Lecture #7.2: Sponsored Search Auctions (Part II)

Prof. Dr. Sven Seuken

22.3.2012

Housekeeping

- Questions? Concerns?
- AGT Homework due next Monday
- Adjusting workload of the course

Outline

1. Recap of Sponsored Search Auctions
2. Today’s topic: The VCG Auction
3. Discussion + Questions

Sponsored Search Auctions

- Bidders have value per click $w_i \in [0, \infty]$
- Bidders submit bids b_i indicating their w_i
- Separability: 1) bidder effect (quality) and 2) slot effect: $eCTR_{i,j} = q_i \times s_j$
- Winner determination: *allocating slots in order of decreasing $q_i w_i$ is efficient.*
- Key assumption: bidders' know each others values
- Simplifying assumption: $\mathrm{q_i} = 1$

Generalized Second-Price Auction (GSP)

Definition 7.2 (Second-Price Rank-by-revenue Auction). *Assume* $q_1 b_1 \geq q_2 b_2 \geq \ldots \geq q_n b_n$. *Allocate slots to bidders in order of decreasing bidder effect* $\times$ *per-click bid price (breaking ties at random), so that slot 1 goes to bidder 1, slot 2 to bidder 2, and so forth. Upon receiving a click on an ad, charge the bidder associated with the ad a price* p_i *such that* $q_i p_i = q_{i+1} b_{i+1}$.

Envy-Free Equilbirium of GSP Auction

Definition 7.6 (Envy-free equilibrium of GSP slot auction)**.** *Assume without loss of generality* $b_1^* \geq \ldots \geq b_n^*$*, and that ties in bid values are broken in this order for the purpose of allocating slots so that advertiser 1 gets slot 1, advertiser 2 slot 2, and so forth. For bid profile* $b^* = (b_1^*, \ldots, b_n^*)$ *to be an envy-free equilibrium of the GSP slot auction we need:*

- *For* $1 \leq i \leq m$ *(agents allocated some slot):*

$$s_i(w_i - b_{i+1}^*) \geq s_{i'}(w_i - b_{i'+1}), \quad \textit{for all } i' \in \{1, \ldots, m\}, i' \neq i \tag{7.10}$$

$$s_i(w_i - b_{i+1}^*) \geq 0 \tag{7.11}$$

- *For* $m < i \leq n$ *(unallocated agents):*

$$b_{m+1}^* \geq w_i \tag{7.12}$$

Results about Envy-Free Equilibria

Theorem 7.3. *Any envy-free equilibrium of the GSP auction is also a Nash equilibrium of the GSP auction.*

- Proof in lecture notes…

Theorem 7.4. *An envy-free outcome in a slot auction is efficient (i.e., allocates slots to maximize total value.)*

- Proof in homework assignment…

Again: why Envy-Free?

- As long as I have envy, I will increase my bid…
- To make you pay as much as possible
- To drive you out of the market

- But, this leads to a threat of retaliation: "jamming"

Example: Envy-Free Equilbrium

slot	w_i	b_i^*	s_i	utility
1	10	10	0.2	
2	4	?	0.18	
3	2	2	0.1	0.1(2-1)=0.1
-	1	1	0	

b_2	utility (for slot 2)	utility if jammed by the agent currently in slot 1	threat?
4	(0.18)(4-2)=0.36	0.2(4-4)=0	yes
3.5	(0.18)(4-2)=0.36	0.2(4-3.5)=0.1	yes
2.5	(0.18)(4-2)=0.36	0.2(4-2.5)=0.3	yes
2.2	(0.18)(4-2)=0.36	0.2(4-2.2)=0.36	no
2.1	(0.18)(4-2)=0.36	0.2(4-2.1)=0.38	no

Buyer-Optimal Envy-Free Equilbrium

- This suggests: advertisers will bid just at the price where they have no envy, but not higher (for fear of retaliation)!

Definition 7.7 (Buyer-optimal Envy free equilibrium)**.** *Bid profile* $(b_1^*, \ldots, b_n^*)$ *is a buyer-optimal envy-free equilibrium of the GSP slot auction if the bid profile is an envy-free equilibrium and the price on each slot is minimized across all possible envy-free equilibria.*

Example: Buyer-optimal Envy-Free Eq.

$slot$	w_i	b_i^*	s_i	$utility$
1	10	10	0.2	0.2(10-2.5)=1.5
2	4	2.5	0.18	0.18(4-2)=0.36
3	2	2	0.1	0.1(2-1)=0.1
-	1	1	0	

- Set $b_4 = 1$. Solve for b_3: $0.1(2-1) = 0.1 = 0.18(2 - b_3^*)$

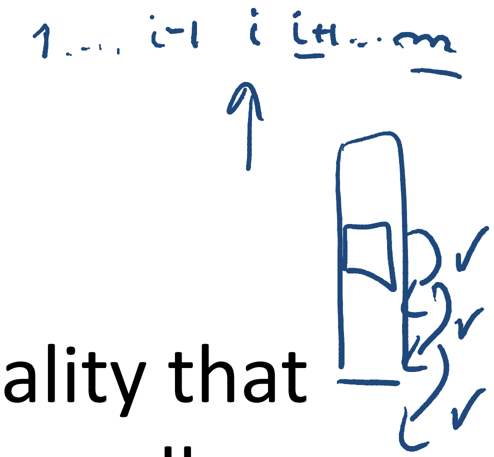
$$\Leftrightarrow \quad b_3^* = \frac{13}{9}$$

- Solve for b_2 $$0.18(4 - \frac{13}{9}) = 0.2(4 - b_2^*)$$

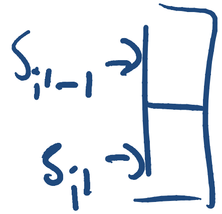
$$\Leftrightarrow \quad b_2^* = \frac{17}{10}$$

- Set $b_1 = 10$

Today: VCG Auction



- Idea: charge each bidder the externality that its presence in the auction imposes on all other bidders



$$t_{\text{vcg},i}(b) = \left(\sum_{i'=i+1}^{m} (s_{i'-1} - s_{i'}) b_{i'} \right) + s_m b_{m+1}$$

Definition 7.8 (Vickrey-Clarke-Groves (VCG) Slot Auction). *Given bids $b = (b_1, \ldots, b_n)$ with $b_1 \geq b_2 \geq \ldots \geq b_n$, the VCG slot auction allocates slots in order of decreasing bid amount and sets a per-click price for bidder i who allocated slot i to:*

$$p_i = \frac{t_{\text{vcg},i}(b)}{s_i} \qquad (7.14)$$

Proof that VCG is Truthful (1/2)

- Asssume agent A with value per click w_A
- Depending on how A bids, he will end up in slot *i*
- Then, agent A's expected payment is:

$$t_i = \underbrace{\left(\sum_{i'=1}^{i-1} s_{i'} \cdot b_{i'} + \sum_{i'=i+1}^{m+1} s_{i'-1} \cdot b_{i'} \right)}_{X} - \underbrace{\left(\sum_{i'=1}^{i-1} s_{i'} \cdot b_{i'} + \sum_{i'=i+1}^{m+1} s_{i'} \, b_{i'} \right)}_{Y}$$

- Thus, agent A's expected utility is:

$$u_i = s_i \cdot w_A - t_i = s_i \cdot w_A - X + Y$$

Proof that VCG is Truthful (2/2)

- Thus, agent A wants to maximize:

$$\max \; s_i \cdot w_A - X + Y$$

$$\Leftrightarrow \max \; s_i \cdot w_A + Y$$

- Remember how VCG allocates slots:

$$\Leftrightarrow \max \; s_i \cdot w_A + \sum_{\substack{i'=1 \\ i' \neq i}}^{m+1} s_{i'} \cdot b_{i'} \Rightarrow b_A^* = w_A$$

$$b_1 \geq b_2 \geq$$

$$s_1 \cdot b_1 \geq s_2 \cdot b_2 \geq \ldots \qquad \max \sum_i s_i \cdot b_i$$

q.e.d. $\square$

GSP vs. VCG

- So, should we use GSP or VCG?

Theorem 7.6. *The outcome of the GSP slot auction in the buyer-optimal envy-free equilibrium is the same outcome as in the truthful dominant-strategy equilibrium of the VCG slot auction.*

- Ok, so let's use VCG?

Theorem 7.7. *Given the same bids, the expected payment by each agent in the GSP auction is at least that in the VCG auction.*

- Well, now what?

Reasons to keep GSP

- Simplicity
- Large market effect + dynamics
- Re-engineering costs
- Ambiguous revenue effects

- But: Facebook uses VCG…