Lecture #6:
Auctions: Theory and Applications

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Housekeeping

- Questions? Concerns?
- BitTorrent homework assignment?
- Posting on NB: do not copy/paste from PDFs
- Game Theory Homework:
  - Formal/mathematical reasoning + write-up
  - Discussing homework assignments
  - Study the standard solution!!
- Registration deadline:
  - Friday, 17:00
  - MSc vs. BSc level
  - If in doubt, talk to me (e.g., today from 2:30-3:00pm)
- Course will be required elective with 99.9%

Outline

1. Recap of Algorithmic Game Theory
2. Today's topic: Auctions
3. Discussion + Questions

Recap: Algorithmic Game Theory

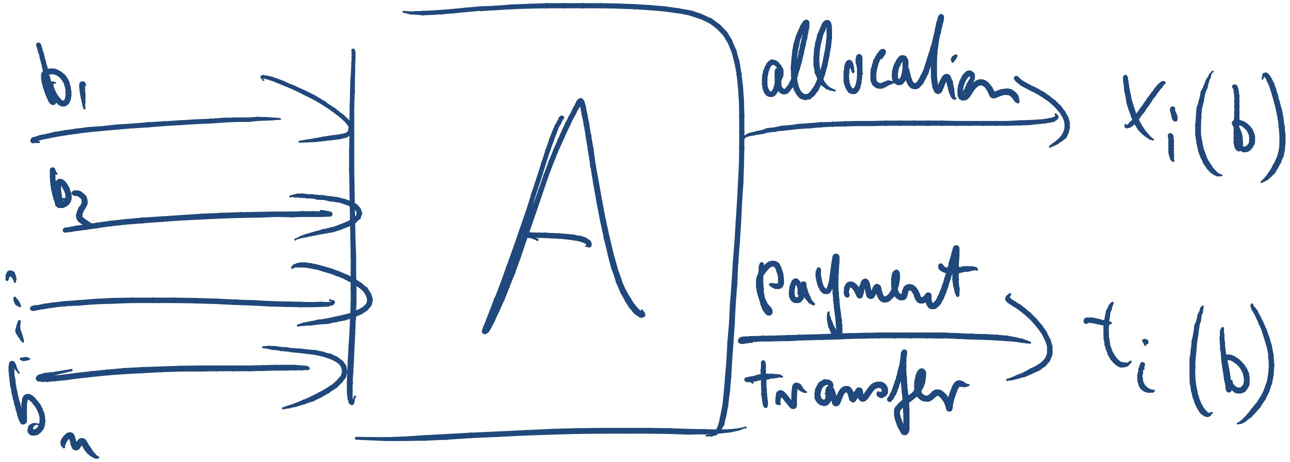
- Why do we care?
- Two-player Zero-Sum Games?
- Two-Player General-sum Games?
- Three-Player Games?
- Complexity of Find-Nash?
- Correlated Equilibrium?

Today: Auctions

What is an Auction

Item, Seller

n Buyers



Auctions: More Formally

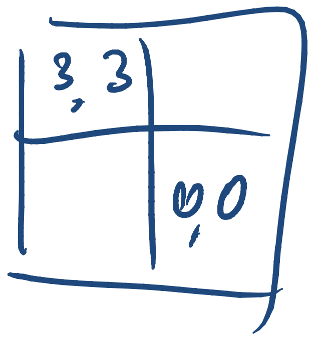
Definition 6.1 (Auction). *Let* $b = (b_1, \ldots, b_n)$ *denote a vector of bids, with* $b_i \in [0, \infty)$. *An auction is defined in terms of:*

- *An* allocation rule $x_i(b) \in \{0, 1\}$ *for each* $i \in N$, *with* $\sum_{i=1}^{n} x_i(b) \leq 1$ *(so that the item is allocated to at most one bidder), where* $x_i(b) = 1$ *if and only if agent* i *is the winner given vector of bids* b
- *A* payment rule $t_i(b) \in \Re$ *for each* $i \in N$, *where payment* $t_i(b)$ *is made by agent* i *to the auctioneer given vector of bids* b *(*t *stands for "transfer").*

Quasi-Linear Utility Function

Given bids $b = (b_1, \ldots, b_n)$, agent i's utility is

$$u_i(b) = v_i x_i(b) - t_i(b)$$



The Independent Private Values Model

- Bidder i's value is a random variable V_i on $[0, \infty]$
- Bidder i knows realization v_i…
- …with only probabilistic information about others
- Each V_i is independently and identically distributed (IID) on $[0, \infty]$ …
- …according to cumulate distribution function $F: [0, \infty] \rightarrow [0,1]$, with $F(v_i) = \Pr(V_i \leq v_i)$.
- $f \equiv F'$, thus $F(v_i) = \int_{w=0}^{v_i} f(w)dw$.

In contrast to…

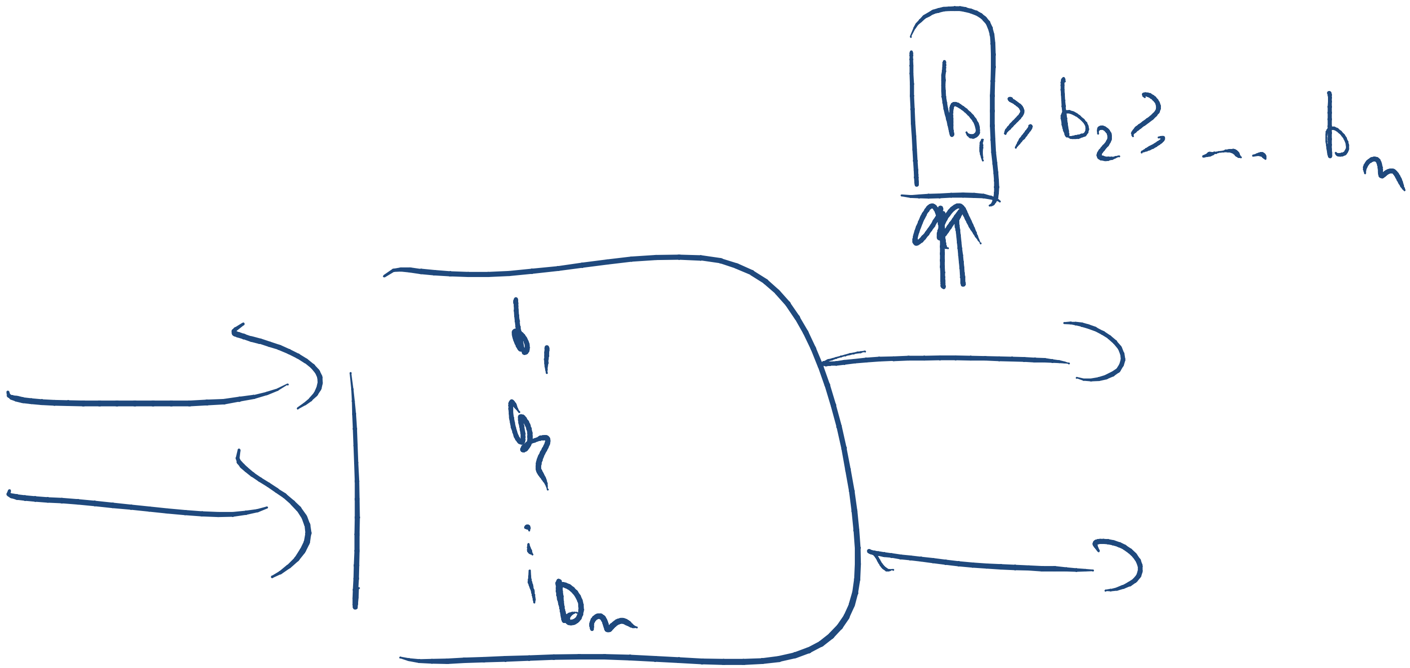
- Not independent…
  - E.g., correlated values
- Not identical
  - Every agent draws from a different distribution
  - …
- Not private values
  - Common value auctions
  - Affiliated value auctions

Questions in Auction Design

- Efficiency

- Revenue

First-Price Auction



Second-Price Auction

$$b_1 \geq b_2 \geq b_3 \geq \dots \geq b_n$$

pay

Equilibrium Analysis

- The outcome is determined by the equilibrium
- A strategy $s_i(v_i) \in [0, \infty]$ defines an agent's bid for every possible v_i.
- Strategy vector:

$$s_{-i}(v_{-i}) = (s_1(v_1), \ldots, s_{i-1}(v_{i-1}), s_{i+1}(v_{i+1}), \ldots, s_n(v_n))$$

- Bid vector:

$$b_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$$

Dominant-Strategy Equilibrium

Definition 6.4 (Dominant-strategy equilibrium). *Strategy profile* $s^* = (s_1^*, \ldots, s_n^*)$ *is a dominant-strategy equilibrium if, for all* i,

$$u_i(s_i^*(v_i), b_{-i}) \geq u_i(b_i, b_{-i}), \quad \text{for all } v_i,\ b_i \text{ and } b_{-i}. \tag{6.2}$$

Equilibrium of the 2nd-price Auction

Theorem 6.1. *Truthful bidding is a dominant-strategy equilibrium in a second price sealed-bid auction. Moreover, the auction is efficient in this equilibrium.*

Definition 6.5. *An auction is truthful if truthful bidding is a dominant strategy equilibrium.*

Bayes-Nash Equilibrium

Definition 6.6 (Bayes-Nash equilibrium). *Strategy profile* $s^* = (s_1^*, \ldots, s_n^*)$ *is a Bayes-Nash equilibrium if, for all* i,

$$\mathbb{E}[u_i(s_i^*(v_i), s_{-i}^*(V_{-i})) \mid V_i = v_i] \geq \mathbb{E}[u_i(b_i, s_{-i}^*(V_{-i})) \mid V_i = v_i], \quad \textit{for all } v_i \textit{ and } b_i. \tag{6.3}$$

Equilibrium of the 1st-Price Auction

Theorem 6.2. *A symmetric Bayes-Nash equilibrium in a first-price sealed-bid auction for agents with independently and identically distributed values uniform on* $[0, 1]$ *is*

$$s^*(v_i) = \left(\frac{n-1}{n}\right) v_i, \quad \textit{for all } i \in \{1, \ldots, n\} \tag{6.4}$$

Moreover, the auction is efficient in this equilibrium.

Relationship of Equilibria

- SPNE $\subseteq$ NE

- DSNE $\subseteq$ BNE

Revenue Equivalence

- What auction design is best for the seller?
- Revenue will be determined by the equilibrium!

→ Need equilibrium analysis

Theorem 6.3 (Revenue equivalence theorem). *For IID private values, any Bayes-Nash equilibrium of any sealed-bid auction in which (i) an agent with the highest value always wins the item, and (ii) the expected payment by an agent with the smallest possible value is zero, provides the same expected revenue to the seller.*

Order Statistics

- Given n independent draws from distribution F, order them as $V_1 \geq V_2 \geq \cdots \geq V_n$. Define the first-order statistic as V_1, the second-order statistic as V_2, and so on.
- Thus, order statistics are new random variables, and we can take their expectations, e.g., $\mathrm{E}[V_1]$
- For a *uniform* distribution on [0,1], the expected value of the k-th order statistic is

$$\frac{n+1-k}{n+1}$$

Revenue Equivalence for Uniform Distribution

- Remember, for 1st-price auction: bidding $\frac{n-1}{n} v_i$ was a symmetric BNE.
- For 2nd-price Auction: expected value of the second highest bid from remaining n-1 bids that are lower than v_i:
- $\frac{(n-1)+1-1}{(n-1)+1} v_i = \frac{n-1}{n} v_i$
- Thus, the auctioneer can expect the same revenue

Revenue-Maximizing Auctions?

- So, what if a seller wants to maximize his revenue?
- Anything he can do…?

English Auction

Definition 6.8 (English auction).

- *The auctioneer starts the auction at some low ask price*
- *At any time, any bidder can bid at or above the current ask price, whereupon he becomes the current winner and the ask price is increased to some increment above his bid*
- *The auction may close at some fixed time ("hard closing rule"), or more typically after some period when no new bids are made ("soft closing rule.")*
- *On closing, the final winner wins the item and pays his final bid price*

Clock Auction

Definition 6.9 (Clock auction). *A (ascending) clock auction is a variation on the English auction in which every agent is initially "in," the ask price increases continuously over time, with agents able to drop out at any price (but without the option to resume bidding again in the future.) The auction closes when there remains a single agent who remains "in." This agent wins and pays the price at which the last agent dropped out.*

Dutch Auction

Definition 6.10 (Dutch auction).

- *The auctioneer starts the auction at some high ask price*
- *The ask price decreases continuously over time*
- *At any time, any bidder can bid (or "buzz in") at the current ask price.*
- *This closes the auction, with the first bidder to bid winning and paying that price.*

Strategic Equivalences

$$clock \approx second-price$$

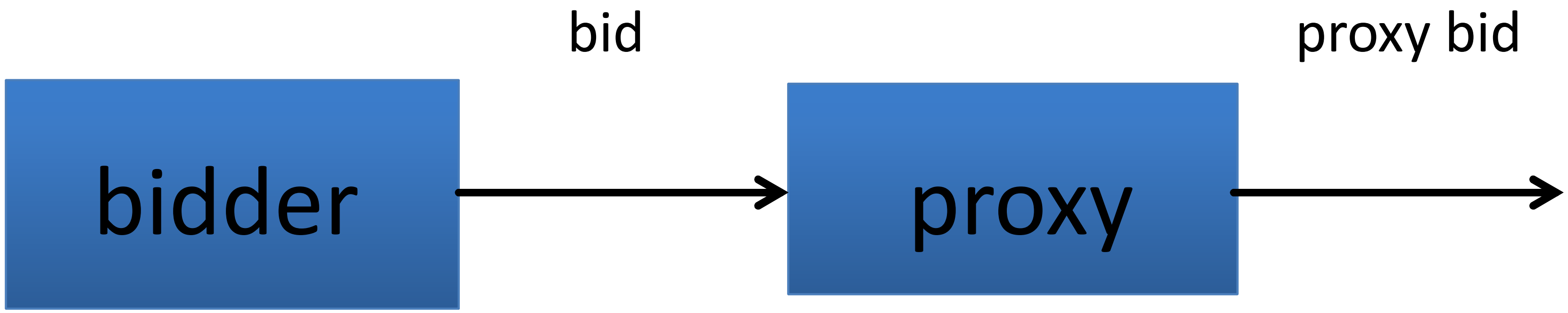
$$Dutch = first-price$$

$$English \quad ?? \quad second-price$$

The eBay Auction

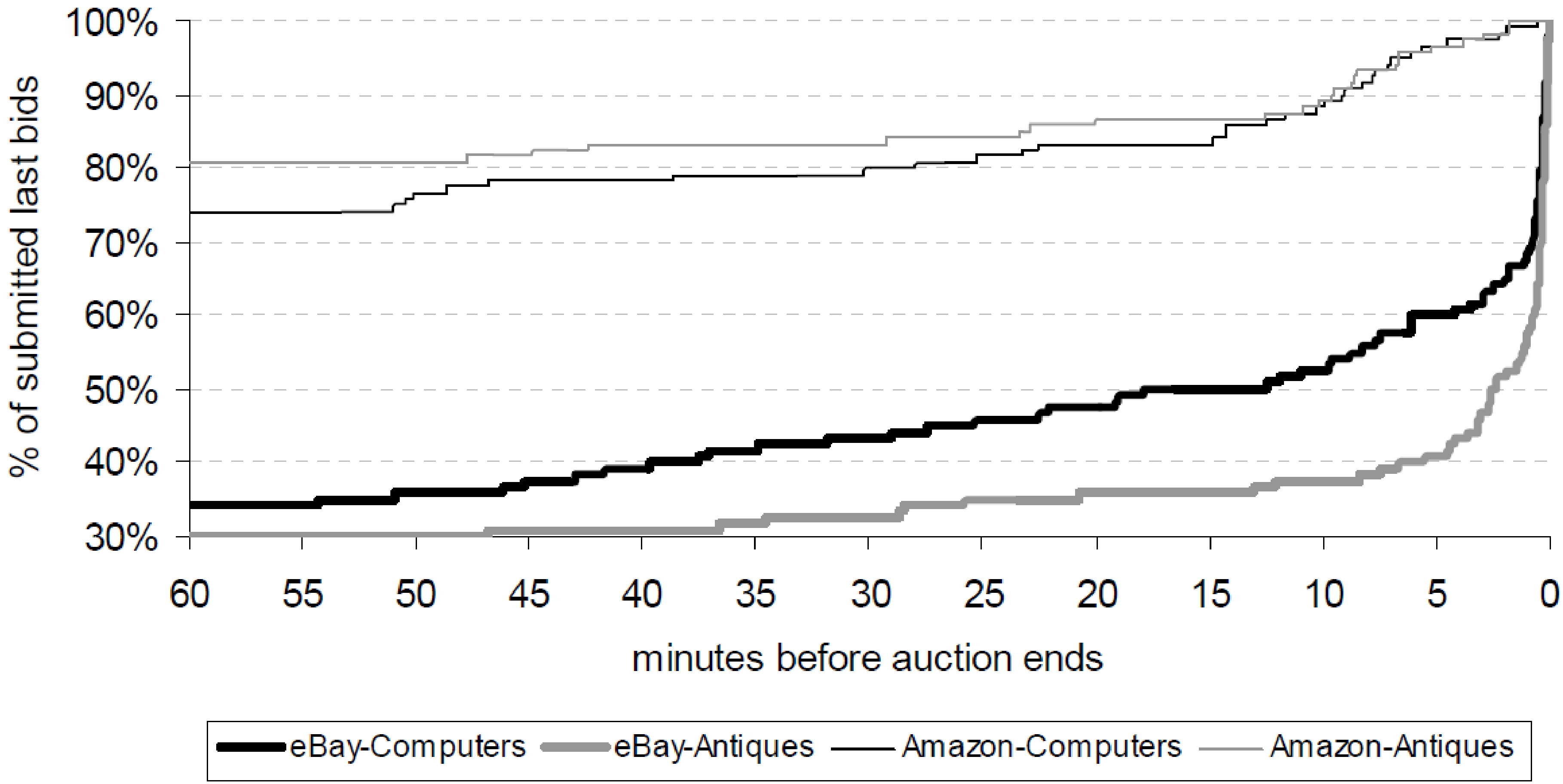
- First-price Auction?
- Second-price Auction?

Proxy Agents



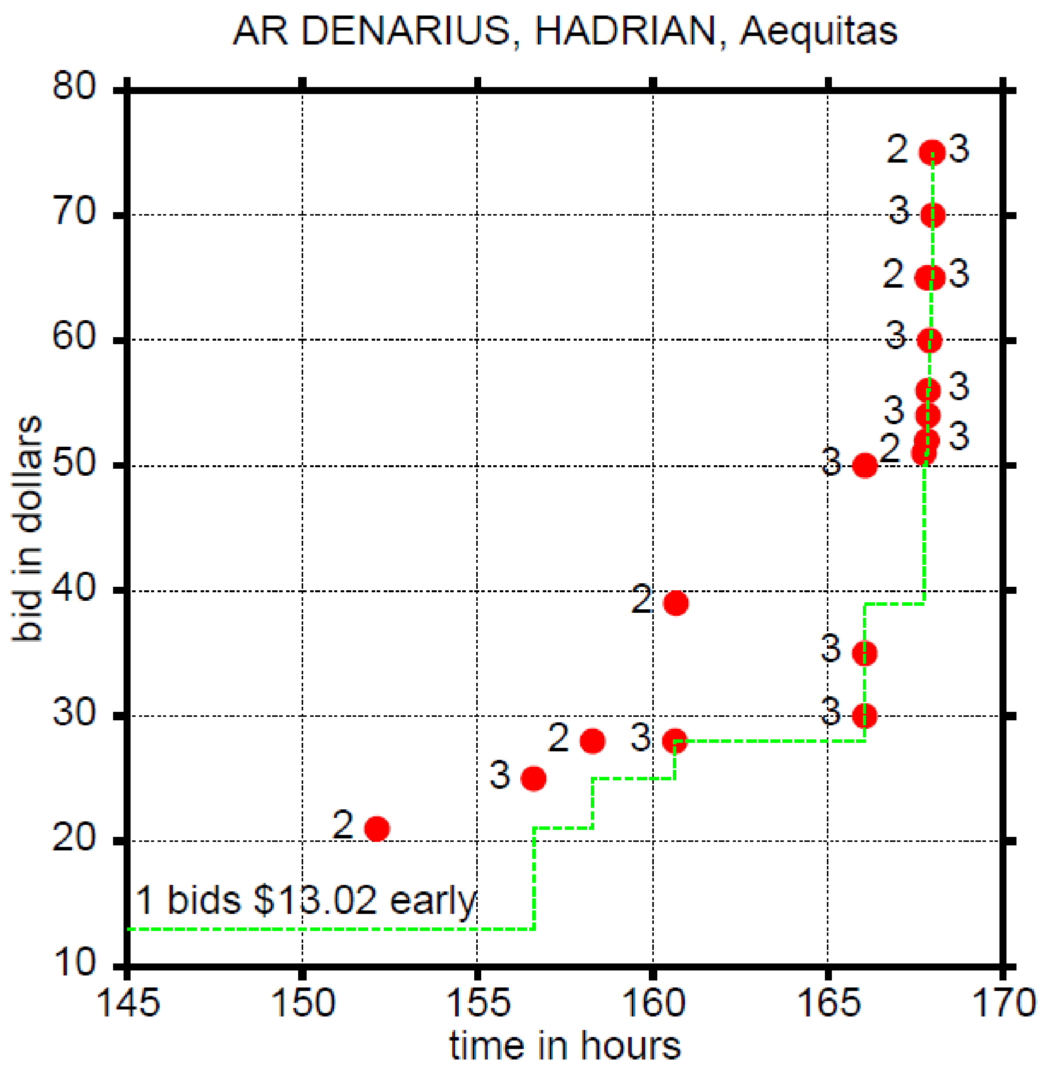
Closing Rule

- Soft vs. Hard Closing Rule
- Effect?
- Irrationality?

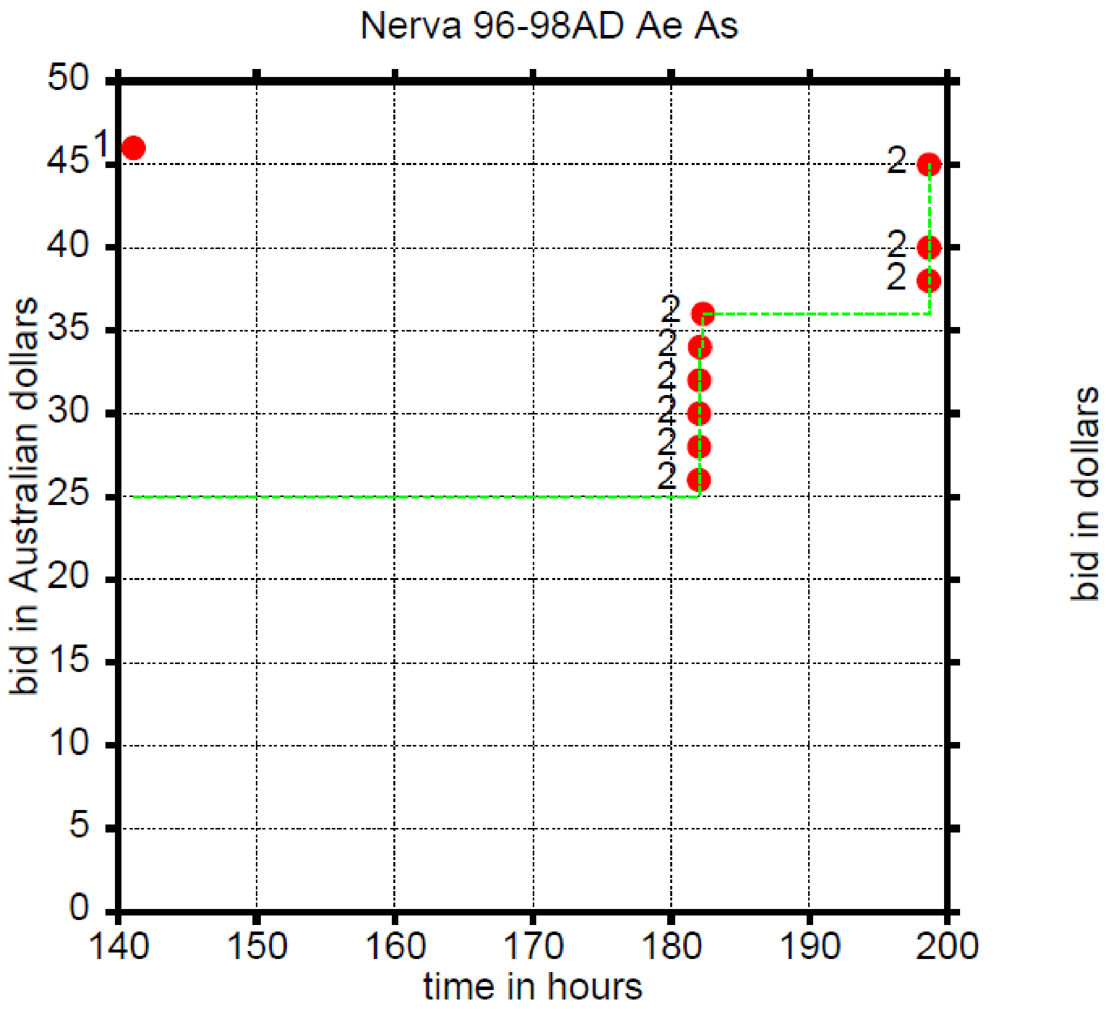


3: eBay vs Amazon auctions: Effect of closing rule. Cumulative distributions over ti

Bidding Dynamics…



: A bidding war towards the end of an auction



c: Potential downside of early bidding. (Note:

If you were the designer…

- What auction rule would you choose for eBay?

Seller Design choices

- start price
- reserve price
- block buyers
- duration, start time
- what to sell (!)
- fixed price vs. auction

Trust

- Who do we need to trust?
- Do you trust the seller? (shill bids)
- Do you trust eBay?