# Computation and Economics - Spring 2012
# Homework Assignment #04: Algorithmic Game Theory and Auction Theory

Professor Sven Seuken
Department of Informatics, University of Zurich
Out Monday, March 19, 2012
Due **14:00** sharp: **Monday, March 26, 2012** (in class)
We strongly encourage typed submissions.

[**BSc 100 Points, MSc 120 Points**] This is a single-person assignment. Points will be awarded for clarity, correctness and completeness of the answers. You are free to discuss the problem set with other students. However, you should not share answers. **Copying will be penalized**.

1. [**15 Points**] Algorithmic Game Theory: Minimax Theorem

   After having arrested all members of Prof. Moriarty's evil criminal gang in London, Sherlock Holmes suddenly fears Moriarty's deadly revenge and boards the train from London to Dover in an effort to reach the continent and so to escape from his antipode. Moriarty can take an express train and catch Holmes at Dover. However, there is an intermediate station at Canterbury at which Holmes may get off the train to avoid such a disaster. But of course, Moriarty is aware of this too and may himself stop instead at Canterbury. The utilities of each outcome are given by the following payoff matrix:

| | | Moriarty | |
|---|---|---|---|
| | | C.bury | Dover |
| Holmes | C.bury | $-10,\ 10$ | $0,\ 0$ |
| | Dover | $5,\ -5$ | $-10,\ 10$ |

   (a) [**5 Points**] Find the minimax strategy for Sherlock Holmes and Prof. Moriarty in the above game using the method described on page 3 of chapter 5 of the lecture notes.
   (b) [**8 Points**] Find the minimax strategies by solving the corresponding linear programs.
   (c) [**2 Points**] Find a Nash equilibrium of the above game and explain how you found it.

2. [**20 Points**] Algorithmic Game Theory: Computational Complexity

   As discussed in the lecture, the problem of finding a Nash equilibrium is trivially in NP because every normal form game is guaranteed to have a mixed-strategy Nash equilibrium. Now consider the following decision problem: given a normal form game G, does it have at least two Nash equilibria?

   Show that this new decision problem is in NP in a non-trivial sense. By non-trivial we mean that there are problem instances for which the answer to the decision problem is "no".

*Hint*: first give an example of a game for which the answer is "no", then construct a polynomial-time verifier for a "yes" instance. A polynomial-time verifier is an algorithm that takes as input a witness (or "proof") that a given game has at least two Nash equilibria and then decides in worst-case polynomial time if the game has indeed two Nash equilibria. So for the verifier you need to provide three things: the input format, the algorithm itself and a run-time analysis of the algorithm.

3. [**BSc 13, MSc 18 Points**] Algorithmic Game Theory: Correlated Equilibrium

   For this exercise, we need the following definition: a *convex combination* of two discrete probability distributions $\{p_1, p_2, ..., p_n\}$ and $\{q_1, q_2, \dots q_n\}$ has the form

$$s_i = \lambda p_i + (1 - \lambda) q_i, \quad i = 1, \dots n,$$

   where $\lambda$ is a real number in the interval $[0, 1]$. The definition can be extended to the convex combination of an arbitrary number of probability distributions

   (a) [**10 Points**] Prove that any convex combination of two correlated equilibria is again a correlated equilibrium.

   (b) [**3 Points**] Prove that any convex combination of two Nash equilibria is a correlated equilibrium (*Hint*: use (a)).

   (c) [**MSc 5 Points**] True or false: Any convex combination of two Nash equilibria is again a Nash equilibrium. Either give a proof of this statement or provide a counter-example.

4. [**22 Points**] Auction Theory: All-pay Auction

   Consider the following sealed-bid auction (which is called all-pay auction in the literature):

   - Allocation rule: the highest bidder wins the item
   - Payment rule: *every* agents pays its bid, regardless of winning or losing

   From the revenue equivalence theorem, we can identify the following candidate equilibrium for this auction (assuming independent private values $v_i$ uniformely distributed in [0,1], see lecture notes, chapter 6.4.1):

$$b_i^{ap} = \frac{(n-1)}{n} v_i^n, \quad i = 1, 2, \dots n$$

   For the special case $n = 2$, verify that this is indeed a Bayes-Nash equilibrium for the all-pay auction

   *Hint*: follow the same method as used in the proof of Theorem 6.2 in the lecture notes, i.e. to check whether this is an equilibrium, assume that player 2 plays $b_2^{ap}$ and then maximize the expected revenue of player 1 w.r.t. $b_1$.

5. [**8 Points**] Auction Theory: English Auction

   Consider an open-out cry ascending-price auction that proceeds as follows: an ask price is maintained, initialized to \$1 (the currency is divided into dollars and cents). In each round, one or more agents can “shout” a new bid price (at least the current ask price) to become the current winner. The ask price is then set to $1 greater than this bid price. Ties are broken at random. The auction closes when there are no new bids, with the final winner paying its bid price.

The *“myopic best-response” strategy* is to bid at the new ask price while (i) losing, and (ii) the ask price is less than the agent’s value.

Now consider such an auction with two bidders, 1 and 2, with values $15 and $10, respectively. Construct an example of a strategy profile that shows that myopic best-response is *not a dominant strategy* for bidder 1.

6. [**10 Points**] Auction Theory: eBay

   (a) [**3 Points**] Provide two reasons for why a single eBay auction is not strategically equivalent to a second-price sealed-bid auction (be concise, 1-2 sentences per reason is fine).

   (b) [**7 Points**] Suppose you want to buy one LCD monitor and face two second-price sealed-bid auctions, one on Monday and one on Tuesday, each selling exactly the same type of monitor. Assume that the set of bidders is fixed from the beginning (i.e. there are no new bidders on Tuesday) and that everyone wants to buy only *one* LCD monitor.

   Construct an example to explain why truthful bidding on both auctions is not your dominant strategy.

7. [**12 Points**] Sponsored Search Auctions: GSP Slot Auction

   Consider the following GSP slot auction setting. There are four bidders with per-click values $w_1 = 15$, $w_2 = 12$, $w_3 = 5$ and $w_4 = 1$ and three slots with slot effects are $s_1 = 0.5$, $s_2 = 0.3$ and $s_3 = 0.2$.

   (a) [**3 Points**] Show that in this specific auction, truthful bidding is not a dominant strategy for all agents.

   (b) [**9 Points**] Find the buyer-optimal, envy-free equilibrium for this auction.

8. [**MSc 15 Points**] Sponsored Search Auctions: Envy-free equilibrium

   Prove Theorem 7.4. of the lecture notes:

   *An envy-free outcome in a slot auction is efficient (i.e. allocates slots to maximize total value).*

   *Hint*: Consider any two agents $i$ and $j$ with $w_i > w_j$. First, show that envy-freeness implies that $s_i \geq s_j$, i.e. $i$ must get a higher slot than $j$. Conclude from this that the slot allocation must be efficient.