Lecture #5:
Algorithmic Game Theory

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Housekeeping

- Questions? Concerns?
- BitTorrent homework assignment?
- …

Outline

1. Recap of Linear Programming
2. Today’s topic: Algorithmic Game Theory
3. Discussion + Questions

Linear Programming/Optimization

- Formulation

$$\text{OBJ: } \max_{x,y} \quad x + y$$

$$\text{s.t.} \quad \begin{aligned} x &\geq 2 \\ y &\leq 5 \\ 2x + 5y &= 0 \\ x, y &\geq 0 \end{aligned}$$

Complexity

$LP \in P$

In Practice

- Simplex Algorithm
- Interior Point Algorithm
- CPLEX

Today: Algorithmic Game Theory

Why do we care?

- Agent Perspective
- Designer's Perspective
- Modeler's Perspective

Zero-Sum Games

		Player 2 H	Player 2 T
Player 1	H	1, −1	−1, 1
	T	−1, 1	1, −1

Figure 5.1: Matching Pennies Game.

Definition 5.2 (maximin strategy). *The maximin strategy of player 1 in a two-player game is*

$$\overline{s}_1 \in \arg\max_{s_1 \in \triangle(A_1)} \left[\min_{a_2 \in A_2} u_1(s_1, a_2) \right] \tag{5.1}$$

Definition 5.3 (minimax strategy). *The minimax strategy of player 1 in a two-player game is*

$$\underline{s}_1 \in \arg\min_{s_1 \in \triangle(A_1)} \left[\max_{a_2 \in A_2} u_2(s_1, a_2) \right] \tag{5.2}$$

Theorem 5.1 (Minimax Theorem). *In any two-player, zero-sum game,*

- *For each player, the set of maximin strategies is identical to the set of minimax strategies*
- *A strategy profile s^* is a mixed-strategy Nash equilibrium if and only if each player plays a maximin (or equivalently, a minimax) strategy*
- *Each player's maximin value is equal to his minimax value.*

LP for solving Zero-Sum Games

$$\textbf{LP}_1: \quad \max_{v_1,\ \{s_1(a_1)\}} \quad v_1$$

$$\text{s.t.} \quad \sum_{a_1 \in A_1} u_1(a_1, a_2)\ s_1(a_1) \geq v_1, \qquad \forall a_2 \in A_2 \tag{5.7}$$

$$\sum_{a_1 \in A_1} s_1(a_1) = 1 \tag{5.8}$$

$$s_1(a_1) \geq 0, \qquad \forall a_1 \in A_1 \tag{5.9}$$

Example: Odd-or-Even Game

		Player 2 1D	Player 2 2D
Player 1	1D	−2, 2	3, −3
	2D	3, −3	−4, 4

Figure 5.2: Odd-or-Even Game.

$$\max_{v_1,\ s_1(1D),\ s_1(2D)} \quad v_1$$

$$\text{s.t.} \quad -2s_1(1D) + 3s_1(2D) \geq v_1$$

$$3s_1(1D) - 4s_1(2D) \geq v_1$$

$$s_1(1D) + s_1(2D) = 1$$

$$s_1(1D),\ s_1(2D) \geq 0$$

$$\max_{v_1,\ p} \quad v_1$$

$$\text{s.t.} \quad -2p + 3(1-p) \geq v_1$$

$$3p - 4(1-p) \geq v_1$$

$$p \geq 0$$

$$\max_p \min\left(-2p + 3(1-p),\ 3p - 4(1-p)\right)$$

By setting $-2p + 3(1-p) = 3p - 4(1-p)$, *this yields optimal solution* $p = 7/12$, *and then* $v = 1/12$.

Rock Paper Scissors

Player 2

Player 1	R 1/3	P 1/3	S 1/3
1/3 R	0, 0	-1, 1	1, -1
1/3 P	1, -1	0, 0	-1, 1
1/3 S	-1, 1	1, -1	0, 0

LP for solving Rock-Paper-Scissors

Solving the Game Using AMPL

Complexity of Solving Zero-Sum Games

- Number of variables?
- Number of constraints?

- Theorem: LPs can be solved in worst-case polynomial time in the number of actions of the game!

$s^* = (s_1^*, s_2^*)$

Two-Player General Sum-Games?

- Pure-Strategy NE? $O(n^2) \cdot O(n) = O(n^3)$
- Mixed-Strategy NE?
  - Not using the same LP as for zero-sum games!
  - Consider simpler problem: find a NE, given a particular support. $2(n-1)$
- Thus, what is to do? $s^* = s_1(a_1), s_1(a_2) \ldots$
- Use LFP to find mixed Nash $s_1(a_i) > 0 \Rightarrow a_i \in S(A)$

LFP for 2-Player General Sum Game

$$\mathbf{LFP}_1: \quad \sum_{a_1 \in A_1} u_2(a_1, a_2)\ s_1(a_1) = v_2, \qquad \forall a_2 \in support_2 \tag{5.10}$$

$$\sum_{a_1 \in A_1} u_2(a_1, a_2)\ s_1(a_1) \leq v_2, \qquad \forall a_2 \notin support_2 \tag{5.11}$$

$$\sum_{a_1 \in A_1} s_1(a_1) = 1 \tag{5.12}$$

$$s_1(a_1) \geq 0, \qquad \forall a_1 \in support_1 \tag{5.13}$$

$$s_1(a_1) = 0, \qquad \forall a_1 \notin support_1 \tag{5.14}$$

$$\mathbf{LFP}_2: \quad \sum_{a_2 \in A_2} u_1(a_1, a_2)\ s_2(a_2) = v_1, \qquad \forall a_1 \in support_1 \tag{5.15}$$

$$\sum_{a_2 \in A_2} u_1(a_1, a_2)\ s_2(a_2) \leq v_1, \qquad \forall a_1 \notin support_1 \tag{5.16}$$

$$\sum_{a_2 \in A_2} s_2(a_2) = 1 \tag{5.17}$$

$$s_2(a_2) \geq 0, \qquad \forall a_2 \in support_2 \tag{5.18}$$

$$s_2(a_2) = 0, \qquad \forall a_2 \notin support_2 \tag{5.19}$$

Support Enumeration Method

- SEM: Enumerate all possible supports

m Action

$\rightarrow |2^m - 1|$ = # supports for one player

- Number of possible supports, if each player has m actions?

$(2^m - 1)(2^m - 1) = O(2^{2m})$

- Worst case running time?

N-Player General Sum Games

- Pure-Strategy Nash Equilibria?

- Mixed-Strategy NE?
  → Generalized SEM Algorithm

Generalized SEM Algorithm

- Enumerate all possible supports, now for n players
- Find mixed strategy, such that each player is indifferent across his actions in his support
- For example: 3-player game…

$$\sum_{a_1 \in A_1} \sum_{a_2 \in A_2} u_3(a_1, a_2, a_3) \cdot s_1(a_1) \cdot s_2(a_2) = v_3, \qquad \forall a_3 \in support_3$$

Complexity of the Generalized SEM

- Non-linear optimization
  → cannot be solved in worst-case polynomial time
- Still effective in practice

- In any case:
  SEM algorithm always has worst-case exponential time complexity!

→ Can we do better?

Complexity of Find-Nash?

- Complexity class? P? NP?
- Every finite games has a mixed-NE!

→PPAD!

Includes:

1. Find-Nash
2. Fixed-Point Problems

Main Complexity Result

- Find-Nash for a general-sum normal-form game is PPAD-Complete, even for two players.
- Conjecture: P $\neq$PPAD
- Implications?

Correlated Equilibrium

- Game of Chicken

		Player 2: Y	Player 2: S
Player 1	Y	0, 0	0, 2
	S	2, 0	−4, −4

Figure 5.10: Game of Chicken.

- Equilibria and resulting distributions:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 4/9 & 2/9 \\ 2/9 & 1/9 \end{pmatrix}$$

Now, coordinate!

- Coordinate actions:
- Use a shared signal!

$$\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

play in a correlated equilibrium, we require, for all actions $a_i' \in A_i$:

$$\sum_{a_{-i} \in A_{-i}} p(a_{-i} \mid a_i) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} p(a_{-i} \mid a_i) u_i(a_i', a_{-i})$$

$$p(a_i, a_{-i}) = p(a_i) p(a_{-i} | a_i)$$

Definition

Definition 5.7 (Correlated equilibrium). *A joint probability distribution p^* on action profiles A such that $p^*(a) \geq 0$ and $\sum_{a \in A} p^*(a) = 1$ is a correlated equilibrium in normal-form game (N, A, u) if, for all i,*

$$\sum_{a_{-i} \in A_{-i}} p^*(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} p^*(a_i, a_{-i}) u_i(a_i', a_{-i}), \quad \textit{for all } a_i, a_i' \in A_i \tag{5.22}$$

Finding a Correlated Eq using LFPs

Theorem 5.4. *A correlated equilibrium of an n-player, general-sum normal-form game can be computed in polynomial time in the size of the payoff matrix.*

For this, we define an LFP to identify a correlated equilibrium. In fact, the definition can be encoded rather directly:

$$\textbf{LFP}_{\text{ce}} \qquad \sum_{a_{-i} \in A_{-i}} u_i(a)\; p(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} u_i(a'_i, a_{-i})\; p(a_i, a_{-i}), \forall i \in N,\ \forall a_i \in A_i,\ \forall a'_i \in A_i \quad (5.23)$$

$$\sum_{a \in A} p(a) = 1 \quad (5.24)$$

$$p(a) \geq 0, \qquad \forall a \in A \quad (5.25)$$