Relations

Laura Kovács
A set is a group of objects;

Objects in a set are called elements or members;

One way of describing a set is by listing its elements inside braces.

For example: \{7,14,21,28\};

- finite set
- set of natural numbers: \(\mathbb{N} = \{0,1,2,\ldots\};\)
- infinite set
- set of integer numbers: \(\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\};\)
- infinite set

The set with 0 elements is the empty set, denoted by \(\emptyset\).

Set membership is denoted by the symbol: \(\in\).

For example: \(7 \in \{7,14,21,28\}\);

Set nonmembership is denoted by the symbol: \(\notin\).

For example: \(8 \notin \{7,14,21,28\}\).
Given two sets A and B.

- A is a **subset** (Teilmenge) of B, written \( A \subseteq B \), if:
  every member of A is also an element of B.
  Thus, \( A \subseteq B \) is logically equivalent to \( \forall x : x \in A \Rightarrow x \in B \).

- A is a **proper subset** of B, written \( A \subsetneq B \), if:
  A is a subset of B and not equal to B.
  Thus, \( A \subsetneq B \) is logically equivalent to \( A \subseteq B \land A \neq B \).

- The **union** of A and B is the set \( A \cup B \) obtained by combining *all* elements of A and B;

- The **intersection** of A and B is the set \( A \cap B \) of elements that are both in A and B;

- The **complement** of A is the set \( A^{\text{c}} \) of all elements that are not in A;

- The **Cartesian product** (karthesischen Produkt) of A and B is the set \( A \times B \) of all pairs \((a,b)\) such that \( a \in A \) and \( b \in B \).
  One may write: \( A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \} \)
Relations (Relationen)

Given the sets $A_1, A_2, \ldots, A_n$.

- An $n$-ary relation $R$ is a subset of the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$:
  $R \subseteq A_1 \times A_2 \times \ldots \times A_n$

- A binary relation $R$ is a subset of the Cartesian product $A_1 \times A_2$:
  $R \subseteq A_1 \times A_2$

  Note: Binary relation is also called a 2-ary relation.

For $(a,b) \in R$ one also writes $aRb$.

Note: The statement $aRb$ means that $aRb$ is True.

If $A_1 = A_2$ we say that $R$ is a relation over $A_1$.

Examples: $<, >, =$ are binary relations over numbers.
Scissors-Paper-Stone game:

- Two players simultaneously select a member from the set \{Scissor, Paper, Stone\};
- If selections are the same, the game starts over;
- If selection differ, one player wins according to the picture.
**Scissors-Paper-Stone game:**

- Two players simultaneously select a member from the set \(\{\text{Scissor, Paper, Stone}\}\);

- If selections are the same, the game starts over;

- If selection differ, one player wins according to the picture.

Relation: \(\text{beats} \subseteq \{\text{Scissor, Paper, Stone}\} \times \{\text{Scissor, Paper, Stone}\}\)

<table>
<thead>
<tr>
<th>(\text{beats})</th>
<th>Scissor</th>
<th>Paper</th>
<th>Stone</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scissor</td>
<td>False</td>
<td>True</td>
<td>False</td>
</tr>
<tr>
<td>Paper</td>
<td>False</td>
<td>False</td>
<td>True</td>
</tr>
<tr>
<td>Stone</td>
<td>True</td>
<td>False</td>
<td>False</td>
</tr>
</tbody>
</table>

\(\text{beats} = \{(\text{Scissor, Paper}), (\text{Paper, Stone}), (\text{Stone, Scissor})\}\)
Binary Relations – Properties

Let $A$ be a set and $R$ a binary relation over $A$ (that is $R \subseteq A \times A$).

- $R$ is **reflexive** if for every $x \in A$ it holds that $xRx$.

  $\forall x : x \in A : xRx$

  That is, every element $x$ of $A$ is in relation $R$ with itself.

Examples:

- $=, \geq$ are ??? binary relations over natural numbers;
- $>$ is ??? binary relation over natural numbers;
- Relation *beats* from the Scissor-Paper-Stone game is ???
Let $A$ be a set and $R$ a binary relation over $A$ (that is $R \subseteq A \times A$).

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  \[ \forall x: x \in A: xRx \]
  That is, every element $x$ of $A$ is in relation $R$ with itself.

Examples:
- $=, \geq$ are reflexive binary relations over natural numbers;
- $>$ is not a reflexive binary relation over natural numbers;
- Relation $beats$ from the Scissor-Paper-Stone game is not reflexive.
Binary Relations – Properties

Let $A$ be a set and $R$ a binary relation over $A$ (that is $R \subseteq A \times A$).

- **$R$ is reflexive** if for every $x \in A$ it holds that $xRx$.
  \[ \forall x: x \in A: xRx \]
  That is, every element $x$ of $A$ is in relation $R$ with itself.

- **$R$ is symmetric** if for every $x,y \in A$ it holds that if $xRy$ then $yRx$.
  \[ \forall x,y: x,y \in A: xRy \Rightarrow yRx \]
  Examples:
  - $=$ is ??? binary relations over natural numbers;
  - $>, \geq$ are ??? binary relations over natural numbers;
  Relation *beats* from the Scissor-Paper-Stone game is ???.
Binary Relations – Properties

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  Examples:
  - $=$ is a symmetric binary relations over natural numbers;
  - $>$, $\geq$ are not symmetric binary relations over natural numbers;
  Relation *beats* from the Scissor-Paper-Stone game is not symmetric
Binary Relations – Properties

Let $A$ be a set and $R$ a binary relation over $A$ (that is $R \subseteq A \times A$).

- $R$ is **reflexive** if for every $x \in A$ it holds that $xRx$.
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  That is, every element $x$ of $A$ is in relation $R$ with itself.

- $R$ is **symmetric** if for every $x, y \in A$ it holds that if $xRy$ then $yRx$.
  $$\forall x, y: x, y \in A: xRy \Rightarrow yRx$$

- $R$ is **transitive** if for every $x, y, z \in A$ it holds that if $xRy$ and $yRz$ then $xRz$.
  $$\forall x, y, z: x, y, z \in A: (xRy \land yRz) \Rightarrow xRz$$

Examples:
- $=$ is binary relations over natural numbers;
- $>$, $\geq$ are relations over natural numbers;

Relation *beats* from the Scissor-Paper-Stone game is ???
Binary Relations – Properties

Let \( A \) be a set and \( R \) a binary relation over \( A \) (that is \( R \subseteq A \times A \)).

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  \[
  \forall x: x \in A: xRx
  \]
  That is, every element \( x \) of \( A \) is in relation \( R \) with itself.

- \( R \) is **symmetric** if for every \( x,y \in A \) it holds that if \( xRy \) then \( yRx \).
  \[
  \forall x,y: x,y \in A: xRy \Rightarrow yRx
  \]

- \( R \) is **transitive** if for every \( x,y,z \in A \) it holds that if \( xRy \) and \( yRz \) then \( xRz \).
  \[
  \forall x,y,z: x,y,z \in A: (xRy \land yRz) \Rightarrow xRz
  \]

Examples:
- \( = \) is a transitive binary relations over natural numbers;
- \( >, \geq \) are transitive binary relation over natural numbers;
- Relation \textit{beats} from the Scissor-Paper-Scissor game is not transitive.
Binary Relations – Properties

Let A be a set and R a binary relation over A (that is $R \subseteq A \times A$).

- **R is reflexive** if for every $x \in A$ it holds that $xRx$.
  $$\forall x: x \in A: xRx$$
  That is, every element $x$ of $A$ is in relation $R$ with itself.

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  $$\forall x,y: x,y \in A: xRy \Rightarrow yRx$$

- **R is transitive** if for every $x,y,z \in A$ it holds that if $xRy$ and $yRz$ then $xRz$.
  $$\forall x,y,z: x,y,z \in A: (xRy \land yRz) \Rightarrow xRz$$

- **R is an equivalence relation** if it is reflexive, symmetric and transitive.

Examples: Are $=, >, \geq, \text{beats}$ equivalence relations?
Binary Relations – Properties

Let A be a set and R a binary relation over A (that is \( R \subseteq A \times A \)).

- R is **reflexive** if for every \( x \in A \) it holds that \( xRx \).
  \[
  \forall x: x \in A: xRx
  \]
  That is, every element \( x \) of A is in relation R with itself.

- R is **symmetric** if for every \( x, y \in A \) it holds that if \( xRy \) then \( yRx \).
  \[
  \forall x, y: x, y \in A: xRy \Rightarrow yRx
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- R is **transitive** if for every \( x, y, z \in A \) it holds that if \( xRy \) and \( yRz \) then \( xRz \).
  \[
  \forall x, y, z: x, y, z \in A: (xRy \land yRz) \Rightarrow xRz
  \]

- R is an **equivalence relation** if it is reflexive, symmetric and transitive.

Examples: = is an equivalence relation; >, ≥, beats are not equivalence relations.
Binary Relations

Let $A$ be a set and $R \subseteq A \times A$ an equivalence relation.

The set of all elements $y$ such that $xRy$ - is called the equivalence class of $x$, - and is denoted by $[x]_R$.

$$[x]_R = \{y \mid xRy\}$$

denotes “the set of all $y$ such that $xRy$”.

Examples: $[1]_e = \ ???$
Binary Relations

Let $A$ be a set and $R \subseteq A \times A$ an *equivalence relation*.

The set of all elements $y$ such that $xRy$ is called the *equivalence class of $x$*, and is denoted by $[x]_R$.

$$[x]_R = \{y \mid xRy\}$$

denotes “the set of all $y$ such that $xRy$”.

Examples: $[1]_e = \{1\}$
Consider the relation \( \equiv_5 \) over the integer numbers \( \mathbb{Z} \) defined as:

\[
i \equiv_5 j \quad if \ and \ only \ if \quad i-j \ is \ a \ multiple \ of \ 5. \quad (\text{where } i,j \in \mathbb{Z})
\]

Is \( \equiv_5 \) an equivalence relation?

If so, what is \([1]_{\equiv_5}\)?
Binary Relations

Let $A=\{a,b,c,d\}$ be a set and $R \subseteq A \times A$ the relation below:
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**Binary Relations**

$R$ with “1 jump”:
Let $A=\{a,b,c,d\}$ be a set and $R \subseteq A \times A$ the relation below:

**R with “1 jump”:**

**R with “2 jumps”:**
Let $A=\{a,b,c,d\}$ be a set and $R \subseteq A \times A$ the relation below:

- R with “1 jump”:

- R with “2 jumps”, “3 jumps”, …:
Binary Relations – Transitive Closure/Hull (Transitive Hülle)

$R^t$ is the transitive closure of $R$:

R with “1 jump”:

R with “2 jumps”, “3 jumps”, …:
Let $A$ be a set and $R \subseteq A \times A$ a transitive relation.

The transitive closure of $R$ is (the smallest) relation $R^t$ such that
- $R^t$ contains $R$: $R \subseteq R^t$;
- it extends $R$ by all those other (indirect) relations among elements that can be obtained using the transitivity of $R$. 

Computing $R^t$:

$$R^1 = R;$$

$$R_i = R_{i-1} \cup \{(a,b) \mid \exists c:: (a,c) \in R_{i-1} \land (c,b) \in R_{i-1}\},$$

for every $i > 1$.

$$R^t = \bigcup_{i \geq 1} R_i = R_1 \cup R_2 \cup R_3 \cup \ldots$$
Let $A$ be a set and $R \subseteq A \times A$ a transitive relation.

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**Computing $R^t$:**

- $R^1 = R$;
- $R^i = R^{i-1} \cup \{(a,b) \mid \exists c:: (a,c) \in R^{i-1} \land (c,b) \in R^{i-1}\}$, for every $i > 1$.

$$R^t = \bigcup_{i \geq 1} R^i = R^1 \cup R^2 \cup R^3 \cup \ldots$$
Let $A$ be a set and $R \subseteq A \times A$.

- $R$ is **irreflexive** if for every $x \in A$ it holds $\neg(xRx)$.
  \[\forall x : x \in A : \neg(xRx)\]

That is, no element $x$ of $A$ is in relation $R$ with itself.

Examples:

- $>$ is **???

Relation *beats* from the Scissor-Paper-Stone game is **???


Binary Relations – Properties

Let $A$ be a set and $R \subseteq A \times A$.

- $R$ is **irreflexive** if for every $x \in A$ it holds $\neg (xRx)$.

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Examples:

- $>$ is irreflexive;

Relation *beats* from the Scissor-Paper-Stone game is irreflexive.
Binary Relations – Properties

Let $A$ be a set and $R \subseteq A \times A$.

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  \[
  \forall x: x \in A: \neg(xRx)
  \]
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- **R is antisymmetric** if for every $x,y \in A$ it holds that if $xRy$ and $yRx$ then $x$ and $y$ are the same.
  \[
  \forall x,y: x,y \in A: (xRy \land yRx) \Rightarrow x=y
  \]

Examples:

Are $\geq$, $=$, $\subseteq$ antisymmetric?
Binary Relations – Properties

Let $A$ be a set and $R \subseteq A \times A$.

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Examples:

Are $\geq, =, \subseteq$ antisymmetric? YES.
Binary Relations – Properties

Let A be a set and $R \subseteq A \times A$.

- **R is irreflexive** if for every $x \in A$ it holds $\neg (xRx)$.
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- **R is antisymmetric** if for every $x,y \in A$ it holds that *if* $xRy$ and $yRx$ *then* $x$ and $y$ are the same.
  $$\forall x,y: x,y \in A: (xRy \land yRx) \Rightarrow x=y$$

- **R is asymmetric** if for every $x,y \in A$ it holds that *if* $xRy$ *then* $\neg (yRx)$.
  $$\forall x,y: x,y \in A: xRy \Rightarrow \neg (yRx)$$
  That is, $xRy$ and $yRx$ cannot hold at the same time.

Examples:

Are $\geq$, $=$, $>$ asymmetric?
Binary Relations – Properties

Let $A$ be a set and $R \subseteq A \times A$.

- $R$ is **irreflexive** if for every $x \in A$ it holds $\neg(xRx)$.
  \[ \forall x: x \in A: \neg(xRx) \]
  That is, no element $x$ of $A$ is in relation $R$ with itself.

- $R$ is **antisymmetric** if for every $x, y \in A$ it holds that if $xRy$ and $yRx$ then $x$ and $y$ are the same.
  \[ \forall x, y: x, y \in A: (xRy \land yRx) \Rightarrow x=y \]

- $R$ is **asymmetric** if for every $x, y \in A$ it holds that if $xRy$ then $\neg(yRx)$.
  \[ \forall x, y: x, y \in A: xRy \Rightarrow \neg(yRx) \]
  That is, $xRy$ and $yRx$ cannot hold at the same time.

Examples:

- $>$ is asymmetric; $\geq, =$ are not asymmetric
Binary Relations – Properties

Let $A$ be a set and $R \subseteq A \times A$.

- **R is irreflexive** if for every $x \in A$ it holds $\neg (xRx)$.
  \[ \forall x: x \in A: \neg (xRx) \]

That is, no element $x$ of $A$ is in relation $R$ with itself.

- **R is antisymmetric** if for every $x,y \in A$ it holds that if $xRy$ and $yRx$ then $x$ and $y$ are the same.
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That is, $xRy$ and $yRx$ cannot hold at the same time.

R is asymmetric if and only if $R$ is antisymmetric and irreflexive.
Binary Relations – Properties

Let \( A \) be a set and \( R \subseteq A \times A \).

- **R** is **non-symmetric** \((\text{unsymmetrisch})\) if it is not symmetric. 
  \[
  \forall x, y: x, y \in A : (xRy) \land \neg(yRx)
  \]

- **R** is a **total relation** if for every \( x, y \in A \) either \( xRy \) or \( yRx \) holds. 
  \[
  \forall x, y: x, y \in A : xRy \lor yRx
  \]
  That is, \( R \) is defined on the entire \( A \).

Note: Total relations are reflexive.

Examples:

- Are \( \geq, =, \text{beats} \) total?
Binary Relations – Properties

Let $A$ be a set and $R \subseteq A \times A$.

- **$R$ is non-symmetric** (unsymmetrisch) if it is not symmetric.
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  \forall x, y : x, y \in A : (xRy) \land \neg(yRx)
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- **$R$ is a total relation** if for every $x, y \in A$ either $xRy$ or $yRx$ holds.
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Examples:

- $\geq$ is total;
- $=, beats$ are not total.
Binary Relations – Properties

Let $A$ be a set and $R \subseteq A \times A$.

- $R$ is **acyclic** (azyklisch) if there is no $x_1, x_2, \ldots, x_n \in A$ such that
  \[ x_1 R x_2 \land x_2 R x_3 \land \ldots \land x_{n-1} R x_n \land x_n R x_1 \]

\[ \forall n: n \in \mathbb{N}: \]
\[ \neg (\exists x_1, x_2, \ldots, x_n: x_1, x_2, \ldots, x_n \in A: x_1 R x_2 \land x_2 R x_3 \land \ldots \land x_{n-1} R x_n \land x_n R x_1) \]

Note: Acyclic relations are irreflexive.

Example: $>$ is acyclic.
Binary Relations – Properties

Let $A$ be a set and $R \subseteq A \times A$.

- $R$ is called a **partial order** (Halbordung, partiale Ordnung) if
  - $R$ is reflexive;
  - $R$ is transitive;
  - $R$ is antisymmetric.

  Example: $\geq$ is a partial order over $\mathbb{N}$.
  Division $/$ is a partial order over $\mathbb{N}$.

- $R$ is called a **total order** or a **linear order** (lineare/totale Ordnung) if
  - $R$ is a partial order;
  - $R$ is a total relation.

  Example: $\geq$ is a total order over $\mathbb{N}$.
  Division $/$ is not a total order over $\mathbb{N}$.

- $R$ is called a **strict partial order** (strenge Halbordnung) if
  - $R$ is irreflexive;
  - $R$ is transitive.

  Example: $>$ is a strict partial order over $\mathbb{N}$. 
Binary Relations – Properties

Let $A$ be a set and $R \subseteq A \times A$.

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  - $R$ is reflexive;
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  Example: $>$ is a strict partial order over $\mathbb{N}$.
Binary Relations – Properties

Let $A$ be a set and $R \subseteq A \times A$ a partial order.

- An element $y \in A$ is an upper bound of a set $X \subseteq A$ if:
  - $xRy$ for every $x \in X$.

- An element $y \in A$ is a least upper bound of a set $X \subseteq A$ if:
  - $y$ is an upper bound of $X$;
  - $yRy'$ for all upper bounds $y'$ of $X$.

Note: By antisymmetry, if $y$ and $y'$ are least upper bounds, then $y = y'$.

Hence, $X$ has a unique least upper bound $y$, and we write $y = \text{lub}(X)$.

- A chain is a set $\{x_0, x_1, x_2, \ldots \} \subseteq A$ with $x_iRx_{i+1}$ for all $i \geq 0$.

- $R$ is a complete partial order (cpo) over $A$ if
  - there is bottom element (denoted by $\bot$);
  - for every chain $X \subseteq A$ there is a least upper bound.
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