{ij} \geq 0$. Note that if $c_{ij} = 0 < l_{ij}$, then $\eta(c_{ij}/l_{ij}) = \eta(0) = 0$ and so the precondition is only satisfied in the trivial case where $\beta = 0$. This holds in particular if $c = (C, \mu)$ is a cycle, $i \in C$, and $(i,j) \notin C$. 

23
Figure 8 Function $\eta$ from Theorem 3

![Graph showing function $\eta$ from Theorem 3](image)

Figure 9 Financial System with a more complex Pareto-improving compression. Let $\beta = 0.5$. $\alpha$ is irrelevant.

<table>
<thead>
<tr>
<th>Uncompressed</th>
<th>Compressed</th>
<th>Partially Compressed</th>
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<tbody>
<tr>
<td><img src="image" alt="Uncompressed Network" /></td>
<td><img src="image" alt="Compressed Network" /></td>
<td><img src="image" alt="Partially Compressed Network" /></td>
</tr>
</tbody>
</table>

(a) Uncompressed  
(b) Compressed  
(c) Partially Compressed

Note how Theorem 3 does not depend on $e$, but only on $c$ and $l$, i.e., on the network structure and the modifications we make to it.

For an example where the above theorem can be useful, consider Figure 9a and the compression $c = (A-B-C, 1) + (B-C-D, 1)$. This is clearly not a single cycle like in the examples before. Figure 9b shows the compressed network. Let $\beta = 0.5$. Whenever $(i, j) \neq (C, A)$ we have $\Delta \pi_{ij} \leq 0$ because $\pi_{ij}^e = 0$. However, we also have $\Delta \pi_{CA} = 1 - 2/3 = 1/3 > 0$. At the same time,

$$\frac{\eta(c_{CA}/l_{CA})}{\eta(c_C/l_{C})} = \frac{\eta(1/2)}{\eta(2/3)} = \frac{1}{2}.$$ 

Thus, for $\beta = 0.5$, this compression will always be a Pareto improvement for any choice of the external assets.

We can use this example to illustrate another phenomenon. Let $b = (A-B-C, 1)$. Note that $b \leq c$, i.e., $b$ corresponds to executing only part of the compression $c$. Figure 9c shows the result of compressing by $b$. Calculation shows that equities correspond to the following table:
Note how partial compression first increases the equity of $A$ and then decreases it again. This implies that even though both $b$ and $c$ are Pareto improvements and $b$ is a partial compression of $c$, this does not imply that $c$ is also a Pareto improvement over $b$. This provides a case for considering compression by arbitrary circulations, not just by cycles. If we had only considered successive compression by Pareto-improving cycles, we would not have been able to reach $c$ even though it is superior to $b$ with respect to social welfare.

## 5 Sufficient Conditions for Incentivized Compression

We now turn to non-local conditions on the network structure under which compression is incentivized or a Pareto improvement. Our main result will be that compression is incentivized if there are no feedback paths. En-route to this result, we develop a methodology that will enable us to decompose the effects due to compression into two separate phases: an immediate effect on only the involved banks and a feedback effect due to reverberations of the immediate effect in the rest of the network. When the compression is relatively simple, e.g., just a single cycle, the immediate effect is easy to estimate while the feedback effect depends on the surrounding network. Our results from this and the following section will exploit the fact that we can sometimes know the direction of one of the two effects upfront.

**Definition 4.** Let $c$ be a compression in a financial system $X = (N, e, l, \alpha, \beta)$. Let $I = \{(i, j) \mid l_{ij} > 0 \land c_{ij} > 0\}$ and $O = \{(i, j) \mid l_{ij} > 0 \land c_{ij} = 0\}$. Observe that $\Psi_{ij}(q) = 0 \ \forall q, (i, j) \notin I \cup O$. We can thus ignore pairs $(i, j) \notin I \cup O$, i.e., pairs where $l_{ij} = 0$. To simplify notation, we leave these pairs out. A feedback path is a path in $O$ from some bank in $N(c)$ to another or the same bank in $N(c)$. A chord is a feedback path of length 1, i.e., an edge $(i, j) \in O$ such that $i, j \in N(c)$. $c$ is called chord-free if there are no chords. We call $c$ normal for $X$ if $c_{ij} > 0 \Rightarrow \Delta \pi_{ij} \leq 0$ for all $i, j \in N$.

The immediate-effect payments $\tilde{p}$ in $X$ due to $c$ are the clearing payments in $X^c$ when all payments in $O$ have been fixed to their values in $X$. That is, $\tilde{p}$ is the

<table>
<thead>
<tr>
<th>$i$</th>
<th>$E_i$</th>
<th>$E_i^b$</th>
<th>$E_i^c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>0.25</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>0.5</td>
<td>1.125</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>*</td>
<td>0.5</td>
<td>1.375</td>
<td>2</td>
</tr>
</tbody>
</table>
point-wise maximal solution to the equation\(^{19}\)

\[
\begin{align*}
\tilde{p} &= \Psi^c(\tilde{p}_I \cup p_O).
\end{align*}
\]

Such a maximal solution exists by monotonicity of the function \(\Psi^c(\cdot \cup p_O)\). Let \(\tilde{E} := E^c(\tilde{p})\) and denote the other values analogously. Define the adjusted immediate-effect payments \(\tilde{p}' := \tilde{p} + c\) and observe that \(E(\tilde{p}') = \tilde{E}\).

The edges in \(I\) are those where any compression takes place. However, as we have discussed at the beginning of Section 4, we should only consider those liabilities truly “part of” the compression where \(\Delta \pi_{ij} \leq 0\). In a normal compression, the two notions are aligned. Note that any cycle \(c = (C, \mu)\) is normal.

Note that if \(i \notin N(c)\), then \(\tilde{p}_{ij} = p_{ij} \forall j\). This is because \(\Psi^c_{ij}(p)\) only depends on the components \((p_{ki})_{k \in N}\) and by assumption we have \((k, i) \in O \forall k\), so \(\tilde{p}_{ij} = \Psi^c_{ij}(\tilde{p}_I \cup p_O)\) only depends on \(p_O\) and by assumption, \(\Psi^c_{ij} = \Psi_{ij}\). Thus, the immediate-effect payments only concern the banks in \(N(c)\) and payments within and leaving this set.

The following lemma shows how our decomposition can be useful.

**Lemma 2.** Let \(c\) be a compression for a financial system \(X\).

1. If \(c\) is normal for \(X\), then \(\tilde{p}'_I \geq p_I\). If \(c\) is normal and chord-free for \(X\), then \(\tilde{E}'_i \geq E_i\) for all \(i \in N(c)\).

2. If \(\tilde{p}_O \geq p_O\), then \(p'_c \geq \tilde{p}\). In particular, \(p'_O \geq \tilde{p}_O \geq p_O\) and \(E'_i \geq E_i\) for all \(i \notin N(c)\). If further, every bank \(i \in N(c)\) has a liability in \(O\), then \(p' \geq \tilde{p}' \geq p\) and thus \(c\) is a Pareto improvement.

3. If \(\tilde{p}' \geq p\), then \(p' \geq \tilde{p}' \geq p\) and thus \(c\) is a Pareto improvement.

The proof can be found in Appendix B.

Part 1 shows that the immediate effects of a (chord-free) compression are always positive to those banks participating in it. Thus, if a compression is not incentivized, this must be due to the feedback effect.

Part 2 and 3 are useful because they tell us that, as far as Pareto improvements (for all banks or the banks not involved in the compression) are concerned, it is sufficient when the immediate effects are beneficial. We will use this fact in Section 6.

Our theorem regarding feedback paths follows from part 1 of the lemma because, if there are no feedback paths, feedback effects do not matter to the banks involved in the compression.

**Theorem 4.** If \(c\) is normal for \(X\) and there are no feedback paths, then \(c\) is incentivized.

\(^{19}\)Recall that we denote restriction of indices to a subset by an index and concatenation along disjoint subsets of indices by a union. Thus, \(\tilde{p}_I \cup p_O\) is a matrix with indices in \(N \times N\) and \((\tilde{p}_I \cup p_O)_{ij}\) is \(\tilde{p}_{ij}\) if \((i, j) \in I\) and \(p_{ij}\) if \((i, j) \in O\).
Proof. By Lemma 2 part 1, it is enough to show that $\tilde{E}_i = E^c_i$ for all $i \in N(c)$. This is for the following reason. Let $G$ be the graph with adjacency matrix $l$. Let

$$A := \{(i, j) \in O \mid \text{there is a path } j \to N(c) \text{ in } G\}$$

$$B := \{(i, j) \in O \mid \text{there is a path } N(c) \to i \text{ in } G\}.$$

Since there are no feedback paths, $A$, $I$, and $B$ are pair-wise disjoint and for all $q$, $\Psi_A(q)$ is independent of $q_I$ and $q_B$, and $\Psi_I(q)$ is independent of $q_B$. The same holds for $\Psi^c$. This implies that $p^c_A = \tilde{p}_A = p_A$ and $p^c_I = \tilde{p}_I$. And $E^c_i$ for $i \in N(c)$ only depends on $p^c_A$ and $p^c_I$ and likewise for $\tilde{E}_i$. \hfill \square

When we look back at our example from Section 3.1 through the lens of Theorem 1, it is now clear that compression will always be incentivized in this example. Note in particular that this does not depend on the external assets, recovery rates or on which banks default. It also does not depend on the compressed amount (cf. Section 3.5). We also learn from this example that this kind of compression is always incentivized, but need not even increase social welfare, let alone be a Pareto improvement. Section 3.6 provides another example where there are no feedback path and thus compression is always incentivized, but need not be socially beneficial.

Remark 5. It is easy to see that not all feedback paths will impair the conclusion of the theorem, but only those where all banks that are part of the feedback path default in $X^c$. By bounding the assets of these banks from below, perhaps similarly to how we did it in Section 4, one may be able to receive a weaker sufficient condition in Theorem 4.

In the real world, it does not seem that banks consider it a complex strategic decision whether or not they should agree to a compression. Rather, it seems that proposed compressions are agreed to in the vast majority of cases. The lemma and the theorem suggest local information or local reasoning as a possible explanation why this might be the case. Feedback paths lie in a part of the network not directly related to the compression. If a bank has limited information about the network structure, it may (incorrectly) assume that feedback paths do not exist. Alternatively, the bank may simply not reason about feedback effects, but only about immediate effects. If it then also assumes that the compression is normal (for example, because it has limited information about the parts of the compression that it is not directly involved in), it may conclude via Theorem 4 of Lemma 2 part 1 that compression is always beneficial. Note that the lemma and the theorem do not depend on the external assets and thus on any shocks, so that this decision can be confidently made ex-ante.
6 Homogeneity

In this section, we consider network structure at an even higher level. Going back to those examples from Section 3 where compression was socially detrimental, we can observe that there was always one bank that was vastly worse off than the others. For example, in Figure 1, bank A was very poorly capitalized while the two other banks in the cycle were very well capitalized. In this section, we show that such a high-level structure is in fact necessary for our examples: if there is a sufficient degree of homogeneity among the balance sheets of involved banks, then compression is always a Pareto improvement.

6.1 Perfect Homogeneity

The following theorem proves our statement analytically for the case of perfect homogeneity.

**Theorem 5.** Let $c$ be a compression for a financial system $X$ and assume that the following values are the same across all $i \in N(c)$: $e_i, p_{Oi} := \sum_{j:(j,i) \in O} p_{ji}, l_{iI} := \sum_{j:(i,j) \in I} l_{ij}, l_{iO} := \sum_{j:(i,j) \in O} l_{ij}, l_{II} := \sum_{j:(i,j) \in I} l_{ij}$, and $c_i$. In this case, we call the pair $(X, c)$ homogeneous. Then $c$ is a Pareto improvement.

The proof of the theorem can be found in Appendix B. Note that, in the previous theorems, none of the individual values $l_{ij}$ or $c_{ij}$ need to be equal; they only need to be equal in aggregate. This is important for compressions that are more complex than a cycle. There are several simple classes of examples for $(X, c)$ pairs that are homogeneous in the sense of Theorem 5:

- Let $M$ be the adjacency matrix of a regular directed graph $G$, let $\gamma \geq 0$, and let $l = \gamma \cdot M$. Let $e_i := \delta \forall i$, where $\delta \geq 0$ is arbitrary. Let $C$ be any cycle in $G$, let $\mu \leq \gamma$ and let $c = (C, \mu)$. Then $X$ and $c$ are homogeneous in the sense of Theorem 5.

- Consider the set of homogeneous $(X, c)$ pairs where $N(c)$ has no incoming liabilities from $O$, i.e., $l_{ji} = 0$ for any $i \in N(c)$ and $(j, i) \in O$. It follows immediately from the definition that this set forms a polytope. In particular, convex combinations of two such homogeneous pairs are homogeneous. Note that Theorem 4 implies that such compressions are always incentivized if they are normal.

- If $(X, c)$ is homogeneous and $Y$ is any financial system, we can consider the disjoint union of the two and connect $N(c)$ to nodes in $Y$ in any way such that $\sum_{j \in Y} l_{ij}$ is the same across $i \in N(c)$. If we call the new financial system $Z$, $(Z, c)$ is homogeneous.
The theorem only makes a statement about \((X, c)\) pairs where the different asset and liability values are *exactly* the same. However, intuition seems to suggest that it is not crucial that these values match exactly; there should be some slack. We study this hypothesis using an example of a particularly simple homogeneous \((X, c)\) pair: a cycle where each bank has the same outgoing liabilities and the same external assets (see Figure 10). We then gradually make the pair less homogeneous by changing the external assets of a single node, say A. Given the role of the \(\beta\) parameter we revealed in Section 4, we hypothesize that more homogeneity is required for a Pareto improvement when \(\beta\) is higher. To isolate this effect, we fix \(\alpha = 0.5\) and consider the effect of compression on the equities under variation of \(e_A\) and \(\beta\).

By Theorem 4, \(c\) is always incentivized, so the equities of banks A, B, and C will always weakly increase. Figure 11 depicts the minimum difference in equity across the other three banks D, E, and F. This number is above zero (i.e., the curve is above the blue zero plane in the figure) iff \(c\) is a Pareto improvement.

The figure reinforces our hypothesis. Consistent with our findings in Section 4, compression is a Pareto improvement for a wide range of \(e_A\) values when \(\beta\) is low. As we increase \(\beta\), however, this region continuously becomes more narrow until it converges to the point 0.5 for \(\beta = 1\): here, we require *exact* homogeneity for compression to be a Pareto improvement.

Further experiments not documented in the present paper show that the value of \(\alpha \in (0, 1]\) affects our finding quantitatively, but not qualitatively. For \(\alpha = 0\), banks are indifferent to compression for all but very high values of \(e_A\) because all banks default, so all value is destroyed. A choice of \(\alpha = \beta\) does not imply a qualitative difference, either. The data is available upon request.

Note, however, that all banks default in this examples unless \(e_A \geq 1\), in which case A does not default in \(X^c\). Only for \(e_A \geq 1.56\) A does not default in \(X\) and compression is detrimental for \(E\).
Our experiment suggests that one may be able to obtain a quantified version of Theorem 5 where an “amount of homogeneity” inversely dependent on $\beta$ may be enough to guarantee that compression is a Pareto improvement. An analytical result in this vein would constitute substantial progress towards a deeper understanding of compression.

7 Conclusion

In this paper, we have studied portfolio compression, a post-trade mechanism which eliminates cycles in the financial network. We have studied the incentives for banks to engage in portfolio compression and its systemic effects in terms of banks’ equity. We have shown that whether or not compression is socially or individually desirable depends on the parameters of the financial system and the compression in a complex and non-monotonic way and the incentives for participating banks to perform compression may be misaligned with the social good. We have presented sufficient conditions under which compression is a Pareto improvement. We have shown that banks always have an incentive to agree to compression unless there are feedback paths. Finally, we have identified homogeneity of participating banks’ balance sheets as an important driver of efficiency of compression.

Our results reveal the default costs on interbank payments, encoded by the $\beta$ parameter in our model, as a central factor for the ex-post desirability of compression. This is intuitive: when payments are lossy, it makes sense to have less of them. Our results on homogeneity suggests that portfolio compression may be particularly effective in a financial system where many actors are exposed to the same (or similar
kinds of) risks, so that if a large shock hits certain assets, then it hits all banks at once. Haldane (2009) describes how the pre-crisis financial system, through a convergence of investment and risk management strategies, had indeed been put into this very situation where “financial sector balance sheets became homogenised.” This may be taken as an indication that compression is indeed helpful to protect from another 2008 crisis; of course, further research is needed before any definite conclusions can be drawn.

An important avenue for future work will be to leave the ex-post model considered in this paper and take on an ex-ante perspective. Assuming a probability distribution of shocks and assuming that banks maximize their expected equity, which compressions are efficient and incentivized, respectively, in terms of expected equity? Incorporating uncertainty about the future in this way will be an important step towards an evaluation of compression from a practical point of view. This perspective may provide another explanation for why banks usually agree to compression and it may reveal another disconnect between the incentives of banks and the interests of society: perhaps compression is beneficial in expectation, but has strong detrimental effects in a small part of the probability space.

The insights from an ex-ante study may also help answer the question regarding regulatory incentives for compression. To date, one of the strongest incentives for banks to engage in compression is relief of regulatory capital requirements. Are these capital reliefs justified relative to the systemic impact of compression? Should we further incentivize or disincentivize compression? Or might the issue of portfolio compression point to a fundamental flaw of prescribing capital requirements relative to a single number, namely risk-weighted assets?

Finally, central clearing and portfolio compression in non-centrally cleared markets both serve the same goal: to reduce notional by rewiring the network. Ultimately, we need to be able to judge which of these approaches is more appropriate to reduce systemic risk in the future: central clearing, compression, or perhaps leaving the network as it is while increasing transparency? Answering these questions requires an understanding of the effect of network modifications on systemic risk, to which we hope to have contributed a first step.

Acknowledgments

We would like to thank (in alphabetical order) Stefano Battiston and Vitor Bosshard for helpful comments on this work.
A Effect of Compression on Prior Work

Researchers have previously studied the effect of network structure on systemic risk. We can examine how compression affects network structure at a grand scale and then apply their results. From this, we receive a first indication what effects we might expect from compression on average. Note that all we can expect from this exercise is an a tendency: apart from Veraart (2019), no prior piece of work has studied the effect of compression on a per-network basis.

Glasserman and Young (2015) studied the effect of a random shock to banks’ external assets on the aggregate payments in the Eisenberg and Noe (2001) model (i.e., our model without default costs) where every bank has an outside liability $b_i \geq 0$ in addition to its interbank liabilities. The authors provided sufficient conditions under which the extent of financial contagion is bounded. These sufficient conditions are based on the following local properties of each bank:

- Its financial connectivity $\beta_i = l_i / (l_i + b_i)$, which measures the share of its interbank liabilities relative to all its liabilities.
- Its leverage of outside assets $\lambda_i = e_i / E_i(l)$, where $E_i(l) = e_i + \sum j l_{ji} - l_i$ is the book value of equity of $i$, i.e., the equity of bank $i$ if all its creditors pay in full.
- Its external assets $e_i$.

Observe that compression reduces financial connectivity and keeps leverage of outside assets and external assets the same. Thus, compression makes the sufficient conditions weaker and thus the overall bounds stronger. These results therefore suggest an overall positive effect of compression, which is however very coarse: unless compression transports a financial system from not satisfying the sufficient conditions to satisfying them, we should not expect any implications.

Bardoscia et al. (2017) assessed the stability of a financial system based on the greatest eigenvalue of the leverage matrix $\Lambda_{ij} = l_{ij} / E_j(l)$. The lower this eigenvalue, the more stable the system, where 1 is an important threshold value. Since compression reduces $l$ point-wise and keeps $E_i(l)$ the same, it reduces the leverage matrix point-wise and thus also reduces its maximal eigenvalue. Thus, compression should make each individual network more stable in the sense of this paper.

Battiston et al. (2012) studied the spread of financial contagion in a model similar to the SIR (susceptible–infected–resistant) class of models from epidemiology. A level of financial distress of $\psi_i \in [0, 1]$ at bank $i$ causes a level of financial distress of $\Lambda_{ij} \psi_i$ at each creditor $j$, where $\Lambda$ is the leverage matrix from above. Since compression reduces $\Lambda$ point-wise, it reduces contagion in this model. Any SIR-style model where

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22Precisely, this statement should be interpreted as follows: consider any of the sufficient conditions in Glasserman and Young (2015) and fix their parameters. If $c$ is a compression for $X$ and $X$ satisfies the condition, then so does $X^c$; there exists an $X$ such that $X^c$ satisfies the condition, but $X$ does not.
infection propagates via liabilities will likely share this property.

Elliott, Golub and Jackson (2014) considered a model of cross-holdings of (essentially) equity cross-holdings and isolate two network measures, both of which must have intermediate levels for contagion to occur: integration, which measures the share of an organization held by financial, compared to external, actors. Diversification measures how spread out the financial part of the cross-holdings is. While there seem to be several options how one could define compression for a cross-holdership model, it seems that any sensible definition would reduce both integration and diversification. The overall effect would therefore be ambiguous.

Demange (2016) studied aggregate payments in the Eisenberg/Noe model like (Glasserman and Young, 2015), but focused on the partial derivatives of this measure with respect to changes in a banks’ external assets. This is what she called the threat index \( \mu_i \) of bank \( i \). The threat index depends on the set of defaulting banks. If the set \( D \subseteq N \) of banks default under the maximal clearing matrix of payments, then \( \mu_i = 0 \) if \( i \notin D \) and for the collection \( \mu_D = (\mu_i)_{i \in D} \) and \( \pi_D := (\pi_{ij})_{i,j \in D} \) we have:

\[
\mu_D = 1 + \pi_D \mu_D
\]

\[
\Leftrightarrow \mu_D = (I - \pi_D)^{-1} \mathbf{1} = \sum_{k=0}^{\infty} \pi_D^k \mathbf{1},
\]

where \( \mathbf{1} := (1, \ldots, 1) \).

Like in the related formula for Katz centrality (cf. the corresponding discussion in Demange (2016)), we thus have for \( i \in D \):

\[
\mu_i = \sum_{P \text{ path starting at } i} \prod_{(j,k) \in P} \pi_{jk}
\]

The risk index of a bank \( i \) is the sum over the products of \( \pi \) values of paths starting at \( i \). Compression transforms the \( \pi \) values in a non-linear and non-monotonic way. It may also move a bank into or out of default. This is why compression may increase or decrease a bank’s risk index.

\section*{B Proofs}

\textit{Proof of Lemma 1.} The key technical step in the proof is that the precondition to the lemma implies the following:

\textit{Claim.} For any \( q \in [0, l - c] \) such that \( q + c \geq p \) we have:

\[
\Psi^c(q) + c \geq \Psi(q + c)
\]

\footnote{For the first line, see Demange (2016, Theorem 1). The second line is by linear algebra and the Neumann series. The series converges because we restricted our attention to the set of defaulting banks.}

33
Proof of the Claim. Consider the inequality for an individual entry \((i, j)\) on both sides. It is easy to see that

\[
a_i^c(q) = a_i(q) = a_i(q + c) - c_i
\]
\[
a_i^{c'}(q) = a_i'(q) = a_i'(q + c) - \beta c_i
\]

and, of course, \(l_i^c = l_i - c_i\). Thus, bank \(i\) defaults under \(q\) in \(X^c\) iff it defaults under \(q + c\) in \(X\). In case of non-default, (3) for \((i, j)\) is of course equivalent to \(l_i^{c'} + c_{ij} \geq l_{ij}\) and trivially true with equality. Thus, assume that \(i\) defaults in these cases. Then (3) for \((i, j)\) is equivalent to:

\[
\pi_i^c a_i'(q) + c_{ij} \geq \pi_i^{c'}(q + c)
\]
\[
\iff \pi_i^c a_i'(q + c) - \pi_i^{c'}(q + c) \geq \pi_i^{c'}(q + c)
\]
\[
\iff \Delta \pi_i a_i'(q + c) + c_{ij} \geq \beta \pi_i^{c'} c_i \tag{4}\]

The second line is by the above identity and the third line is by simple algebra. We now distinguish the cases \(\Delta \pi_{ij} \leq 0\) and \(\Delta \pi_{ij} > 0\).

If \(\Delta \pi_{ij} \leq 0\), then we can use the identity \(c_{ij} = \pi_i^{c'} c_i - \Delta \pi_{ij} l_i\) to see that (4) is equivalent to:

\[
\Delta \pi_{ij} (l_i - a_i'(q + c)) \leq (1 - \beta) \pi_i^{c'} c_i
\]

Since \(i\) defaults under \(q + c\) in \(X\), the parenthesis on the left-hand side is \(\geq 0\). Since \(\Delta \pi_{ij} \leq 0\), the left-hand side is non-positive and the right-hand side is positive, so the inequality holds trivially.

If \(\Delta \pi_{ij} > 0\), we observe that, if \(q\) is as specified in the claim, then \(q + c \geq p\) and (since \(q \geq 0\)) also \(q + c \geq c\). Thus, \(q + c \geq p \lor c\) and this implies that i) \(i\) also defaults under \(p \lor c\) in \(X\) and ii) \(\Delta \pi_{ij} a_i'(q + c) \geq \Delta \pi_{ij} a_i'(p \lor c)\). Now (4) follows from the precondition of the lemma.\(^{24}\)

With the claim shown, we can prove the statement of the lemma by induction on the iteration sequence that converges to the clearing payments. More in detail, let:

\[
p^0 = l \quad p^{c,0} = l - c \quad p^n = p^{c,n} + c
\]
\[
p^{n+1} = \Psi(p^n) \quad p^{c,n+1} = \Psi^c(p^{c,n})
\]

We know that \(p^n \to p\), \(p^{c,n} \to p^{c}\), and thus \(p^n \to p^{'}\) for \(n \to \infty\), all from above (see Section 2.2). We now prove by induction that \(p^n \geq p^n \forall n\). This immediately implies \(p^' \geq p\).

\(^{24}\)Note that it is in fact equivalent to the precondition of the lemma that (4) hold for all specified \(q\) in the \(\Delta \pi_{ij} > 0\) case lemma because \(\bigwedge \{ q + c \mid q \in [0, l - c], q + c \geq p \} = p \lor c\).
For \( n = 0 \), we trivially have \( p^0_0 = l = p^0 \). Assuming \( p^n \geq p^n \), we have

\[
p^{n+1} = p^{n+1} + c = \Psi^c(p^n + c) \geq \Psi(p^n) = p^n.
\]

The first inequality is by the above claim. The claim is applicable because \( p^{n+1} + c = p^n \geq p^0 \geq p \) by induction hypothesis. The second inequality is by the induction hypothesis and monotonicity of \( \Psi \).

\[\textbf{Proof of Lemma 2.}\]

2: By assumption and monotonicity of \( \Psi^c \), we have

\[
\Psi^c(\tilde{p}) = \Psi^c(\tilde{p}_I \cup \tilde{p}_O) \geq \Psi^c(\tilde{p}_I \cup p_O) = \tilde{p}.
\]

By the Knaster-Tarski fixed point theorem (see, e.g., Granas and Dugundji (2003)), we have

\[
p^c = \bigvee \{ q \in [0, l - c] \mid \Psi^c(q) \geq q \},
\]

where “\( \bigvee \)" denotes the point-wise supremum. By the above, \( \tilde{p} \) is a member of the set on the right-hand side and thus \( p^c \geq \tilde{p} \). Now trivially by assumption \( p^c_0 \geq \tilde{p}_O \geq p_O \).

If \( i \notin N(c) \), then \( (j, i) \in O \) whenever \( l_{ji} > 0 \) for all \( j \). Thus, since \( p^c_0 \geq p_O \), we have \( a^c_i(p^c) \geq a_i(p) \). At the same time, since \( i \notin N(c) \), \( l^c_i = l_i \). This implies \( E^c_i \geq E_i \).

For the second sentence, we show that under the preconditions also \( \tilde{p} \geq p^c \). Then the statement follows by part 3. Let \( (i, j) \in I \). If there is a \( k \) such that \( (i, k) \in O \), then we have \( \tilde{r}_i l_{ik} = \tilde{p}_{ik} \geq p_{ik} = \tilde{r}_i l_{ik} \) and thus \( \tilde{r}_i \geq r_i \) and

\[
\tilde{p}^c_{ij} = \tilde{r}_i (l_{ij} - c_{ij}) + c_{ij} = \tilde{r}_i l_{ij} + (1 - \tilde{r}_i) c_{ij} \geq r_i l_{ij} = p_{ij}.
\]

3: The statement implies the precondition to the first sentence of part 2 since \( \tilde{p}^c_O = \tilde{p}_O \) and \( p^c_O = p_O \). Therefore, \( p^c = \tilde{p} + c \geq \tilde{p} + c = \tilde{p}' \). And \( \tilde{p}' \geq p \) by assumption.

1: Similarly to the proof of Lemma 1, the result is driven by the following statement, which only holds if \( c \) is chord-free:

\[\textbf{Claim.}\] If \( c \) is normal for \( X \) and \( q_I \in [0, (l - c)I] \) be arbitrary, then:

\[
\Psi^c_I (q_I \cup p_O) + c \geq \Psi^c_I ((q_I + c_I) \cup p_O) \tag{5}
\]

\[\textbf{Proof of the Claim.}\] Let \( (i, j) \in I \) and define for convenience \( q^c_I := q_I + c_I \). We
distinguish two cases based on the default of \(i\). Note that

\[
a_i^c(q_I \cup p_O) = c_i + \sum_{k:(k,i) \in I} q_{I,k,i} + \sum_{k:(k,i) \in O} p_{O,k,i}
= c_i + \sum_{k:(k,i) \in I} (q_{I,k,i} + c_{ki}) + \sum_{k:(k,i) \in O} p_{O,k,i} - c_i = a_i(q_I' \cup p_O) - c_i
\]

\[
a_i^c(q_I \cup p_O) = a_i(q_I' \cup p_O) - \beta c_i
l_i' = l_i - c_i.
\]

Thus, \(i\) defaults (i.e., assets are below liabilities) in \(X^c\) under \(q_I \cup p_O\) iff it defaults in \(X\) under \(q_I' \cup p_O\). In case of non-default, we have

\[
\Psi_{ij}^c(q_I \cup p_O) + c = l_{ij}' + c = l_{ij} = \Psi_{ij}(q_I' \cup p_O),
\]

and in particular (5) holds for \((i, j)\). In case of default, we have

\[
\Psi_{ij}^c(q_I \cup p_O) + c = \pi_{ij}^c a_i^c(q_I \cup p_O) + c_{ij} = \pi_{ij}^c a_i(q_I' \cup p_O) - \pi_{ij}^c \beta c_i + c_{ij}
\]

\[
\Psi_{ij}(q_I' \cup p_O) = \pi_{ij} a_i(q_I' \cup p_O)
\]

Taking the difference, we have \(\Psi_{ij}^c(q_I \cup p_O) + c \geq \Psi_{ij}(q_I' \cup p_O)\) iff

\[
\Delta \pi_{ij}^c a_i^c(q_I' \cup p_O) + c_{ij} \geq \beta \pi_{ij}^c c_i.
\]

Via the elementary identity \(c_{ij} = \pi_{ij}^c c_i - \Delta \pi_{ij} l_i\), this is equivalent to

\[
\Delta \pi_{ij} (l_i - a_i(q_I' \cup p_O)) \leq (1 - \beta) \pi_{ij}^c c_i.
\]

On the left-hand side, since \(c\) is normal, the first factor is \(\Delta \pi_{ij} \leq 0\), and since \(i\) defaults, the second factor is \(\geq 0\). Thus, the left-hand side is non-positive. Since the right-hand side is always non-negative, the inequality holds.

With the claim proven, we can show our original statement \(\tilde{p}_I' \geq p_I\). Define two sequences of vectors

\[
\tilde{p}_I^0 := l_I - C_I
\]

\[
\tilde{p}_I^{n+1} := \Psi_{I}(\tilde{p}_I^n \cup p_O)
\]

\[
p_I^0 := l_I
p_I^{n+1} := \Psi_{I}(p_I^n \cup p_O).
\]

By definition, \(\tilde{p}_I\) is the maximal fixed point of the map \(q_I \in [0,(l - c)_I] \mapsto \Psi_{I}(q_I \cup p_O)\). Also, \(p_I\) is the maximal fixed point of the map \(q_I \in [0,l_I] \mapsto \Psi_{I}(q_I \cup p_O)\) because \(p\) is the maximal fixed point of \(\Psi\).\(^{25}\) Thus, by the well-known extension of

\(^{25}\)This follows via the Tarski-Knaster fixed point theorem if we consider \(\Psi\) as a function of two variables \(\Psi(q_I, q_O)\). See Lemma 3 in Appendix C.

36
the Tarski-Knaster fixed point theorem (see Remark 1), we have that \( \tilde{p}^n_I \to \tilde{p}_I \) and \( p^n_I \to p_I \) from above for \( n \to \infty \).

Note that \( \tilde{p}^0_I = p^0_I \) and via the claim, \( \tilde{p}^n_I + c_I \geq p^n_I \) \( \Rightarrow \tilde{p}^{n+1}_I + c_I \geq p^{n+1}_I \). Thus, by induction, \( \tilde{p}^n_I + c_I \geq p^n_I \) and by taking the limit, \( \tilde{p}_I = p_I + c_I = \lim_n (\tilde{p}^n_I + c_I) \geq \lim_n p^n_I = p_I \).

Towards the second part of the statement, if \( c \) is chord-free and \( i \in N(c) \), then for all \( j \in N \) we either have \( (i,j) \in I \) and thus \( \tilde{p}^n_{ij} \geq p_{ij} \) or \( c_i = c_{ij} = 0 \) and thus \( \tilde{p}^n_{ij} = \tilde{p}_{ij} = p_{ij} \). Therefore, \( a_i(\tilde{p}') \geq a_i(p) \) and thus \( B^c_i(\tilde{p}) = B_i(\tilde{p}') \geq B_i(p) \). In particular, \( \tilde{E}_i \geq E_i \).

**Proof of Theorem 5.** Consider the immediate-effect payments \( \tilde{p} \) and consider the recovery rates \( \tilde{r}_i = \frac{\tilde{p}_i}{\tilde{p}^+_i} \). We show that \( \tilde{r}_i \geq r_i \) \( \forall i \in N(c) \). This implies the statement via Lemma 2 because then

\[
\tilde{p}^n_{ij} = \tilde{r}_i(l_{ij} - c_{ij}) + c_{ij} = \tilde{r}_i l_{ij} + (1 - \tilde{r}_i) c_{ij} \geq \tilde{r}_i l_{ij} = p_{ij} \ \forall i \in N(c), j \in N.
\]

The recovery rates \( \tilde{r}_i =: \tilde{p} \) and \( r_i =: p \) are the same across all \( i \in N(c) \). To see this, consider the sequences

\[
\begin{align*}
\tilde{r}^0 := & (l - c)_I \cup p_O \\
\tilde{r}^{n+1} := & \Psi^c(\bar{p}^n_I \cup p_O) \\
p^0 := & I_I \cup p_O \\
p^{n+1} := & \Psi(p^n_I \cup p_O).
\end{align*}
\]

We know that \( \tilde{r}^n \to \tilde{p} \) and \( p^n \to p \) monotonically decreasing for \( n \to \infty \) (see the proof of Lemma 2). We show by induction that \( \tilde{r}^n_i := \tilde{r}^n_{ii} = \frac{\tilde{p}^n_{ii}}{\tilde{p}^+_i} \) and \( \rho^n_i := r^n_i = \frac{p^n_{ii}}{l_i} \) are equal, respectively, across \( i \in N(c) \). This implies that also \( \tilde{r}_i \) and \( r_i \) must be equal, respectively, across \( i \in N(c) \), via continuity from above. For \( n = 0 \), the two statements are trivial because \( \tilde{r}^0_i = r^0_i = 1 \). Given the statement for \( n \), we have:

\[
\begin{align*}
a_i^c(\tilde{p}^n_I, p_O) = & \ e_i + p_O i + \sum_{j:(j,i) \in I} \tilde{p}^n_{ji} = e_i + p_O i + \tilde{p}^n (I_{ii} - c_i) \\
a_i^c(\bar{p}^n_I, p_O) = & \alpha e_i + \beta p_O i + \beta \sum_{j:(j,i) \in I} \bar{p}^n_{ji} = \alpha e_i + \beta p_O i + \beta \tilde{p}^n (l_{ii} - c_i) \\
l^+_i = & l_i O + l_i I - c_i.
\end{align*}
\]

By assumption, all of these values are the same across \( i \in N(c) \) and thus,

\[
\tilde{r}^{n+1}_i = \begin{cases} 
1 & \text{if } a_i^c(\tilde{p}_I, p_O) \geq l^+_i \\
\frac{a_i^c(\bar{p}^n_I, p_O)}{l^+_i} & \text{if } a_i^c(\tilde{p}^n_I, p_O) < l^+_i
\end{cases}
\]

are the same across \( i \in N(c) \). An analogous argument holds for the \( r^n_i \).

The above-described symmetry implies that either all or no bank in \( N(c) \) default
and this is independent of compression. More in detail:

- If, equivalently across $i \in N(c)$, $l_i \leq a_i(p^0_1, p_O) = c_i + p_{Oi} + l_{Ii}$, then $p^1 = p^0$ and thus $p^a = p^0 \forall a$, $p = p^0$, and $\rho = 1$. Further, $l_i' = l_i - c_i \leq a_i(p^0_1, p_O) - c_i = a_i(p^0_1, p_O)$ and thus $\tilde{\rho} = \rho^0$ and $\tilde{\rho} = 1$.
- If $l_i > c_i + p_{Oi} + l_{Ii}$, then $\rho \leq \rho^1 < 1$ and $\tilde{\rho} \leq \tilde{\rho}^1 < 1$ by the same argument.

If $\tilde{\rho} = \rho = 1$, our statement is of course trivially true. If $\tilde{\rho}, \rho < 1$, we have by symmetry and the clearing identity (1) (let $i \in N(c)$ be arbitrary):

\[
\rho = \frac{\alpha e_i + \beta p_{Oi} + \beta l_{Ii}}{l_i} \\
\tilde{\rho} = \frac{\alpha e_i + \beta p_{Oi} + \beta l_{Ii} - \beta c_i}{l_i - c_i}
\]

Solving for $\rho$ and $\tilde{\rho}$, respectively, and simplifying yields:

\[
\rho = \frac{\alpha e_i + \beta p_{Oi}}{l_i - \beta l_{Ii}} \\
\tilde{\rho} = \frac{\alpha e_i + \beta p_{Oi}}{l_i - \beta l_{Ii} - (1 - \beta)c_i}
\]

And this obviously implies $\tilde{\rho} \geq \rho$. Note that both fractions are well-defined since, if $l_i - \beta l_{Ii} - (1 - \beta)c_i = 0$, then in particular $l_i - c_i \leq l_{Ii}$ and thus, by the above discussion, $i$ would not default. Likewise for $l_i - \beta l_{Ii}$.

### C Fixed Points of Monotonic Functions

**Lemma 3.** Let $K, L$ be complete lattices and let $F : K \times L \to K \times L$ be monotonic. Let $(x^+, y^+)$ be the (unique) maximal fixed point of $F$. Then $x^+$ is the maximal fixed point of the function $x \mapsto F_1(x, y^+)$. 

Note that the converse of the lemma does not hold. For example, let $K = L = [0, 1]$ and $F(x, y) = (y, x)$. Then 0 is the unique (and therefore maximal) fixed point of $F_1(\cdot, 0)$ and of $F_2(0, \cdot)$, but $(0, 0)$ is not the maximal fixed point of $F$.

**Proof.** Let $x^*$ be the maximal fixed point of $x \mapsto F_1(x, y^+)$. We will show that $x^* = x^+$. As $x^+$ is a fixed point of $F_1(\cdot, y^+)$ by choice of $(x^+, y^+)$, we have $x^+ \leq x^*$. It remains to show “$\geq$”.

We have $(x^*, y^+) \leq F(x^*, y^+)$. To see this, consider the two components of $F$ separately. We have $x^* = F_1(x^*, y^+)$ by choice of $x^*$. For the second component, note that $y^+ = F_2(x^+, y^+) \leq F_2(x^+, y^+)$, where the equality is by choice of $(x^+, y^+)$ and the inequality is because $x^+ \leq x^*$. Now, by Tarski’s fixed point theorem, we have

\[
(x^+, y^+) = \bigvee \{(x, y) \mid (x, y) \leq F(x, y)\},
\]
where “$\bigvee$” denotes the supremum. By the above, $(x^*, y^+)$ is a member of the set on the right-hand side and thus $(x^+, y^+) \geq (x^*, y^+)$, i.e., $x^+ \geq x^*$.

References


Duffie, Darrell. 2018. “Compression Auctions With an Application to LIBOR-SOFR Swap Conversion.”


