Cellular Automata - Dynamical Systems

November 1, 2013

Rolf Pfeifer

Rudolf M. Füchslin

Zurich University of Applied Sciences

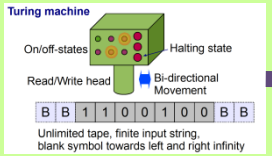
School of Engineering

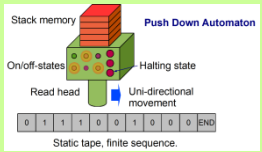
ZAMP Center for Applied Mathematics and Physics

Cellular Automata

The Quest for Infinity

Discrete systems

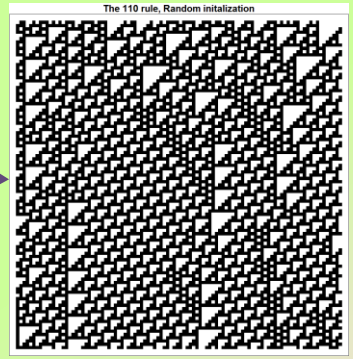


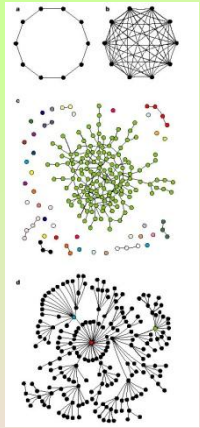


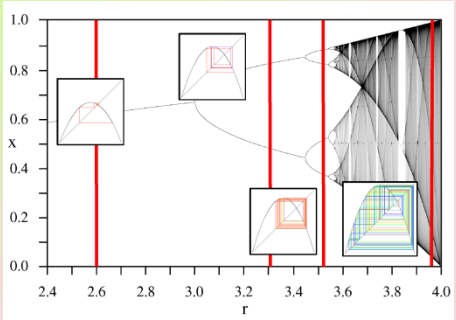
EQUIVALENCE!

Infinity via memory or tape

Infinity by space and number of automata

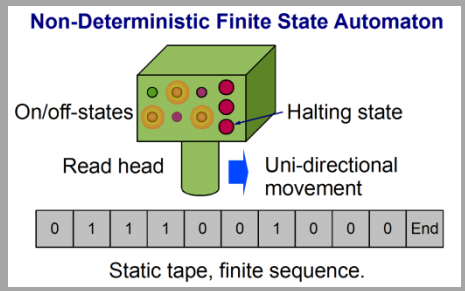


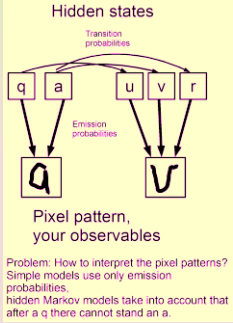


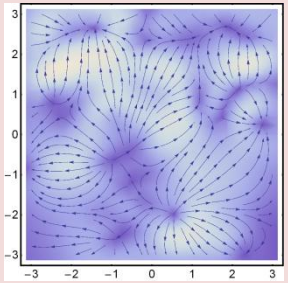


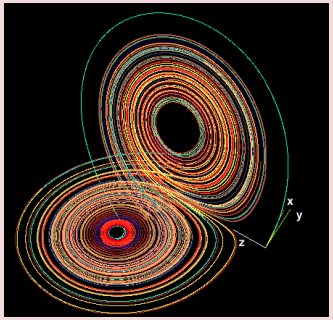
Continous parameter(s)

Infinity by probability

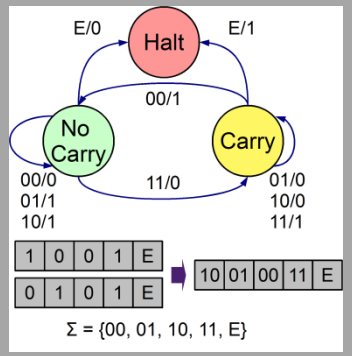




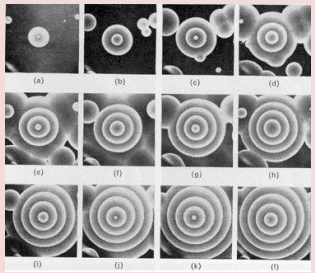




Finite systems



Continuous systems



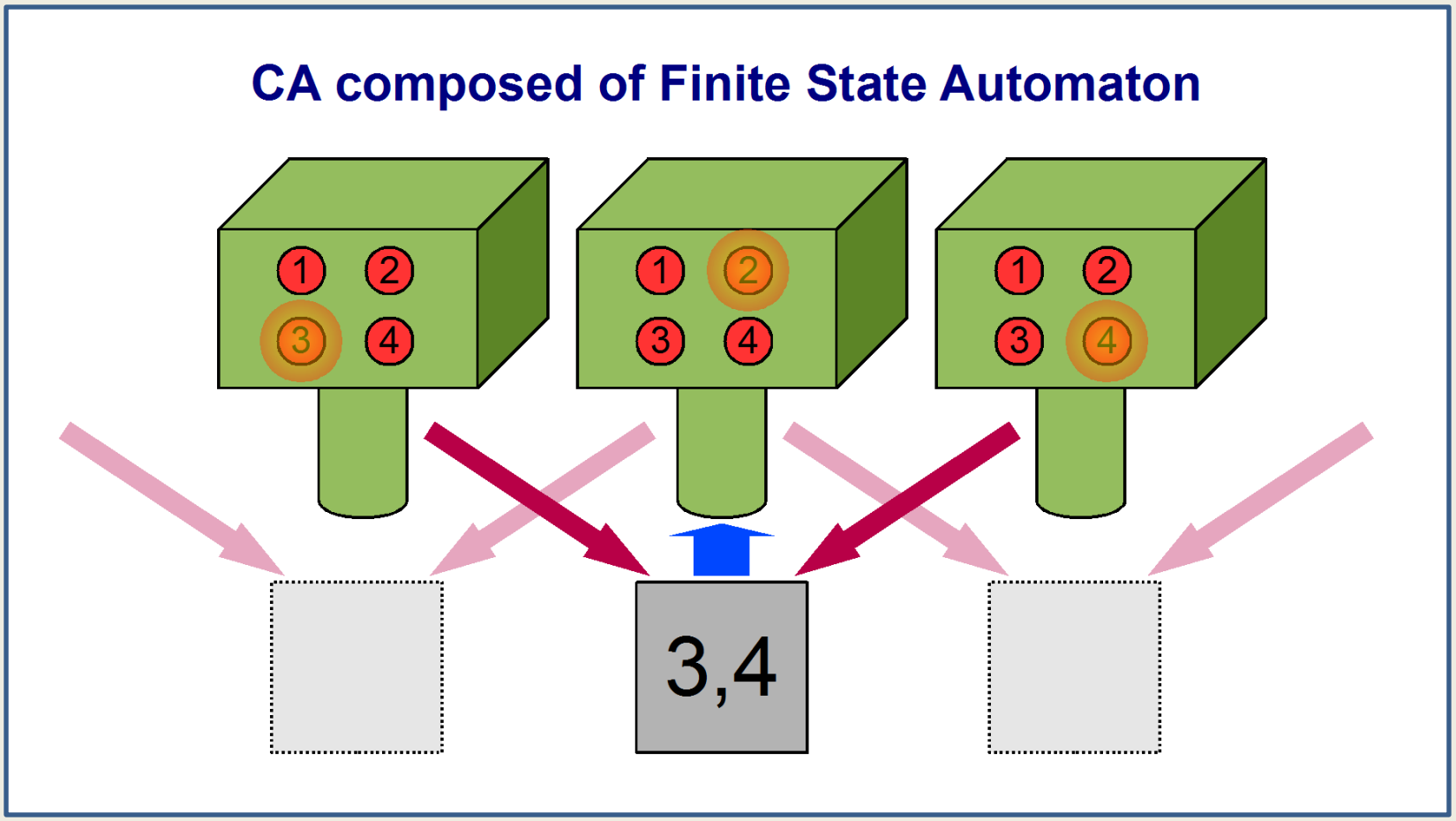
Infinity by spatial and temporal continuity.

Cellular Automata in General

What are Cellular Automata?

- CA can be regarded as a **potentially unlimited collection of finite state automata.**
- Some types of CA are based on finite state automata, but not all.
- The cells are arranged on a lattice → With CA, we get a **notion of space**.

Cellular Automata - Visualized



Why Cellular Automata?

- **Virtual World perspective**: **Virtual universe with much simpler laws than those we observe in our world**. Nevertheless, they are capable of exhibiting complex collaborative phenomena and give a simple notion of space.
- **Support perspective**: Some types of CA are able of **simulating specific aspects of real physico-chemical systems**.

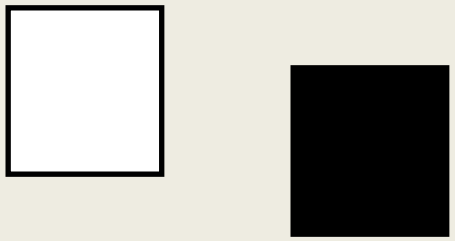
Components of a FSA-CA

- Automata are FSA
  - States
  - Transition rules
  - Initial conditions
- Lattice
  - Geometry
  - Neighborhood
  - Boundary Conditions

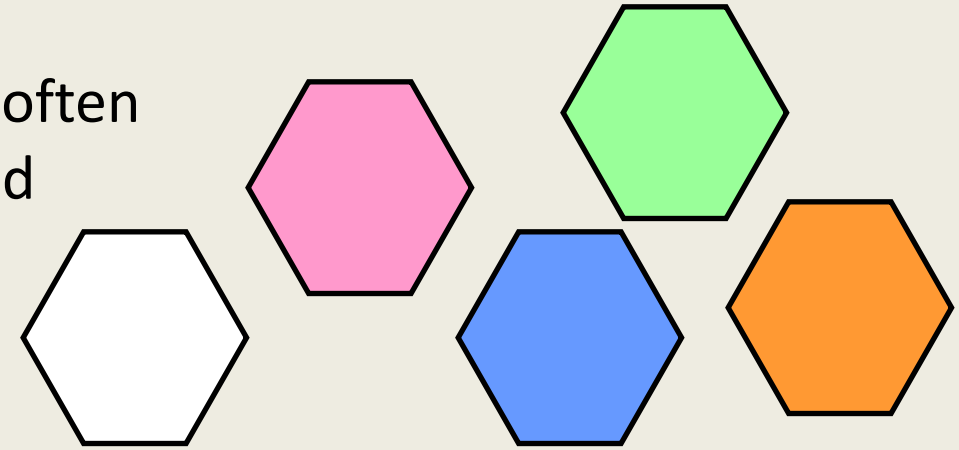
The quite many ingredients allow a broad variation of types of CA.

Cell and States

- A cell is the basic element of Cellular Automata
- A cell is a kind of memory element and stores a state like an FSA
- Cell *i* at time *t* is in one of a finite set Σ of *k* states
  - often binary, $\Sigma = \{0,1\}$
  - 0 is most often regarded as the "quiescent" state.



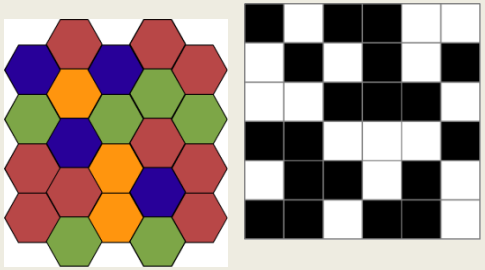
States are often color coded

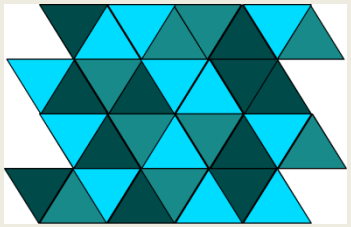


Lattice: Structure

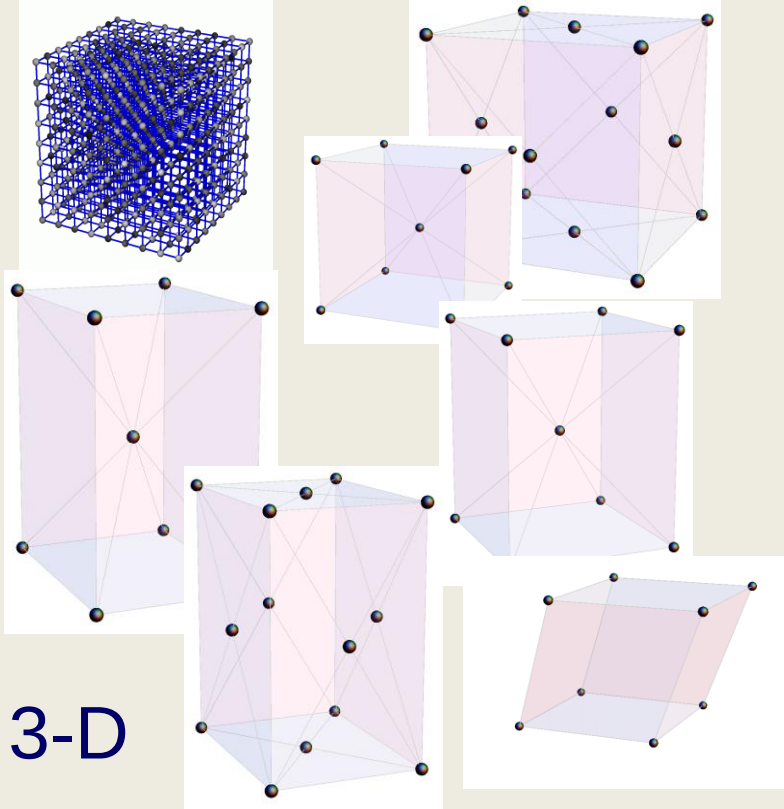
- A lattice is an (1-D, 2-D, 3-D) array in which all the cells are arranged
- A lattice is regular: Structure is the same everywhere.

1-D





2-D

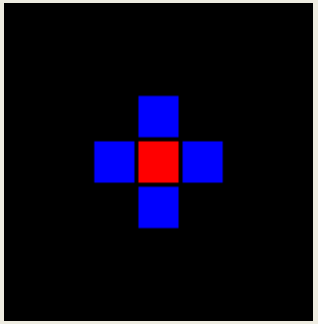


3-D

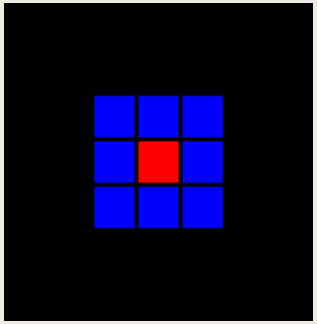
Lattice: Neighborhood

- On square lattices, two definitions of neighborhood are common.
- More exotic definitions may occur.

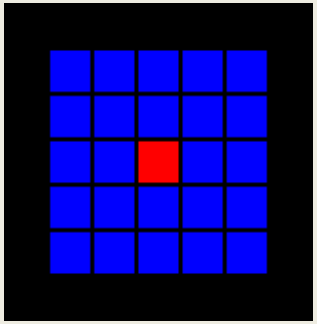
Von Neumann neighborhood



Moore - neighborhood



Extended Moore - neighborhood

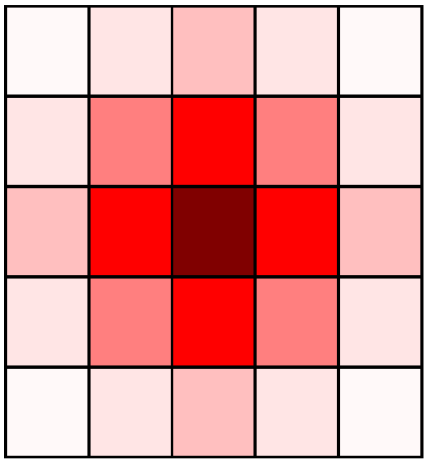


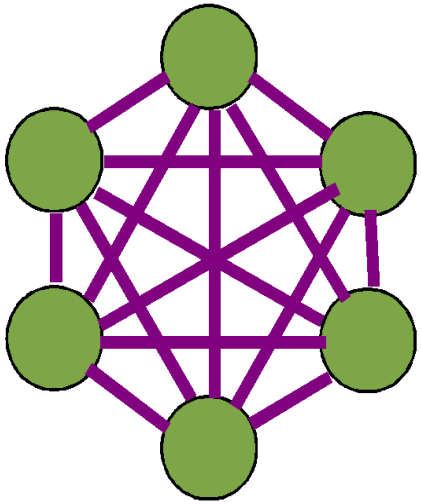
Lattice: Space, Distance, Neighborhood

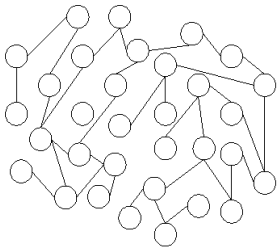
- Space enables **heterogeneity**. Here it looks differently than there.
- Space defines **neighborhoods**. This is closer than that. Space gives you a concept of distance.

PS: Space gives you also a concept of "angle" but we don't care.

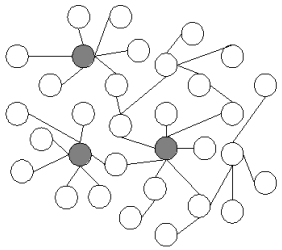
Lattice: Space, Distance, Neighborhood







(a) Random network



(b) Scale-free network

Space as we know it: Heterogeneity and distance

Simplex space: Heterogeneity but NO distance (orr only one distance, without the possibility to be closer or farer.)

Graphs and networks: Another concept of distance.

Lattice: Boundary Conditions

- Infinite/adaptive grid
  - The grid grows as the pattern propagates
- Finite grid
  - Hard boundary (edge cells have a fixed state, usually zero)
  - Soft boundary (periodic boundary conditions)
    - Edge wraps around
    - 1D is a ring
    - 2D is torus
    - Weirder topologies with a twist: Moebius bands, Klein bottles

Boundary Conditions

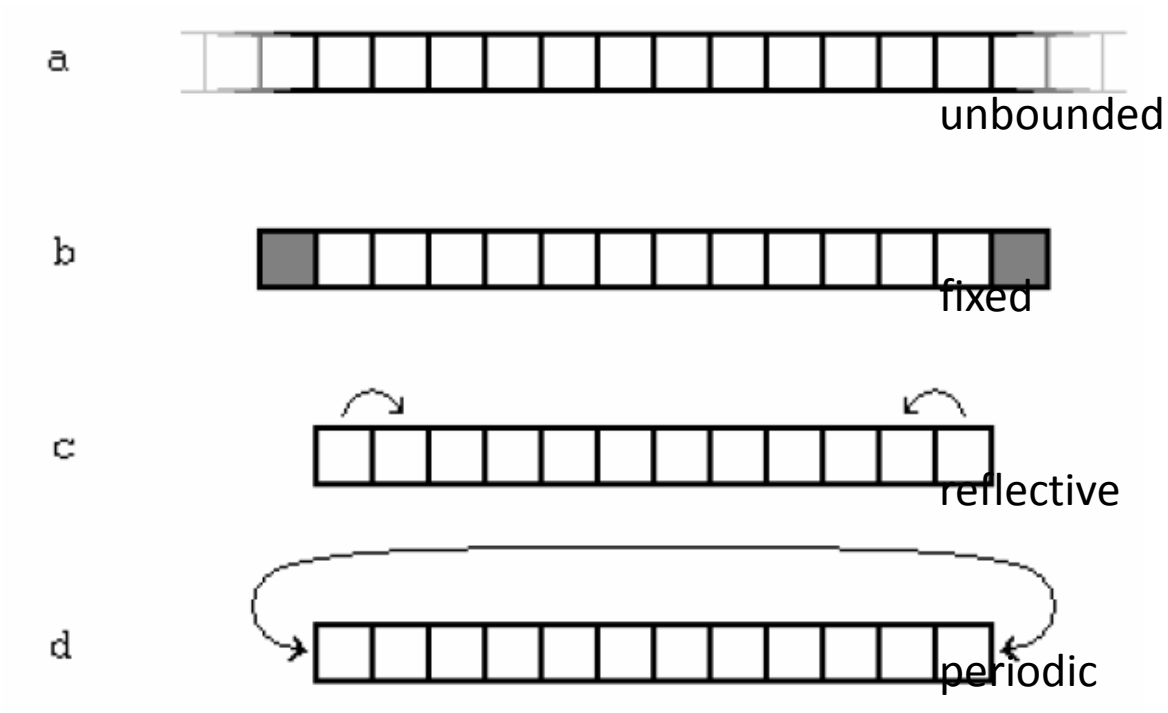


Figure 2.3: *Four possibilities for boundary conditions in a 1-dimensional CA. a) infinite (unbounded) array of cells. b) finite array of cells with fixed boundaries. The end points have cells in their neighborhood with a fixed value. c) finite array of cells with reflective boundary. The leftmost cell can only diffuse to the right. d): finite array of cells closed to a circle, periodic boundary. The leftmost cell becomes a neighbor of the rightmost cell.*

1-D Lattice

From (Wuensche and Miller, 1992)

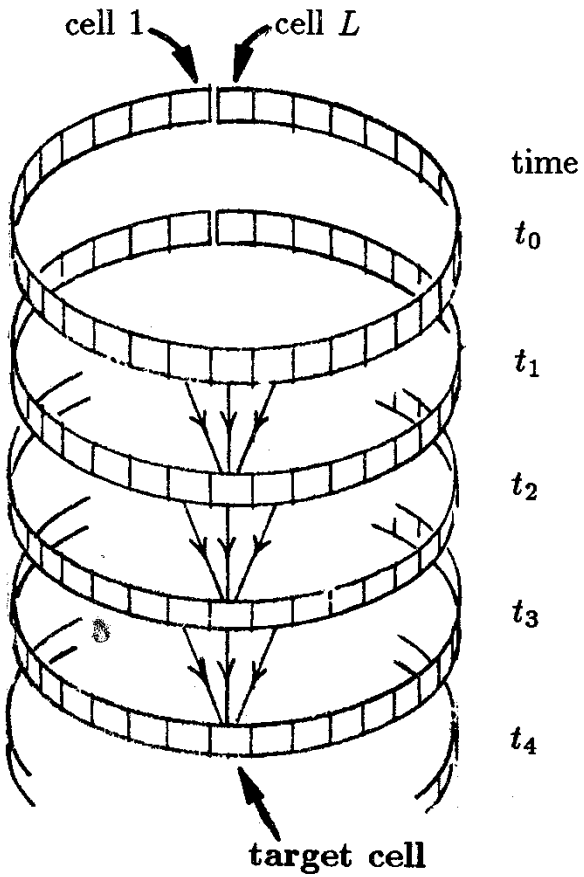
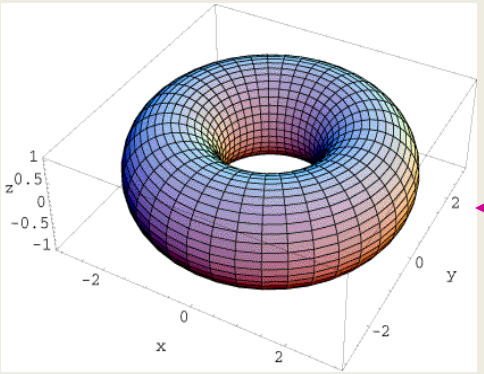
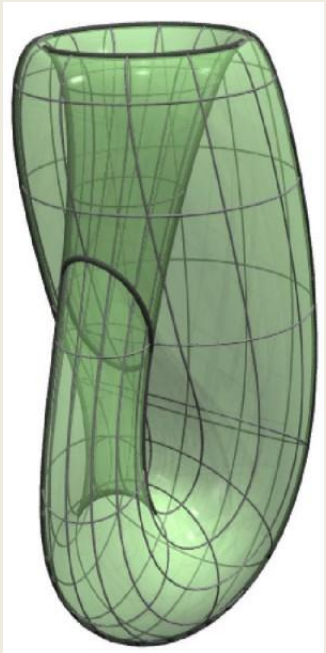
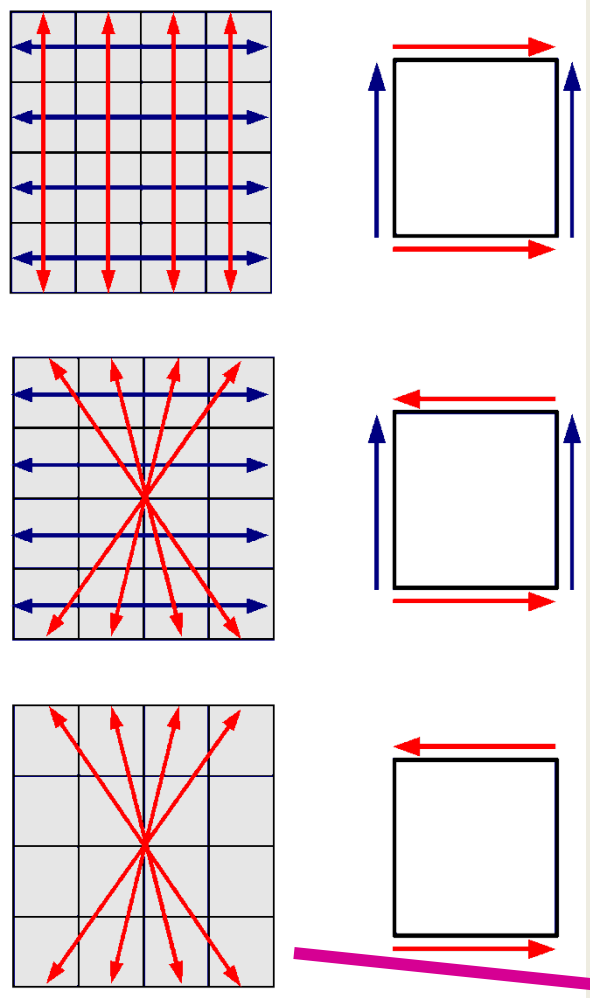


FIGURE 2.1 1-D, local binary CA with periodic boundary conditions, neighbourhood 3 (elementary rules), array length L.

2-D: Different Periodic Boundary Conditions



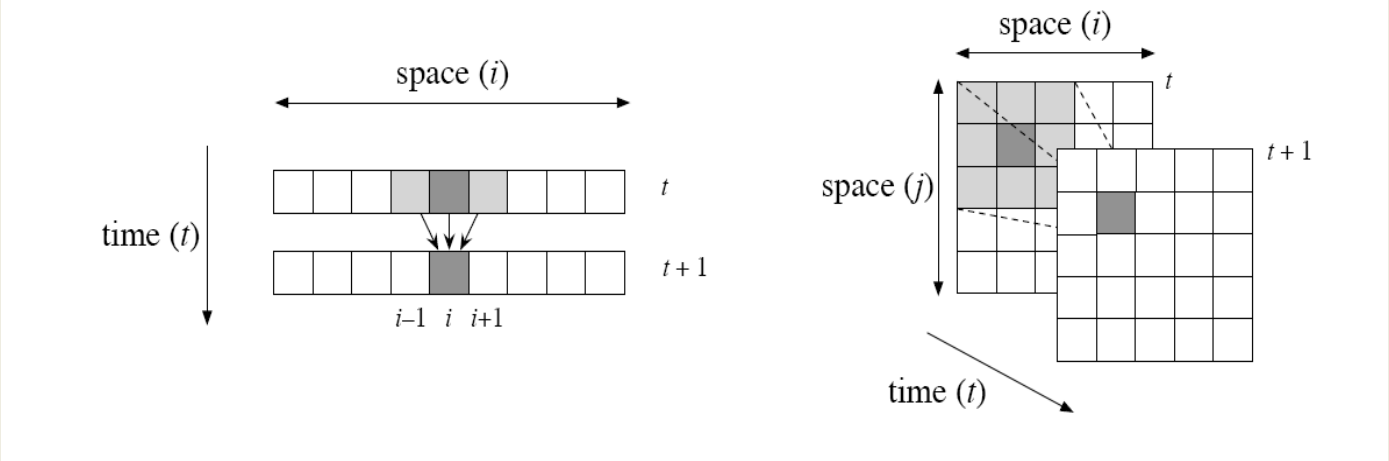




In higher dimensions, one has a multitude of possibilities of gluing together boundaries.

FSA-CA Transition Rules

- Transition rules determine the next states of the cells as a function of
  - Cell state
  - Cell states in neighborhood
- Typically rules are uniform, i.e. they are the same everywhere.



Transition Rules: 1-D

- k: Number of states
- r: range
- For k = 2 and r = 1 (n = 3) 1-D CA there are:
  - 8 possible neighbor-configuration: k^(2r+1)
  - 256 different rules: k^(k^(2r+1))

→ Exhaustive enumeration is possible

Representation: Rule Table

$a_{i-1}(t)$	$a_i(t)$	$a_{i+1}(t)$	$a_i(t+1)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

$a_i(t)$: state of cell i at time t

Representation: Rule Table

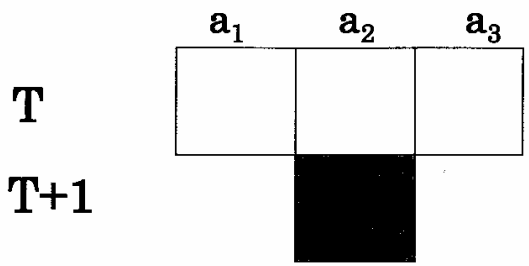
$a_{i-1}(t)$	$a_i(t)$	$a_{i+1}(t)$	$a_i(t+1)$
0	0	0	0
		1	1
		0	0
		1	0
		0	1
1	0	1	1
1	1	0	0
1	1	1	0

The rules are named according using the output: eight bits can be read as an integer, using as least significant bit the output of 000. The rule given by this table is then named **rule 50**

$a_i(t)$: state of cell i at time t

Transition Rule Table: Representation 2

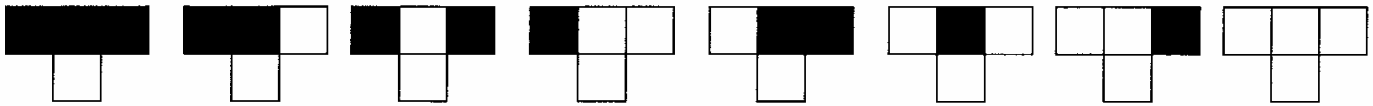
Elementary Cellular Automata



k = 2, r = 1

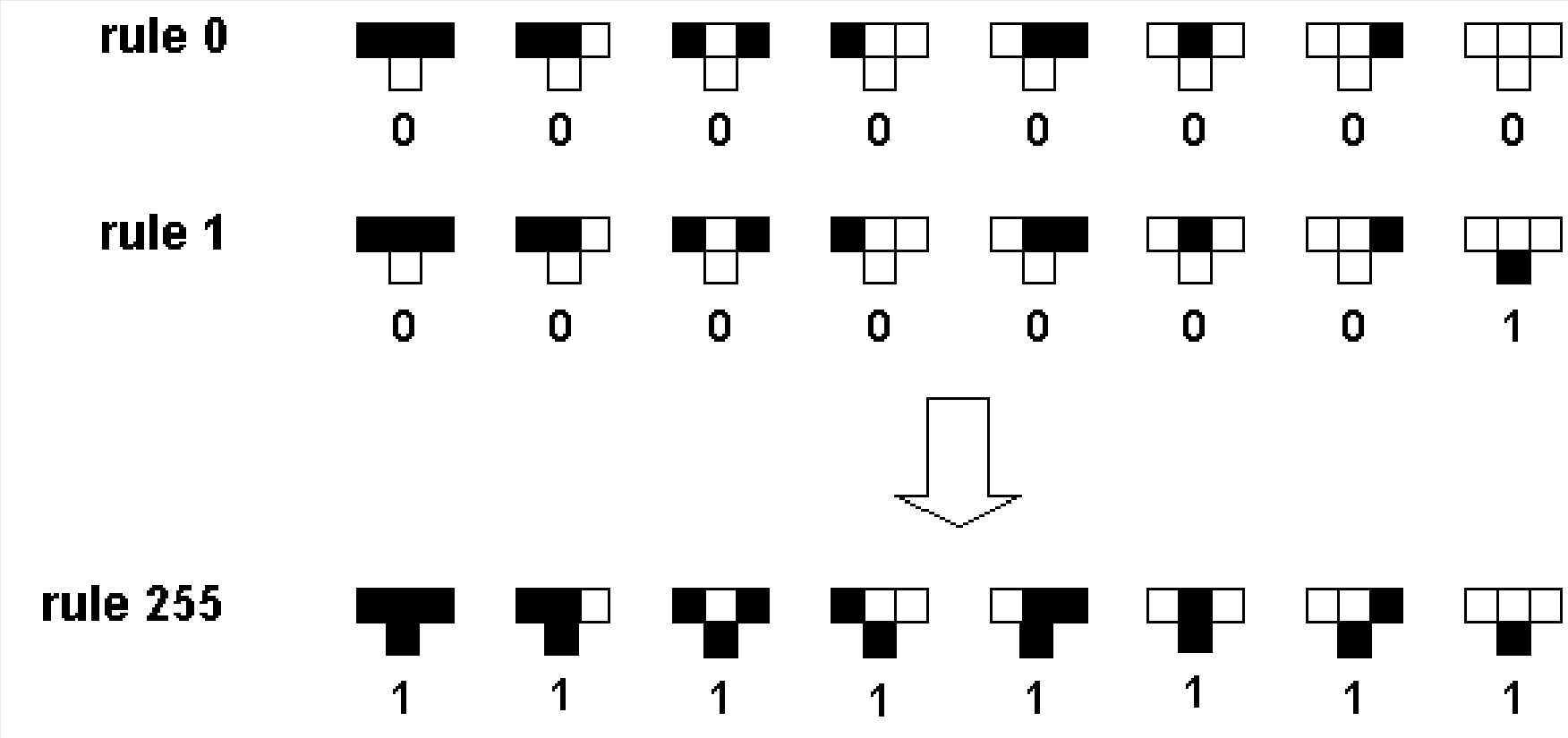
k^(2 r + 1) = 8

k^(k^(2 r + 1)) = 256



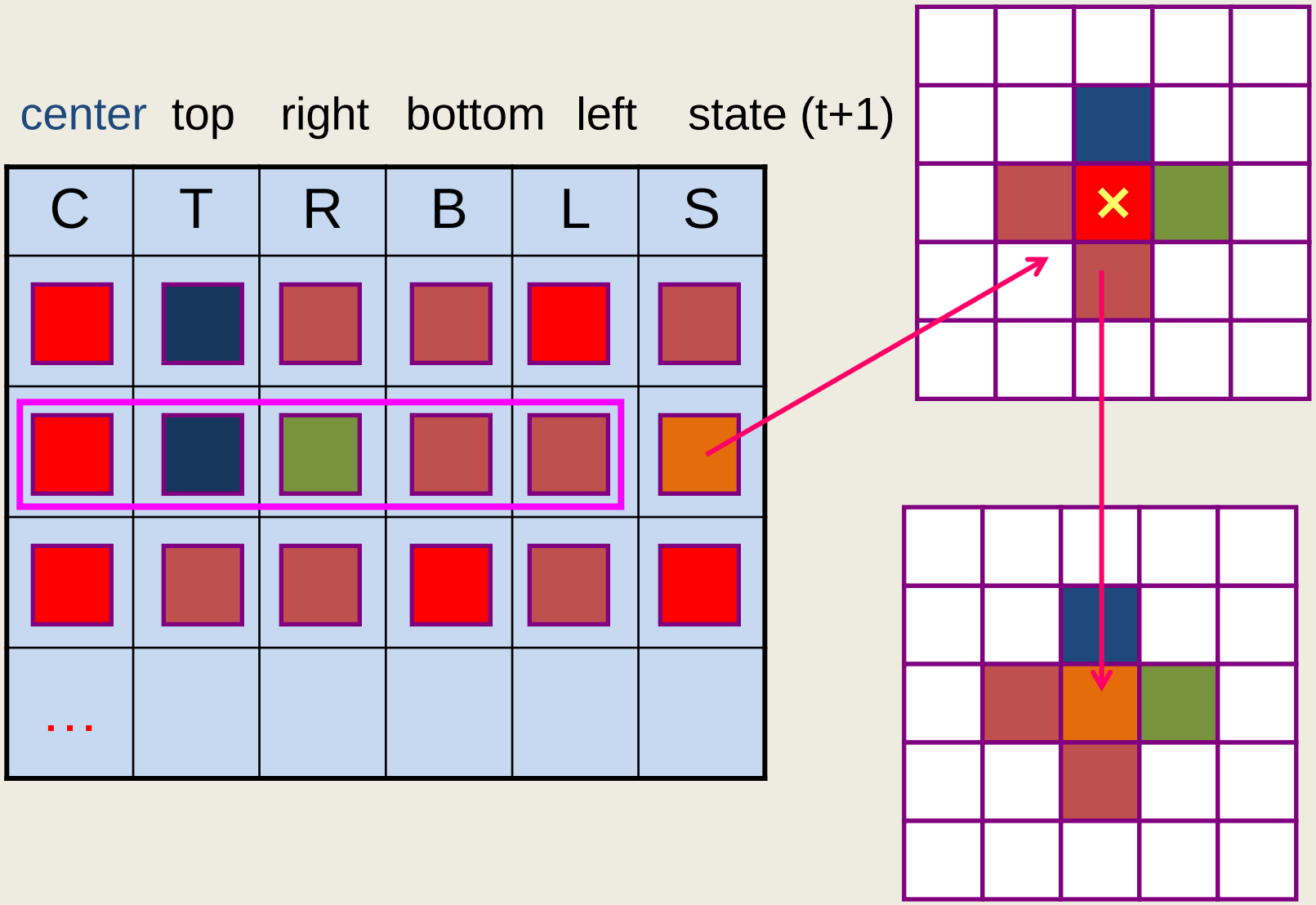
Given a binary CA and a 1-dim neighborhood of size 1. Then, there are $2^3 = 8$ possible configurations for $a_{i-1}(t)a_i(t)a_{i+1}(t)$. Consequently, we need eight bits for defining the dynamics.

Transition Rule Table: Exhaustive



Simple CA's (k = 2, r = 2) are encoded by a number between 0 and 255.

Transition Rule Table: 2-D



Transition Rule Table: 2-D (rectangular)

- There are 2^(2^9) = 2^512 different *k*=2, *r*=1 (*n*=9) 2-D CA
- Size of a rule table: k^((2r+1)^2)
- # number of possible rule tables: k^(k^((2r+1)^2))

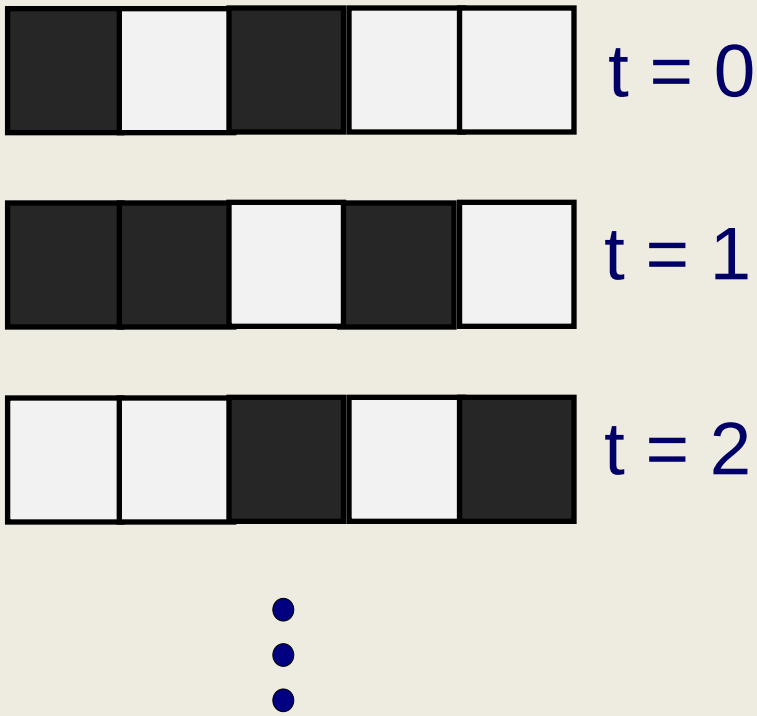
Initial Configuration

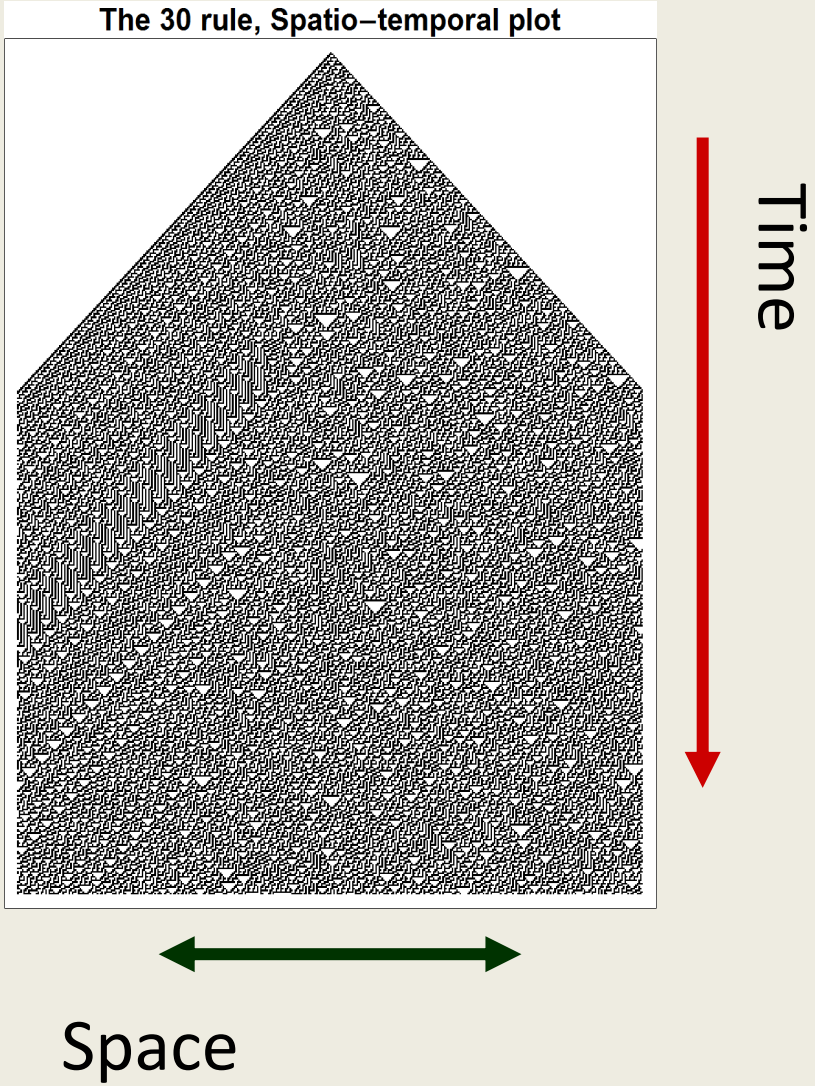
- Initial configuration defines the initial states of all the cells
- Depending on the boundary conditions, there can be potentially infinitely many cells. If not stated differently, cells are assumed to be in the quiescent state at t = 0..
- Usually, one of the three following possibilities is used:
  - All 0, except for one cell
  - Random
  - Handcrafted.

Elementary CA – A Case Study

Zurich University
of Applied Sciences
School of Engineering
ZAMP Center for Applied
Mathematics and Physics

Lattice and Time for 1-D Systems





Taxonomy of Elementary CA

Physica 10D (1984) 1–35
North-Holland, Amsterdam

UNIVERSALITY AND COMPLEXITY IN CELLULAR AUTOMATA

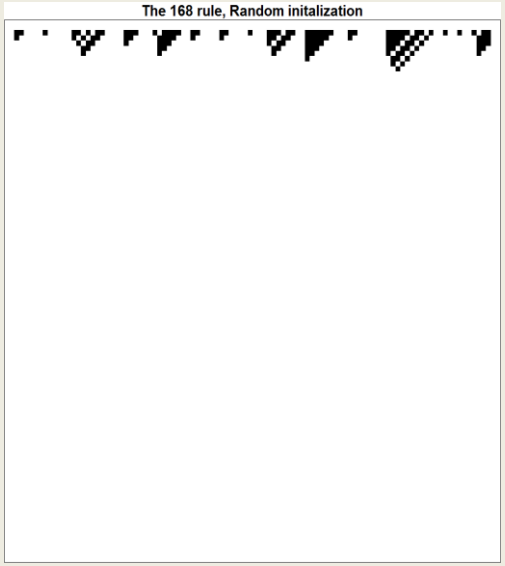
Stephen WOLFRAM*
The Institute for Advanced Study, Princeton NJ 08540, USA

Cellular automata are discrete dynamical systems with simple construction but complex self-organizing behaviour. Evidence is presented that all one-dimensional cellular automata fall into four distinct universality classes. Characterizations of the structures generated in these classes are discussed. Three classes exhibit behaviour analogous to limit points, limit cycles and chaotic attractors. The fourth class is probably capable of universal computation, so that properties of its infinite time behaviour are undecidable.

"THE" paper

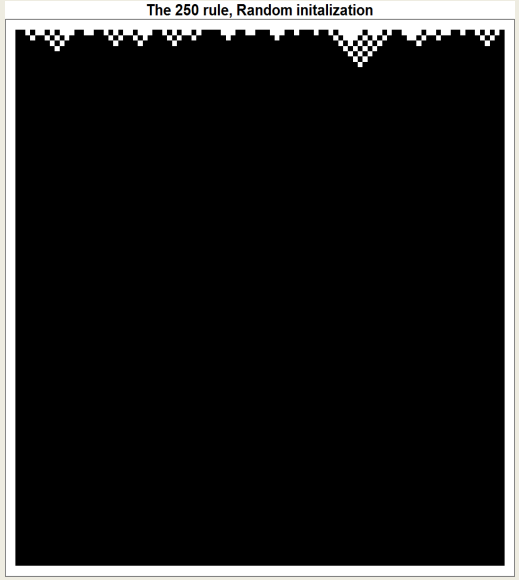
Class 1 - Constant

- Degenerate, single color, homogenous
- Nothing really interesting or surprising, all cells are either on/off



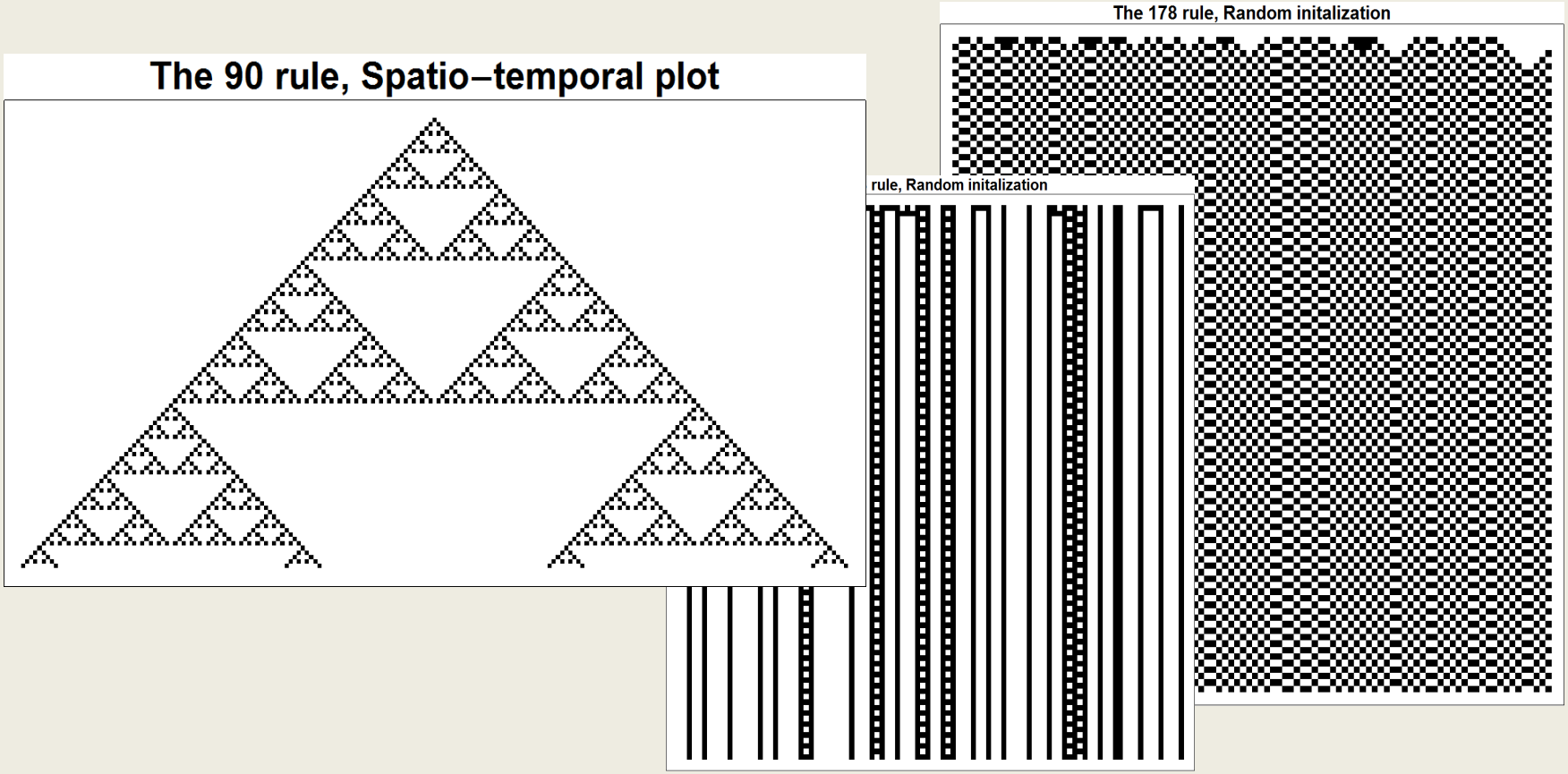
Rule 168

Rule 250



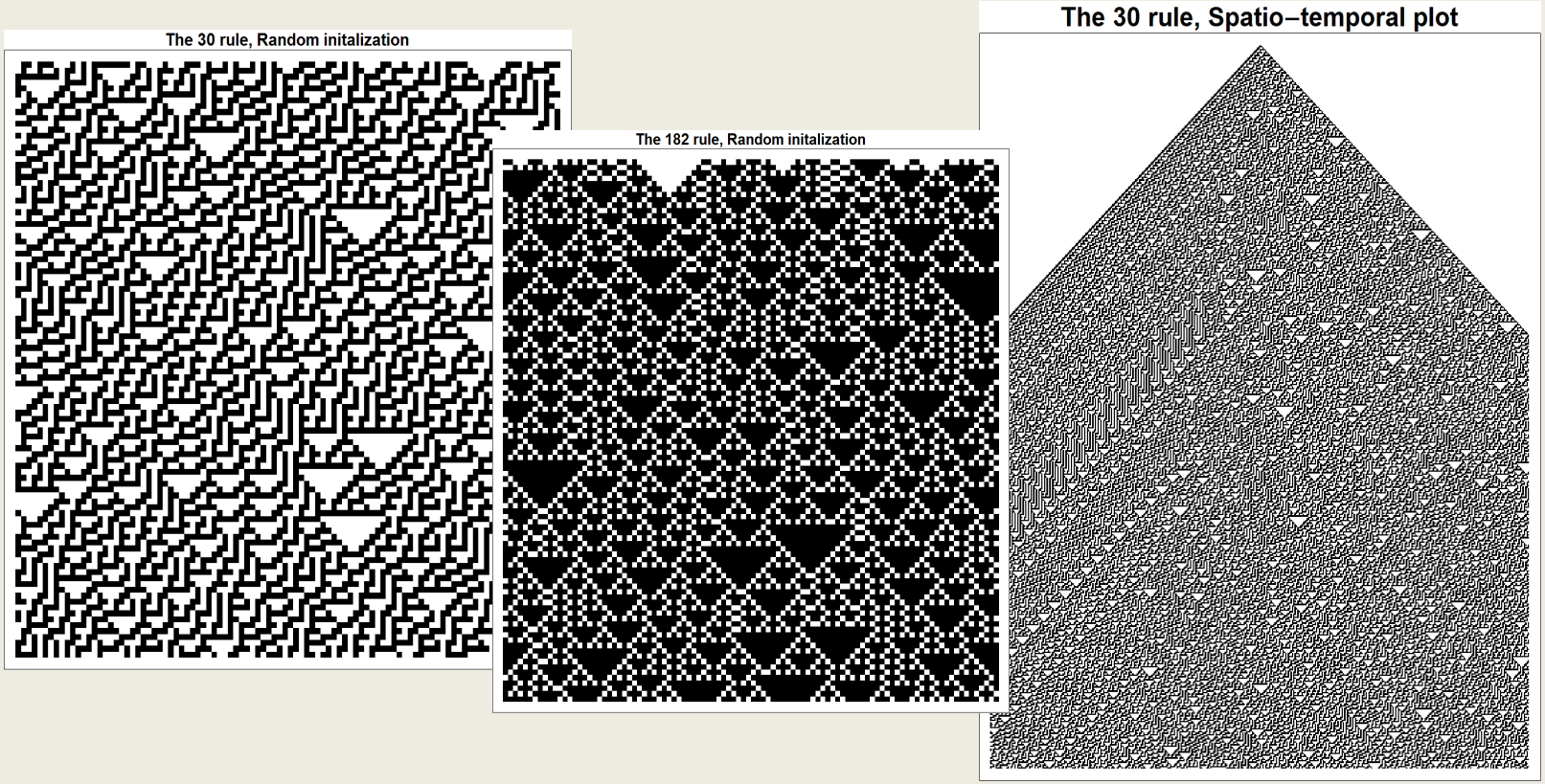
Class 2 – Repeats, Local Structures

- Periodic structures, nested patterns, e.g. rules 90,108,170



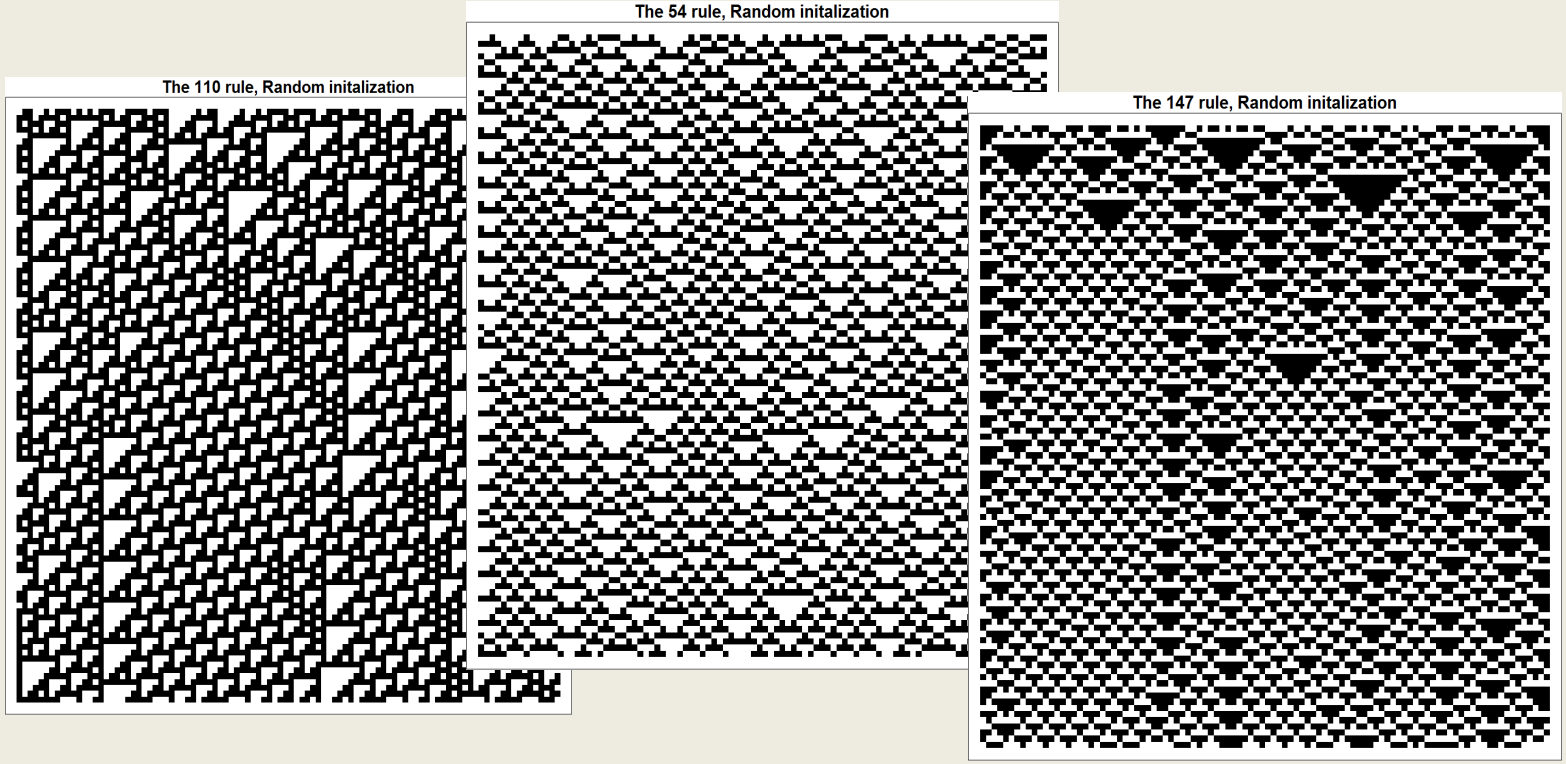
Class 3 - Pseudo Random

- Statistical analysis show randomness



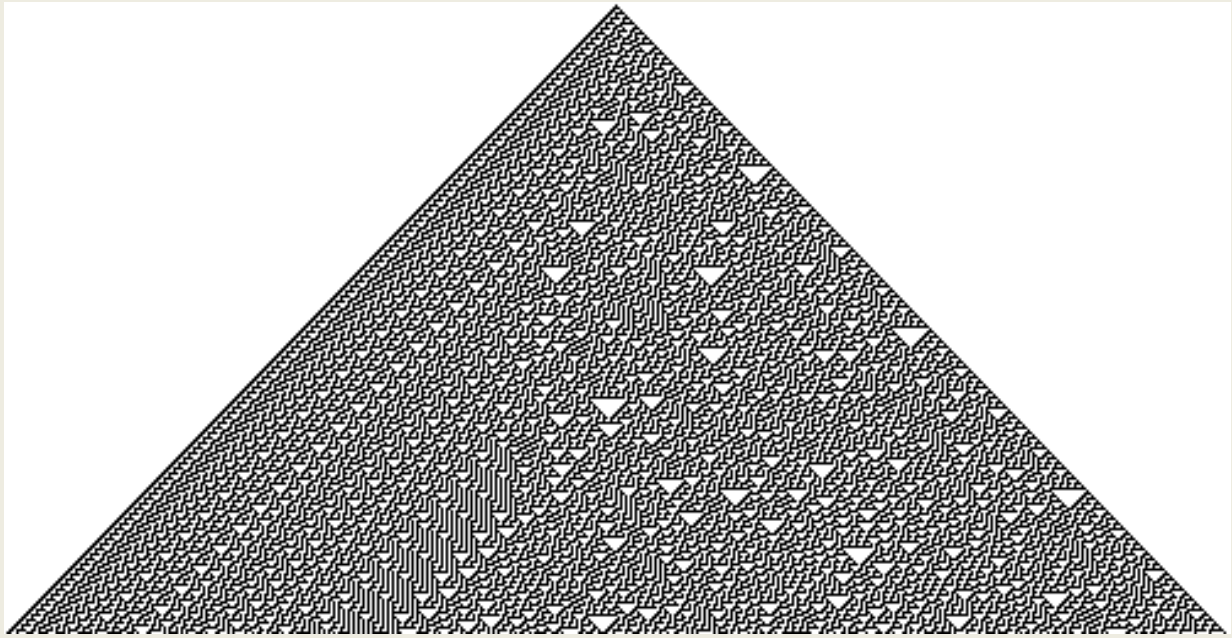
Class 4 - Complex

- Beyond “randomness” (Wolfram)
- Neither regular nor completely random



Elementary CA – Rule 30

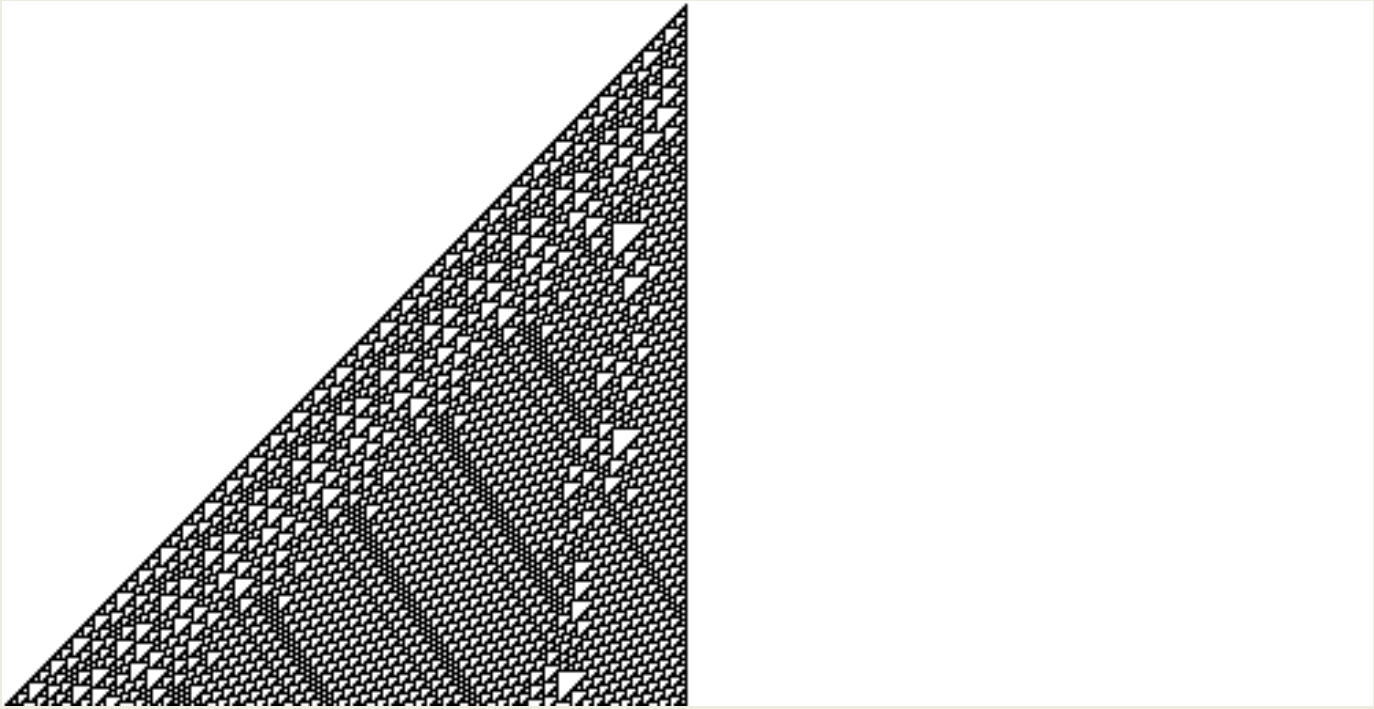
Rule 30 CA, (00011110, 8 possible patterns for a neighborhood), starting with 1 in center.



Generates randomness without random input → can be used as a PRNG.

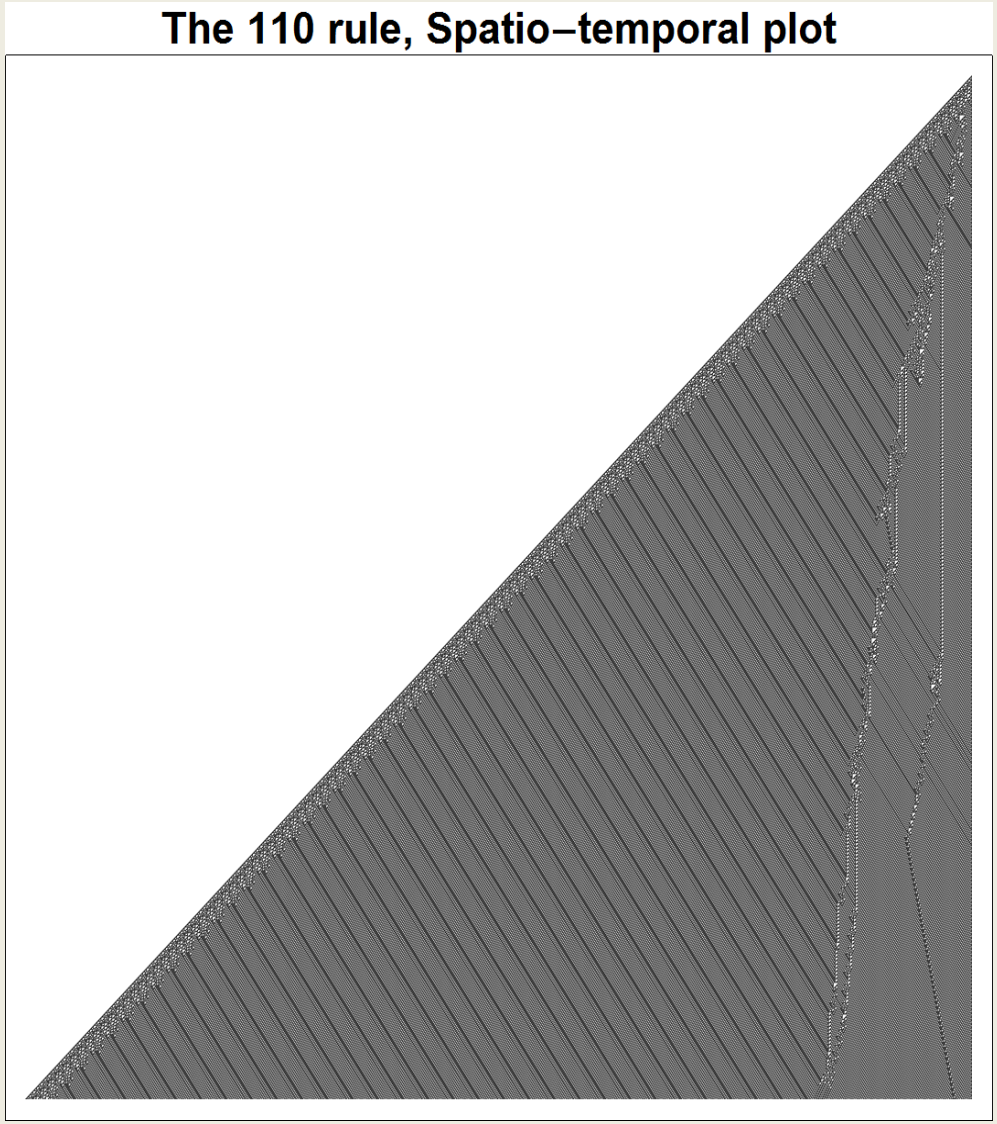
Elementary CA - Rule 110

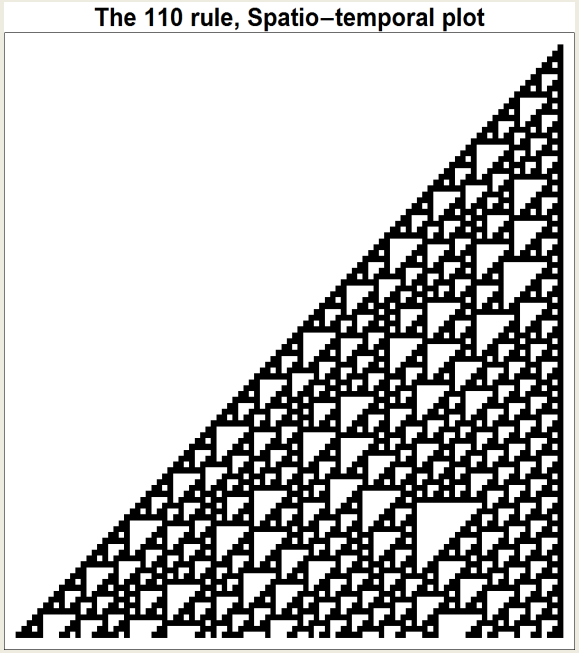
Rule 110, starting from a 1 in the center



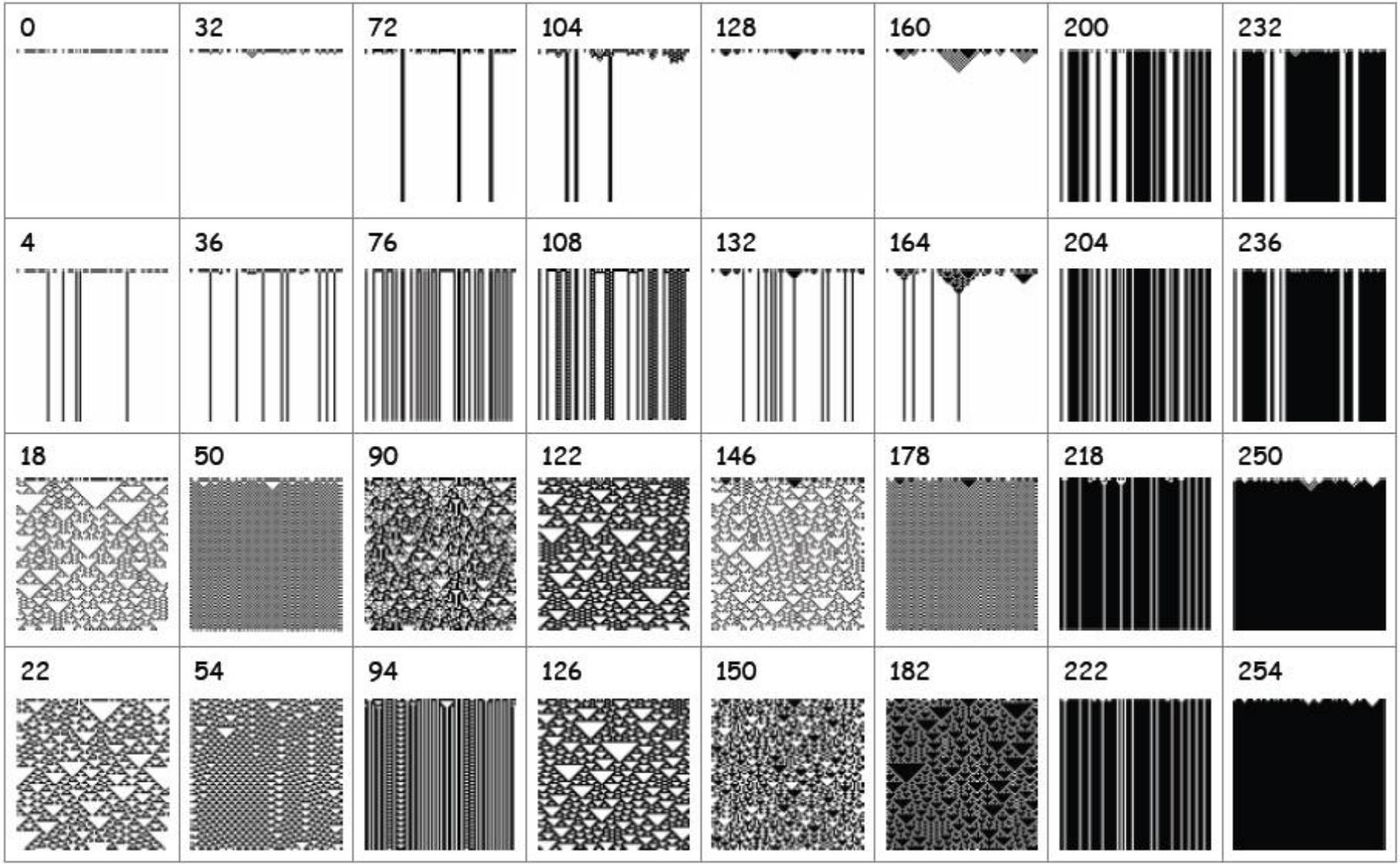
Neither completely random nor completely periodic → class IV CA

Elementary CA - Rule 110





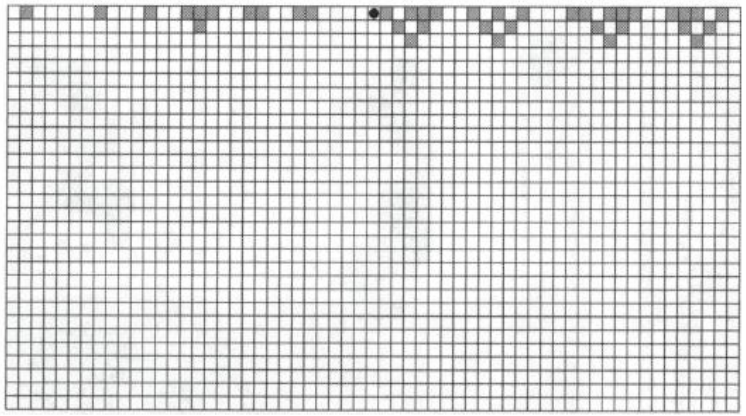
Space-Time Diagrams of 32 CA



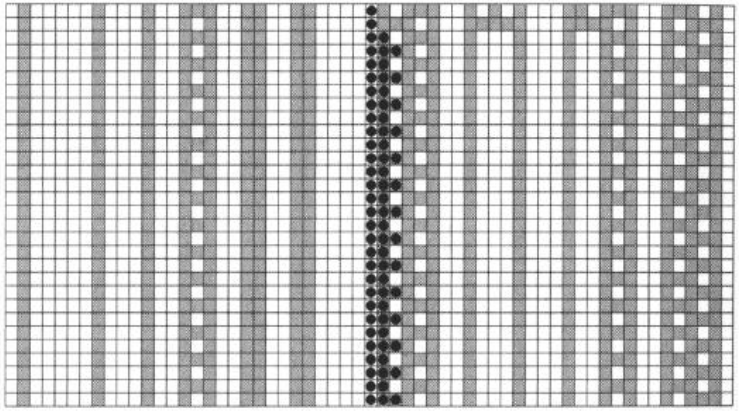
Elementary Cellular Automata: Summary

- small persistent patterns (~170)
  - stationary
  - moving
- growing patterns (~85)
  - repetitive patterns (~45)
  - more complex patterns (36)
    - fractal, nested patterns (24)
    - random patterns (10)
    - complex mixture of regular and irregular (1)

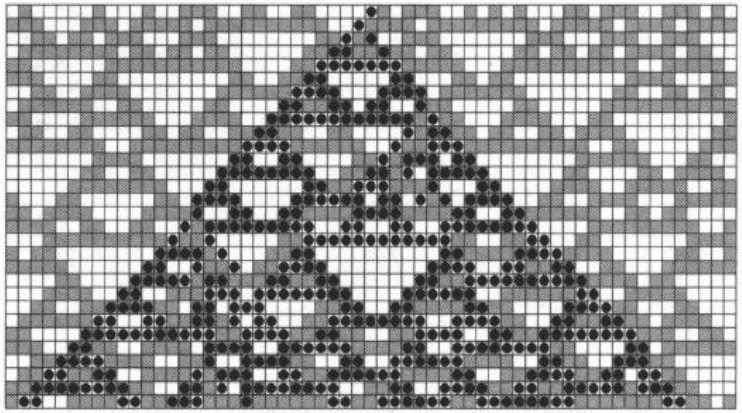
Dependence on Initial State



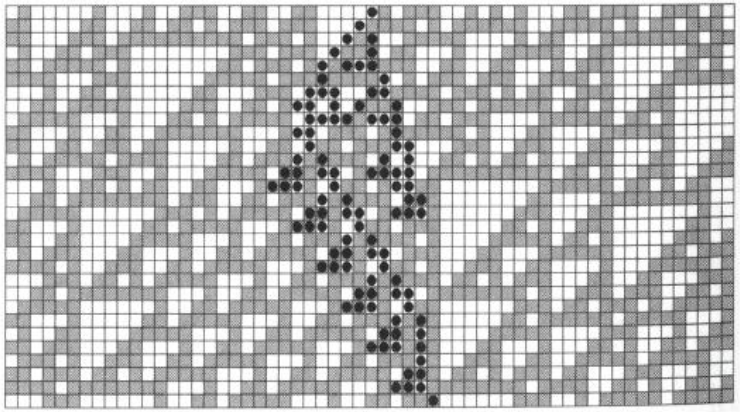
(a) Class 1



(b) Class 2



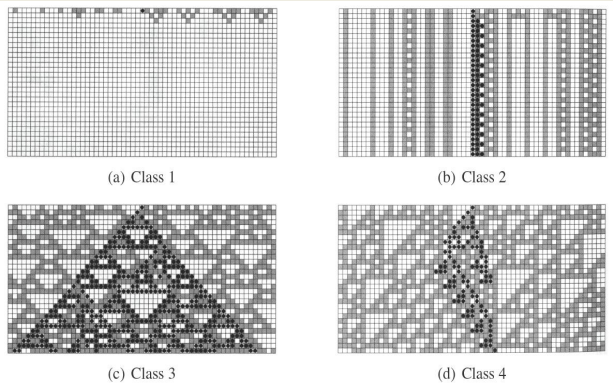
(c) Class 3



(d) Class 4

Dependence on Initial State

- Class 1: Small changes eventually die out, the final state is not affected.
- Class 2: Small changes may persist, but effect remains local.
- Class 3: Small changes spread out, and eventually regions arbitrary far away are affected.
- Class 4: Small changes may or may not spread out via complicated, but sometimes highly regular dynamics.



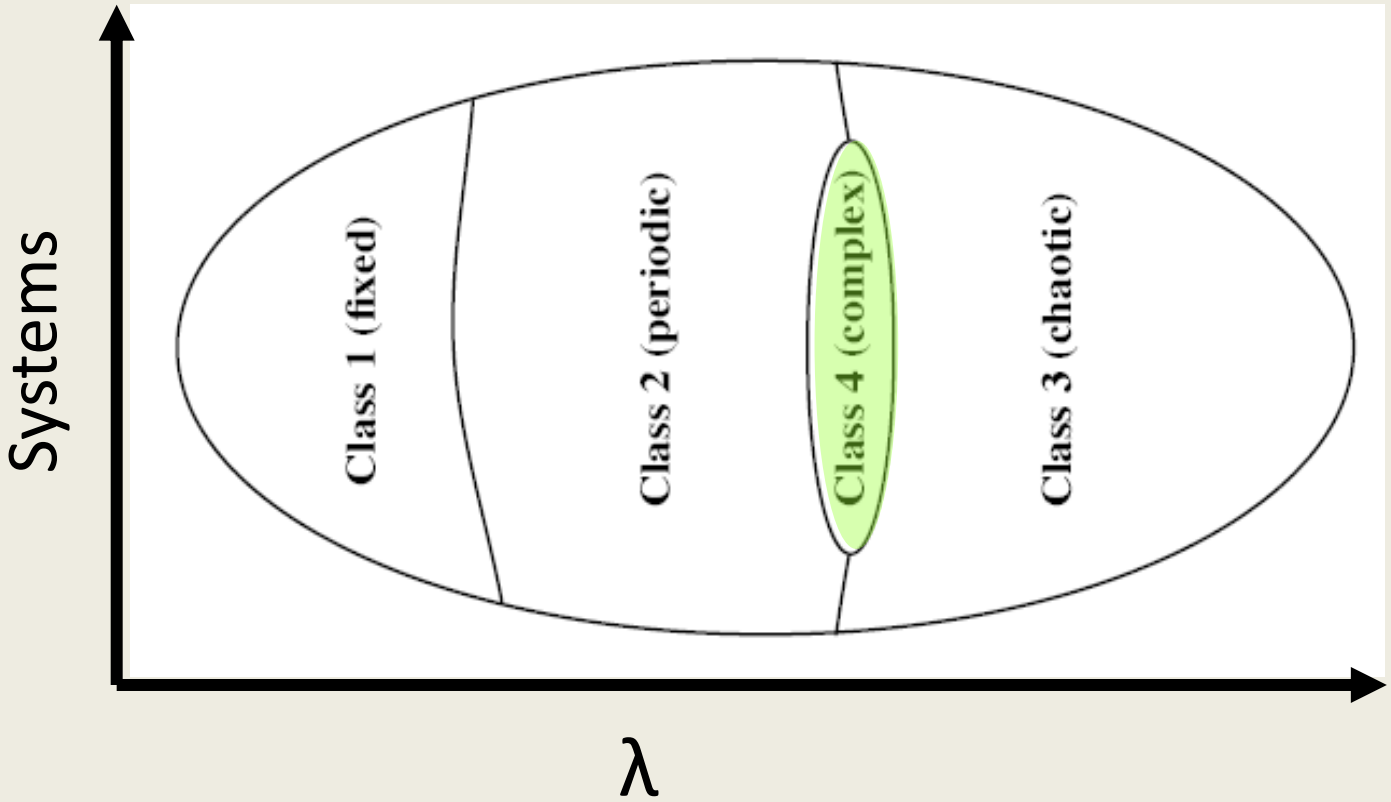
The Lambda Parameter λ

- Introduced by Chris Langton in 1986
- Observation: some CA display interesting, complex behavior. But: Wolfram's classification scheme is phenomenological
- λ is an attempt to measure complexity of a CA
- Idea: Regard one type of states (say zero) as quiescent, the other as active. Try to predict whether a system containing initially some random distribution of active states remains active.
- Intuitively: the probability that a neighborhood in a particular rule is mapped to an active (non-quiescent) state

N: # inputs in transition tables
n_0: # quiescent outputs

$$\lambda = \frac{N - n_0}{N}$$

The Lambda Parameter λ



Norman Packard: Life is at the edge of chaos.

Is λ A Good Measure?

- Rather good correlation between λ and other interesting properties for extreme values of λ, near 0 and 1
- Results based on “average” behavior, but there is no “average” behavior for intermediate λ → there is great deal of variation

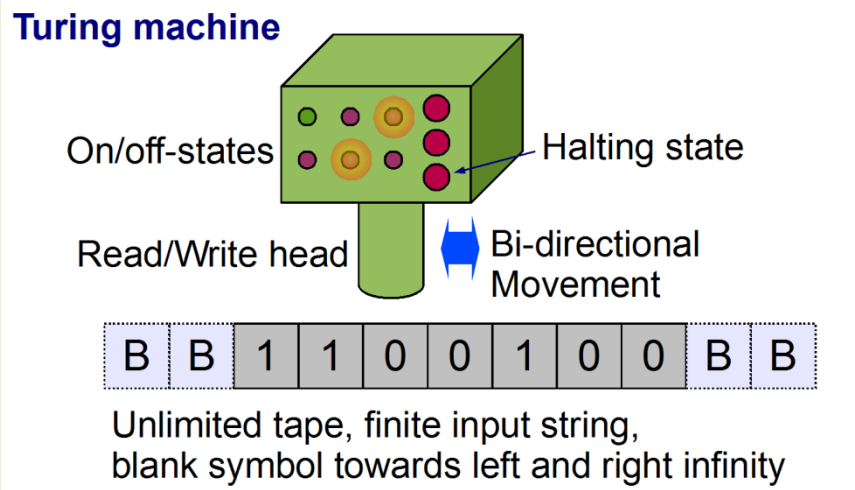
CA and Turing Machines

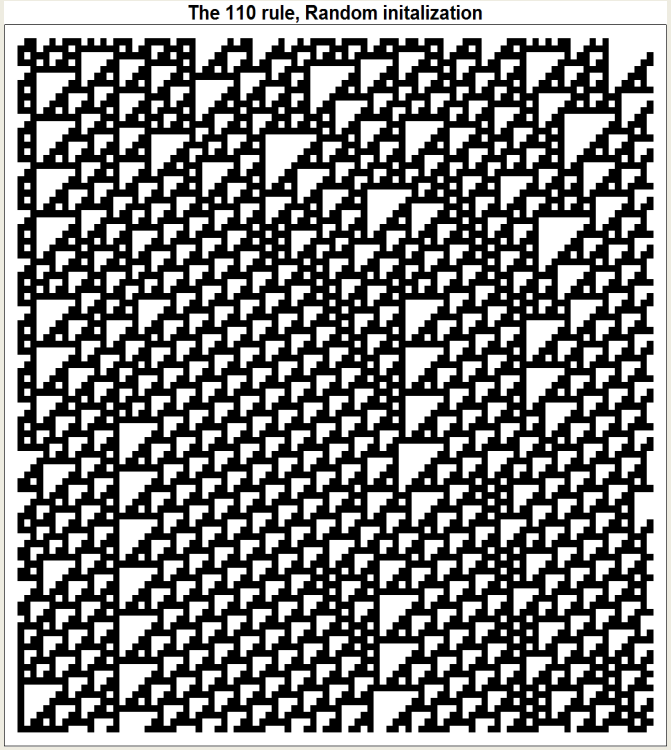
Zurich University
of Applied Sciences

School of
Engineering

ZAMP Center for Applied
Mathematics and Physics

The Rule 110: A Universal Computer





The Rule 110: A Universal Computer

Claimed by Wolfram and proven by Cook:

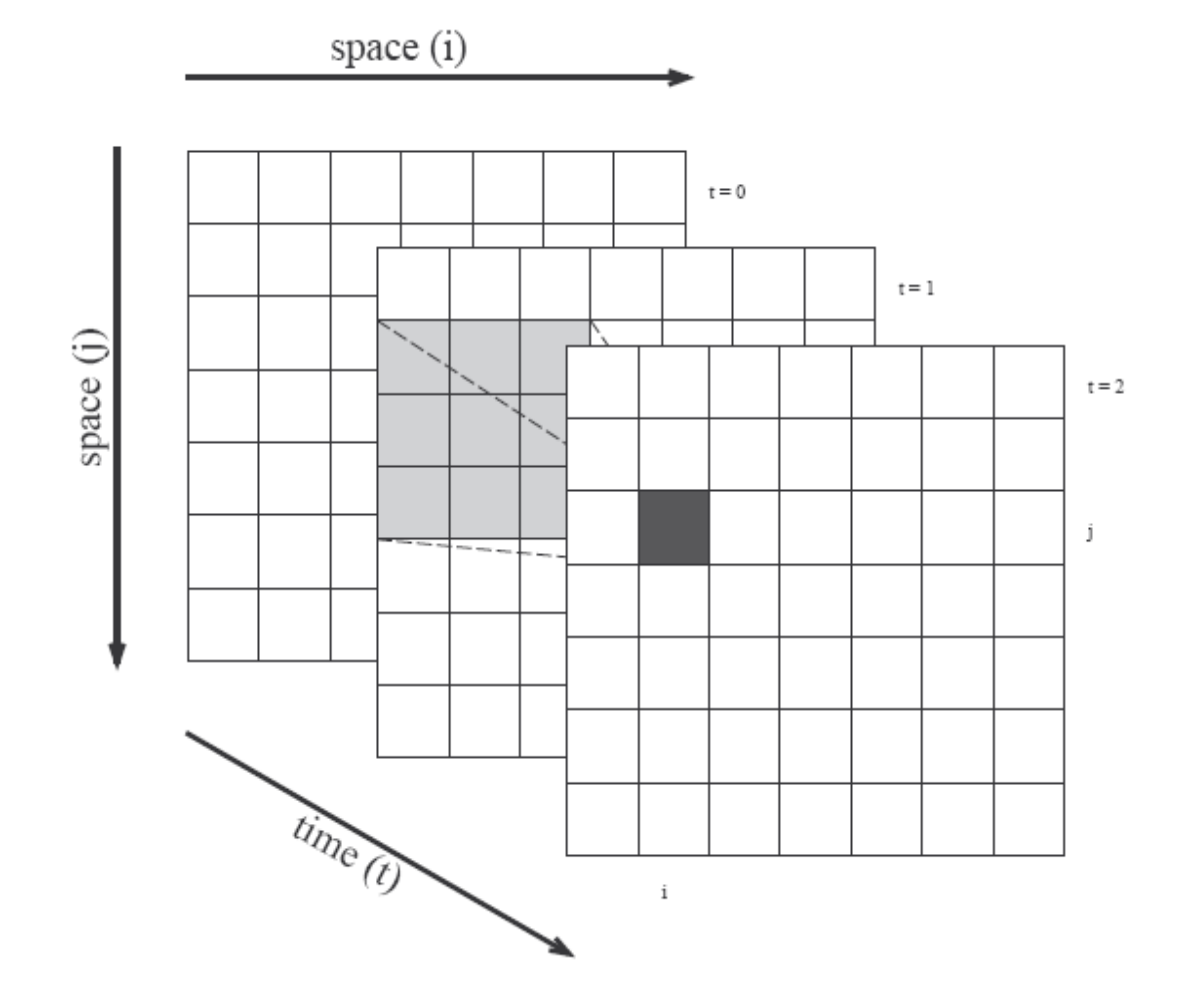
The rule 110, as a function of its input, has a dynamic equivalent to a universal Turing machine.

This is an important result, becasue it show that the definition of a computation via Turing machines is a concept that is deeper than the Turing machine itself. In other words: Two different views of computation, either via a TM or via a (potentially) infinite number of FSA leads to the same set of computable problems!

Conway's Game of Life
A Second Case Study

Zurich University
of Applied Sciences
School of
Engineering
ZAMP Center for Applied
Mathematics and Physics

2D Cellular Automata

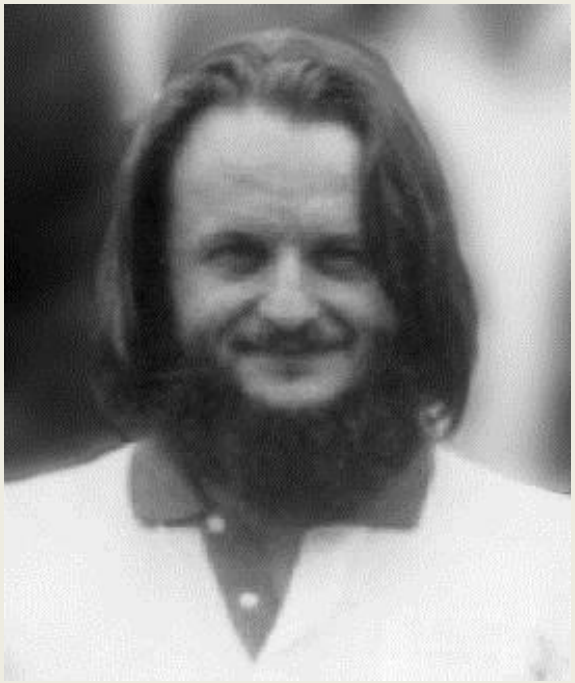


Totalistic Cellular Automata

T**otalistic cellular automaton** Rules depend only on the total (*sum* or *average*) of the values of the cells in a neighborhood

Conway's "Game of Life"

- The Game of Life is a CA (of class IV) devised by the British mathematician John Horton Conway in 1970 and popularized by Martin Gardner.
- Original "game" was played with pieces on a Go board.

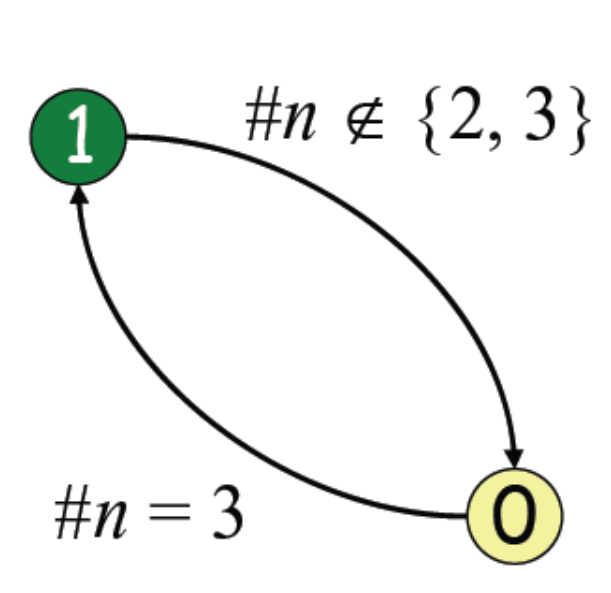


John Horton Conway

Conway's "Game of Life"

- Essentially, a 2-D, k = 2, r = 1 (n = 9) CA
- **Rules:** Next state depends on: sum of the 8 neighbor cell states, and on state of central cell ("outer totalistic")

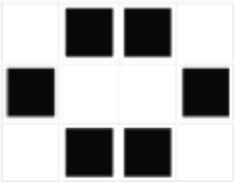
$\#n \setminus s_{i,t}$	0	1	
<2	0	0	dies of "loneliness"
2	0	1	"survives"
3	1	1	"born" / "survives"
>3	0	0	dies of "overcrowding"



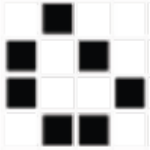
Stable Patterns

- block

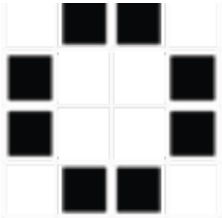
- beehive



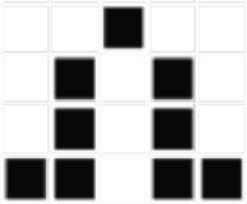
- loaf



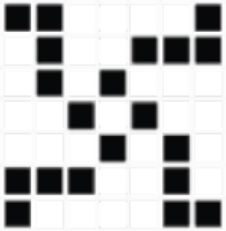
pond



hat



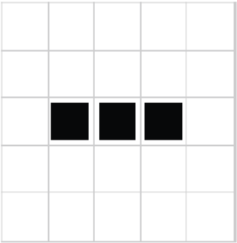
spiral



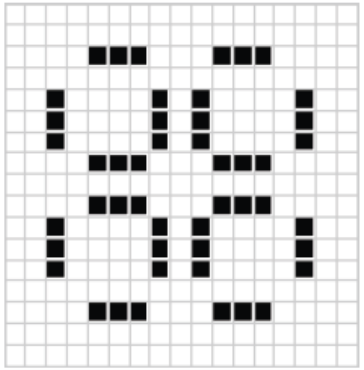
Oscillators

repeating pattern

- blinker, p=2

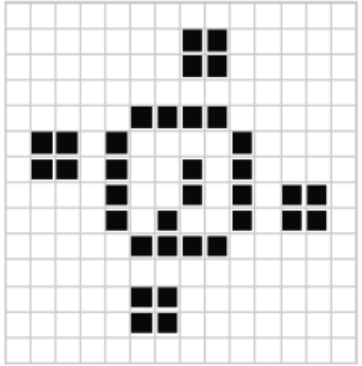


pulsar, p=3



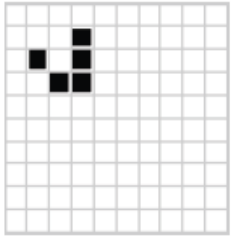
- blocker, p=8

clock II, p=4

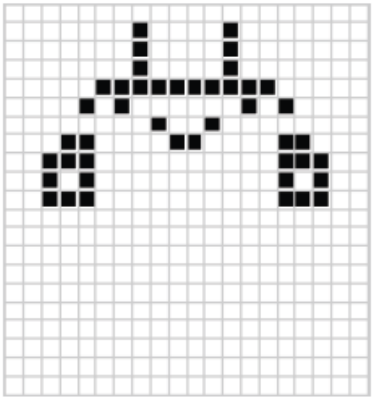


Moving Patterns

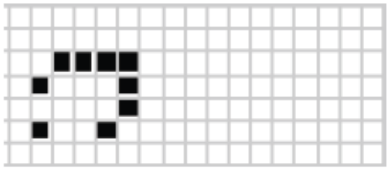
- glider



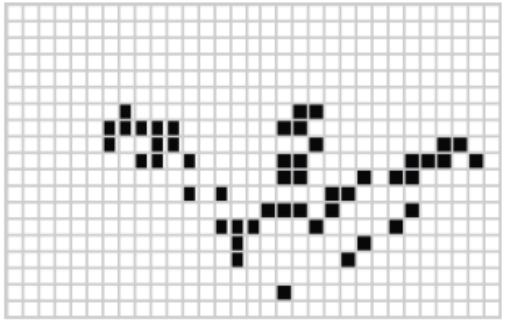
weekender



- lightweight spaceship



swan



Logic Gates

Glider gun to produce string of gliders

Carefully arrange streams to intersect and annihilate, to produce “gates”

NOT XOR gate example
First step to building a computer in Conway’s game of life.

Applications of CA

A Comment on Simulation

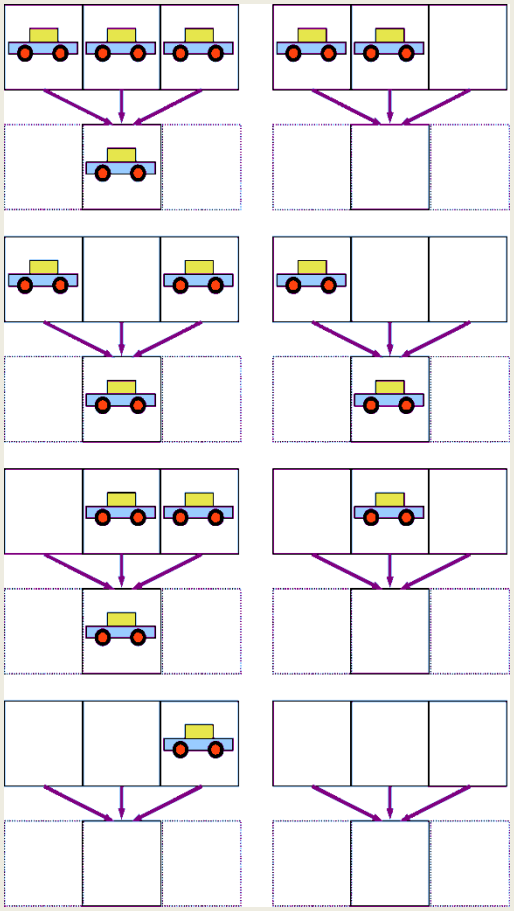
- Sometimes, one is interested in a specific type of dynamics, say e.g. spatially resolved chemical kinetics or the dynamics of a replicating population.
- Many questions concerning these dynamics are caused **by structure of the interactions of and relations between** the members of the population or chemical mixture and not so much by the physical details of their dynamics.
- CA offer than a way to explore the behavior of such a system in a very simple world, a world with only discrete physics.
- In many cases, e.g. in the study of biological tissues, this turns out to be sufficient for capturing and modeling essential processes.

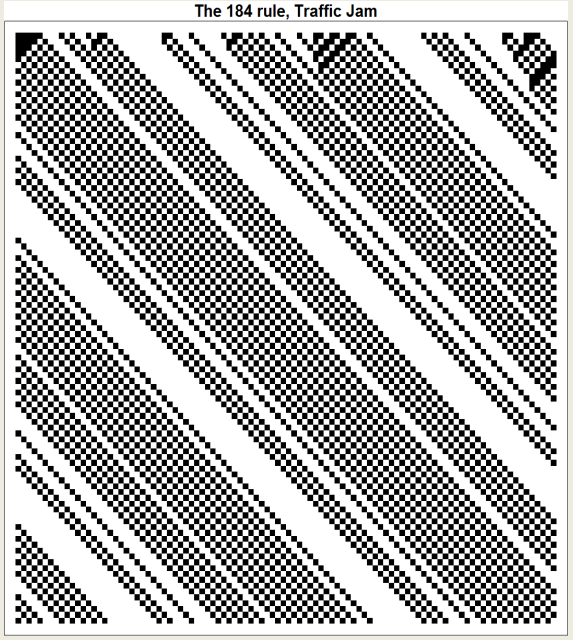
A CA – Based Model of Traffic Jams

Zurich University
of Applied Sciences
School of
Engineering
ZAMP Center for Applied
Mathematics and Physics

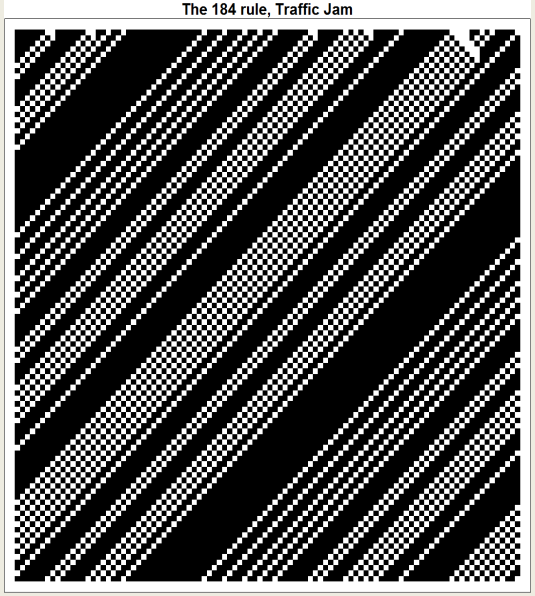
An Application: Traffic Jams

- A very simplistic model of traffic



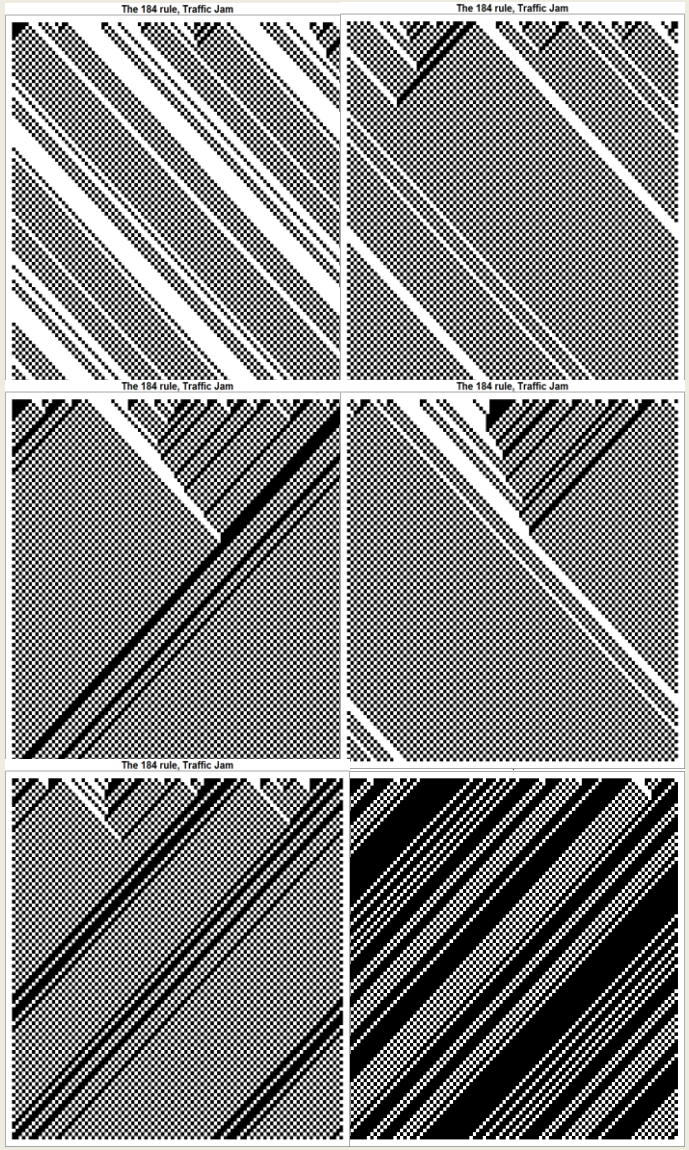


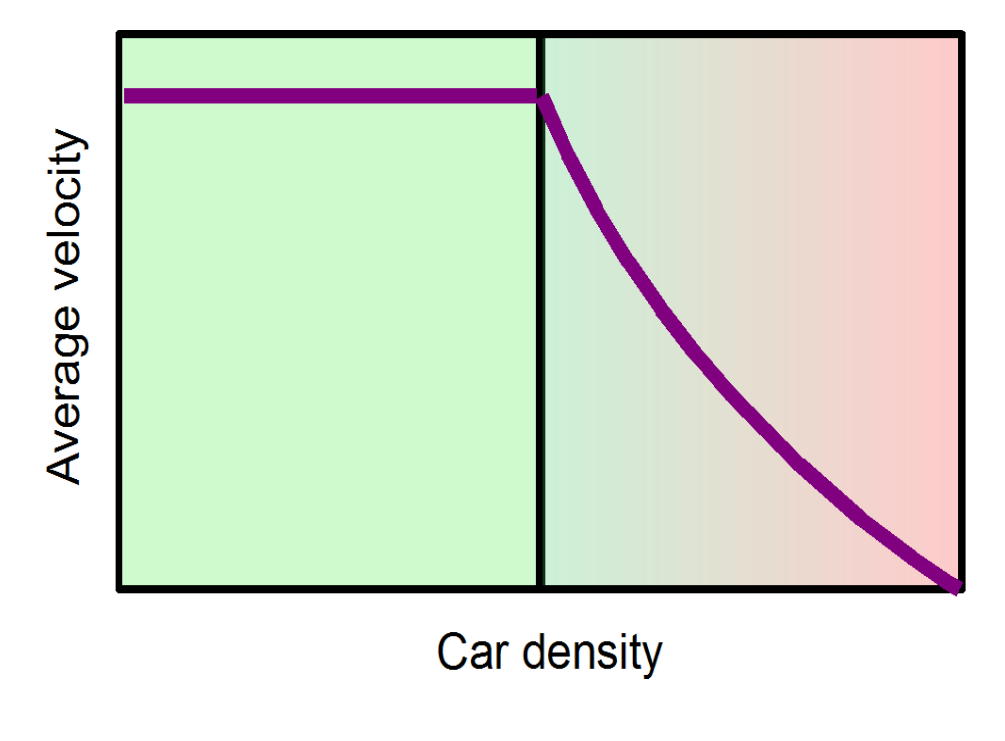
ρ= 0.3



ρ= 0.7

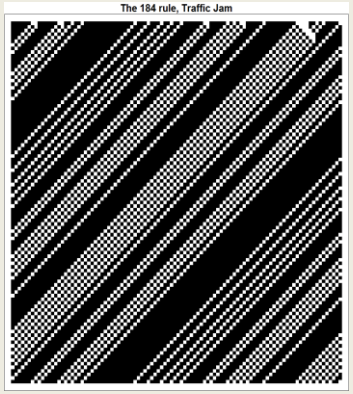
An Application: Traffic Jams

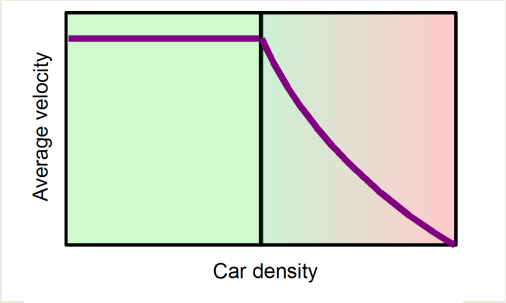




At ρ= 0.5, we observe a transition from flowing traffic to nose-to-tail traffic

An Application: Traffic Jams





Already simple model shows:

- Traffic jams are dynamic correlation structures and not caused by individuals
- Observing the statistics of the traffic flow may help to prevent traffic jams.

Ants can

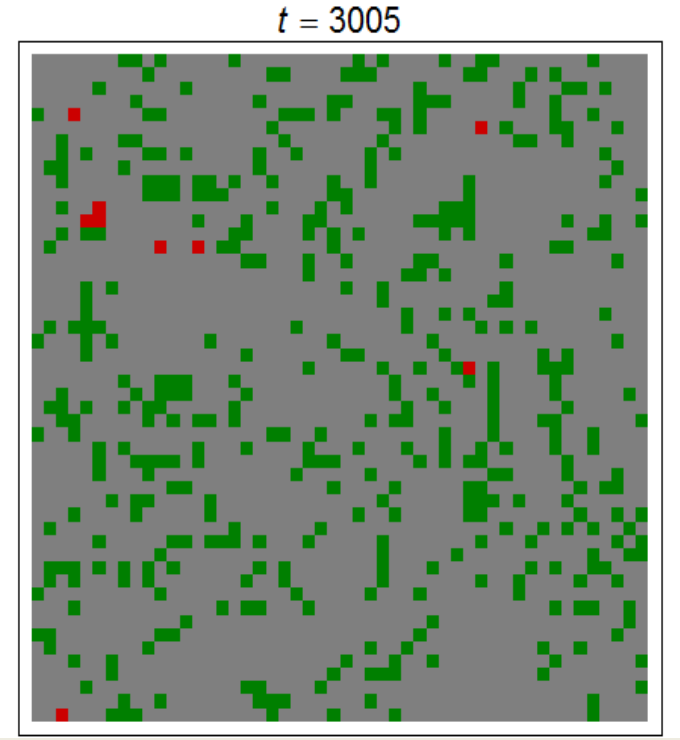
We can't

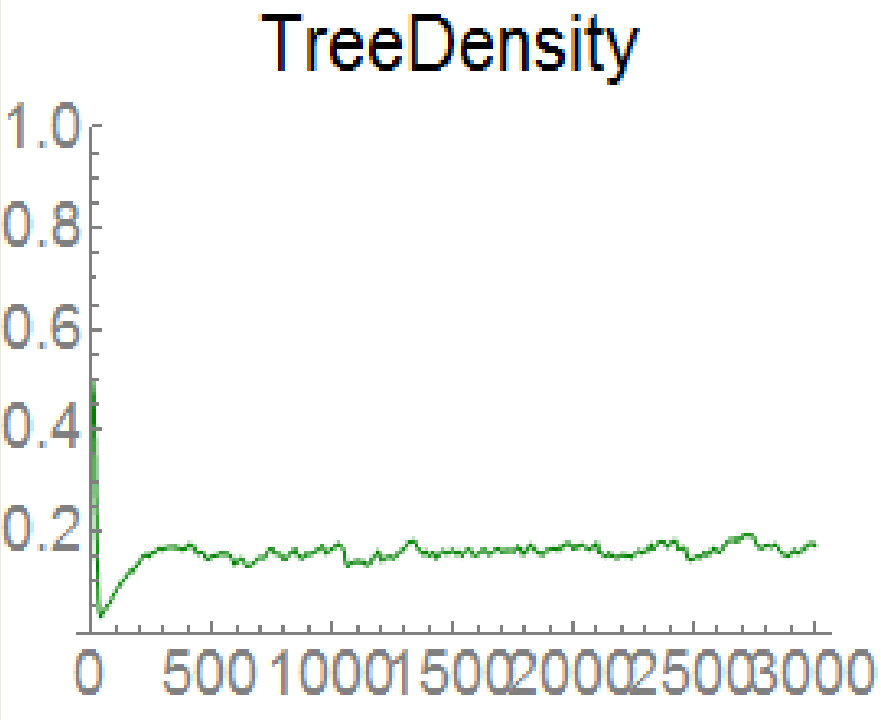
Forest Fire Model

- Forest Fire Model is a stochastic *3*-state cellular automaton defined on a *2*-dimensional lattice of size L
- Each site is occupied by a tree, a burning tree, or is empty
- During each time step the system is updated according to the rules:

1) empty site (state 0) $\rightarrow$ tree (state 1): with the growth rate probability p_{growth}
2) tree (state 1) $\rightarrow$ burning tree (state 2): with the lightning rate probability $p_{lightning}$
3) tree (state 1) $\rightarrow$ burning tree (state 2): with the probability $1-(1-p_{catchfire})^{\#\text{burning neighbors}}$
4) burning tree (state 2) $\rightarrow$ empty site (state 0) witch extinction probability p_{ext}

Forest Fire Model

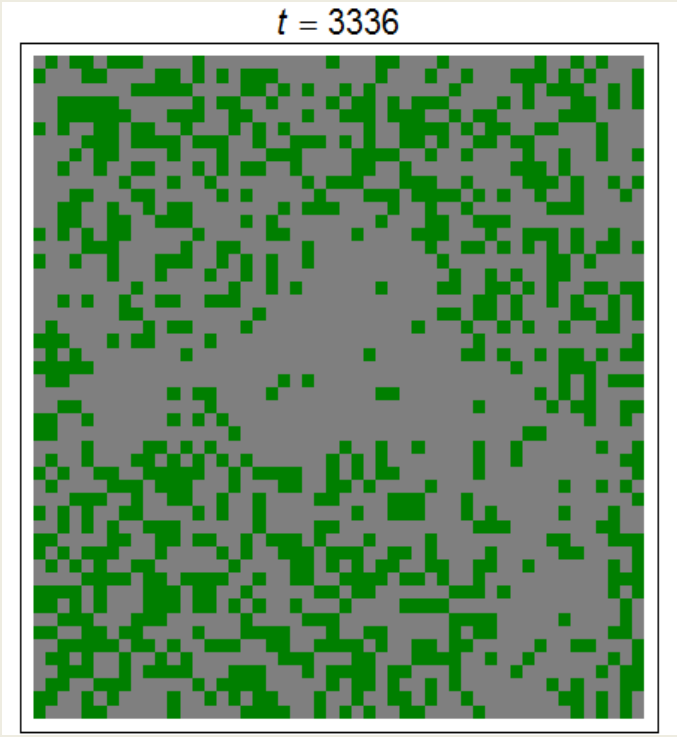


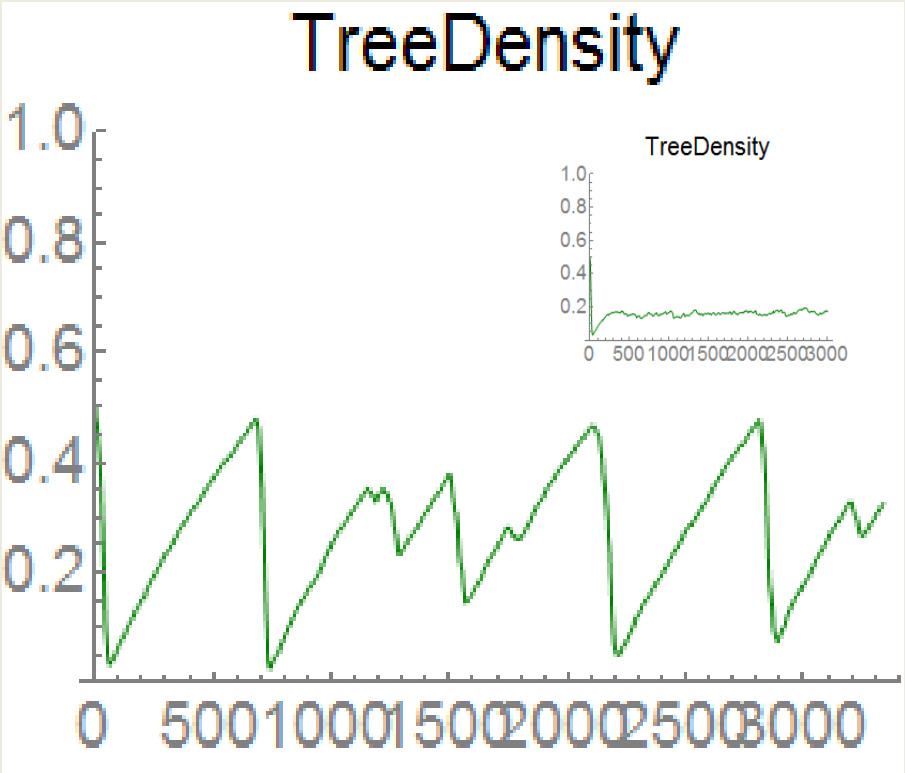


$$p_{lightning} = 10^{-3}$$
$$p_{growth} = 0.001$$
$$p_{catch} = 0.5$$
$$p_{extinct} = 0.2$$

What happens, if the lightning probability is reduced?

Forest Fire Model





$$p_{lightning} = 10^{-5}$$

$$p_{growth} = 0.001$$

$$p_{catch} = 0.5$$

$$p_{extinct} = 0.2$$

➔ Small lightning probablities lead to higher ecological dammage!

Generalization of CA

Cellular Systems

- Cellular automata are characterized by a **discrete set of states** and a **discrete time variable**.
- **Identical automata** are coupled via a dynamics determined by the state of the cell and its neighbors.
- **Neighborhoods are identically structured** for all cells.
- The update happens in a **deterministic** and **synchronous** manner.

General cellular systems relax these requirements.

Non-Homogeneous CA

Non-Homogeneity caused by:

- Non-homogeneous transition tables.
  - in space
  - in time
- Non-homogeneous neighborhoods.
- Obviously, such CA can reflect the effect of a non-homogenous environment.

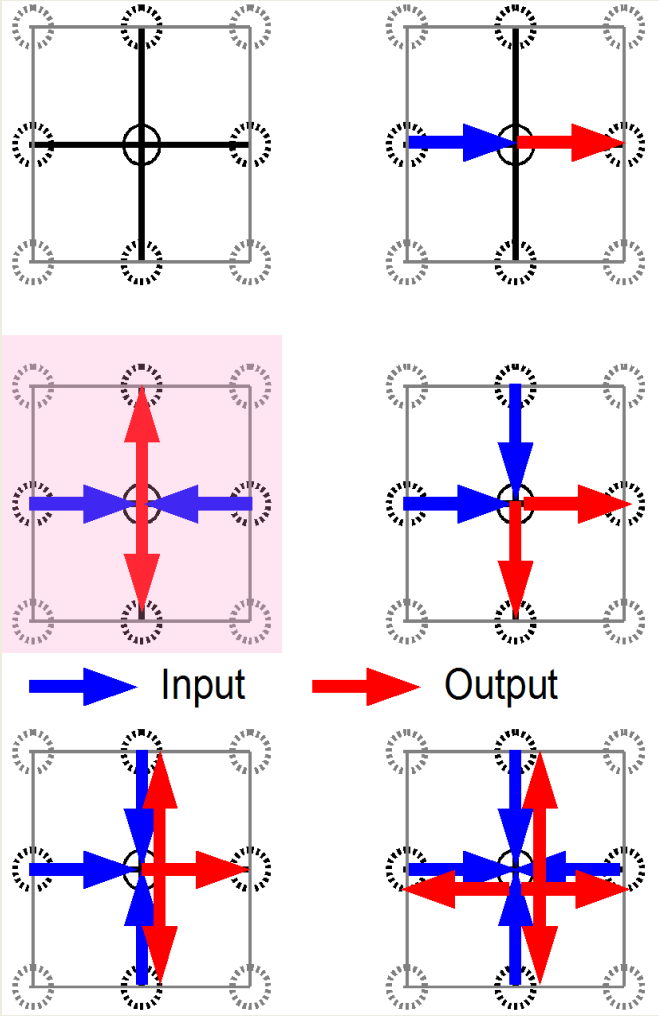
Asynchronous CA

- In CA, an "old lattice" is used to produce a new lattice", which after completion of the update becomes the new "old lattice".
- Strict synchronicity sometimes leads to artifacts. **Randomized choice** of cells to be updated is a strategy to avoid this.
- **Cyclic update** (as your screen does it) is technically interesting but heavily prone for producing artifacts.

Probabilistic CA

- Transition tables have a probabilistic element.
- Probablistic CAs are well suited for simulations.

Particle CA



(and all rotations)

- Lattice sites (cells) can host up to four particles. All particles have always a velocity which determines to which cell they will jump next.
- On a site, collisions take place (Here, only one configuration changes velocities).
- Momentum and particle number is conserved.

Coupled Map Lattices

- In contrast to CA, in **coupled map lattices** (CML) the cells are characterized not by one out of a finite number of states, but by a vector of continuous values. Time is, however, still discrete.
- CML's are especially suited for the simulation of chemical and/or physical processes. A prominent type of CML's are so called **Lattice Boltzmann models**.
- CML's still suffer from the limitations given by a fixed background (no Galilean invariance).

Simulation of Reaction Diffusion Systems

Zurich University
of Applied Sciences
School of
Engineering
ZAMP Center for Applied
Mathematics and Physics

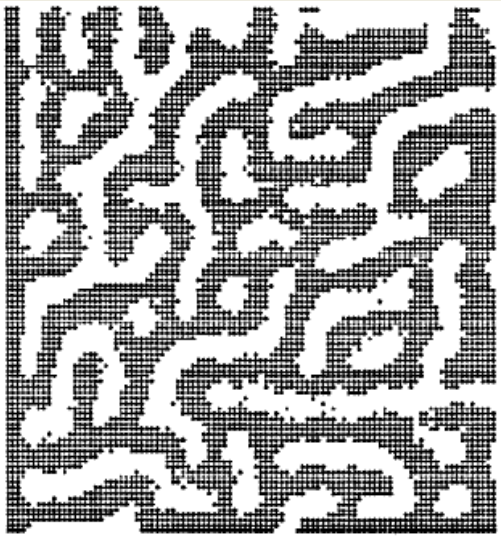
Reality and Formal Languages

- CA sometimes model reality quite closely.
- CA are also perfect instances of systems described by often simple formal languages
- ➔ CA help us to derive formal descriptions of the world, though these descriptions may be only caricatures of what is really going on (Remember: Good caricatures capture the essences of what they describe)

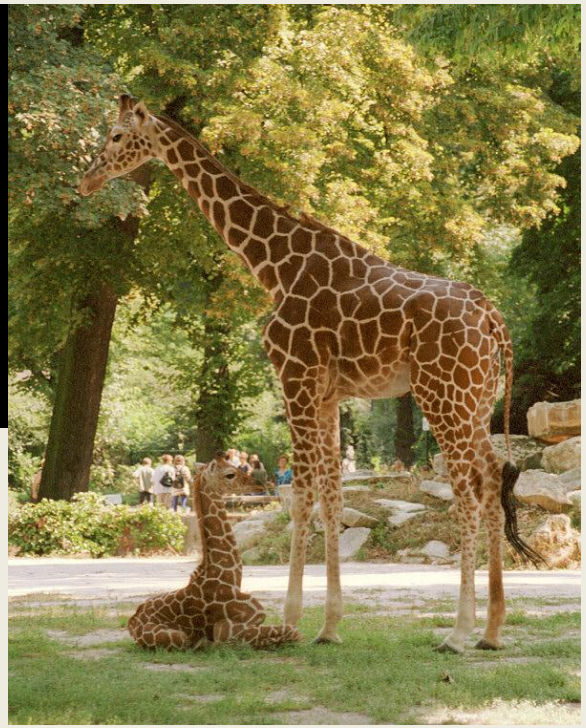
Analogy to Reaction-Diffusion Models

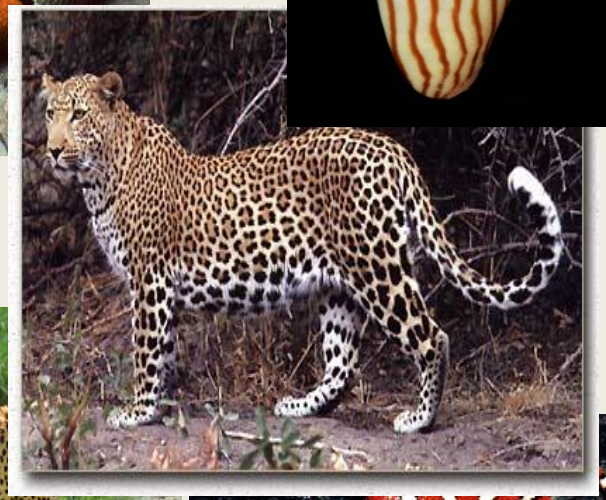
CA can be considered a discrete-space counterpart of Reaction Diffusion models. The space is represented by a uniform grid, with each site or cell characterized by a state chosen from a finite set

For example, Young (1984) proposed a CA model of animal coat patterns using only two states: pigmented or not



Striped Patterns and Spotted Bodies





Morphogenetic Model

[Classic] "The Chemical Basis of Morphogenesis," 1952, *Phil. Trans. Roy. Soc. of London*, Series B: Biological Sciences, 237**:37—72.**

THE CHEMICAL BASIS OF MORPHOGENESIS

By A. M. TURING, F.R.S. *University of Manchester*

(*Received* 9 *November* 1951—*Revised* 15 *March* 1952)

It is suggested that a system of chemical substances, called morphogens, reacting together and diffusing through a tissue, is adequate to account for the main phenomena of morphogenesis. Such a pattern or stru off by rando he case of an system. The ir are six essenti waves appear patterns on *Hydra* and for whorled leaves. A system of reactions and diffusion on a sphere is also considered. Such a system appears to account for gastrulation. Another reaction system in two dimensions gives rise to patterns reminiscent of dappling. It is also suggested that stationary waves in two dimensions could account for the phenomena of phyllotaxis.

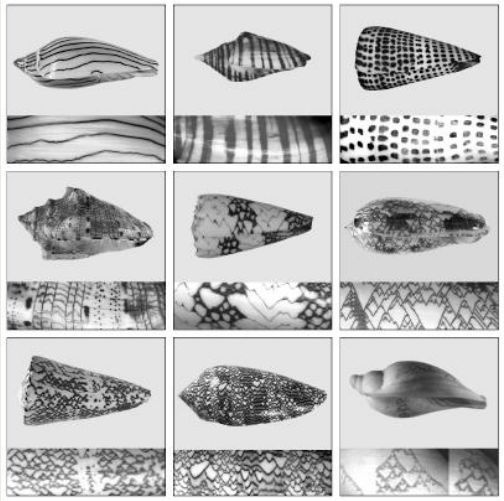
The purpose of this paper is to discuss a possible mechanism by which the genes of a zygote may determine the anatomical structure of the resulting organism. The theory does not make any new hypotheses; it merely suggests that certain well-known physical laws are sufficient to account for many of the facts. The full understanding of the paper requires a good knowledge of mathematics, some biology, and some elementary chemistry. Since readers cannot be expected to be experts in all of these subjects, a number of elementary facts are explained, which can be found in text-books, but whose omission would make the paper difficult reading.

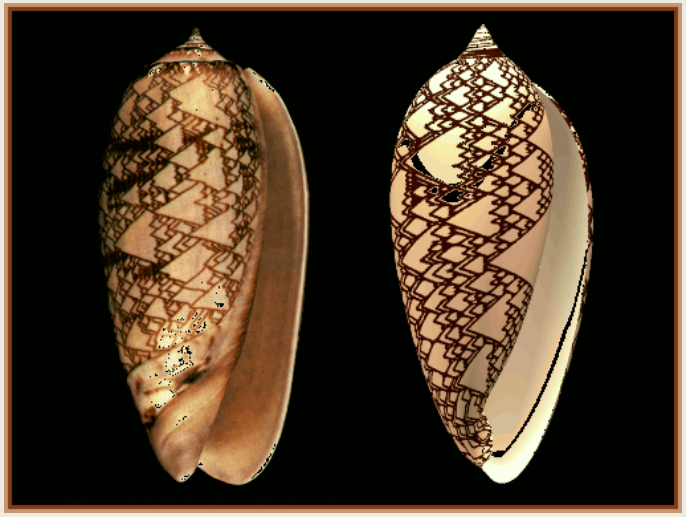
It's the Turing machine Turing!

The Classical Meinhardt Model

Turing delivered a very mathematical analysis, using Fourier decomposition. Meinhardt was already able to use computers for his investigations.

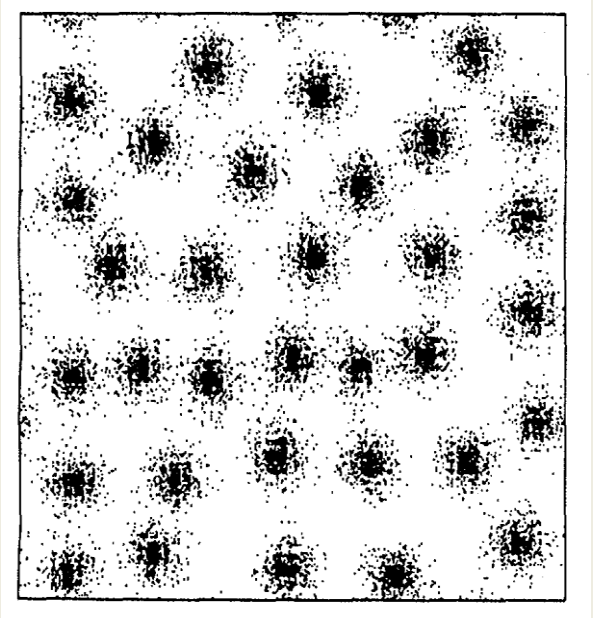
His most popular work is a comparably simple model explaining the patterns on seashells.

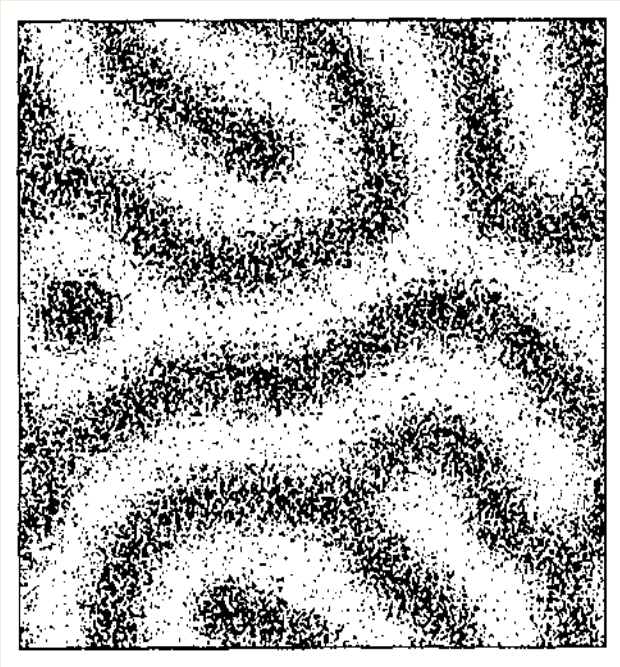




Patterns in a Chemical Medium

Patterns arise spontaneously through competition between localized autocatalytic chemical reaction and long-ranged diffusion of a substance inhibiting the reaction

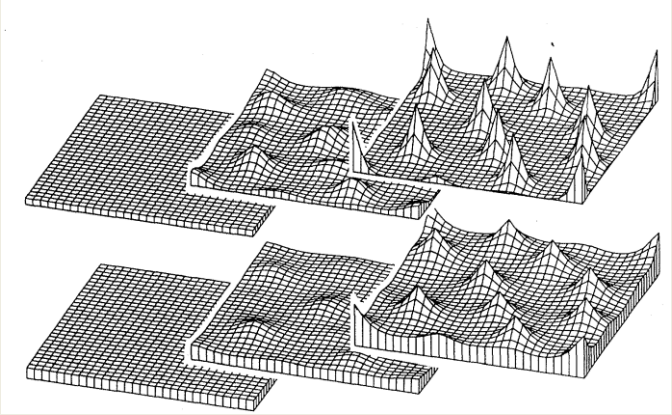


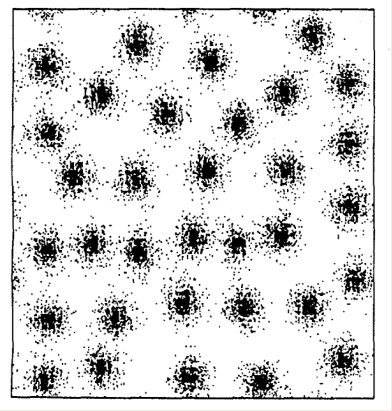


CA – simulation of pattern formation

Patterns in a Chemical Medium

Patterns arise spontaneously through competition between localized autocatalytic chemical reaction and long-ranged diffusion of a substance inhibiting the reaction

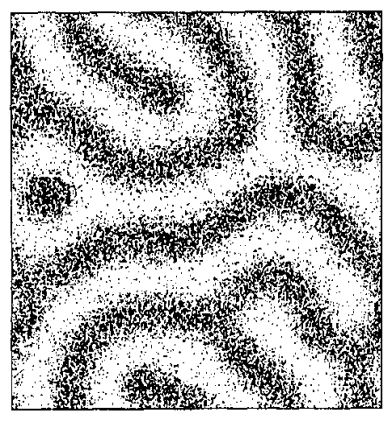




$$\kappa_a = 0$$

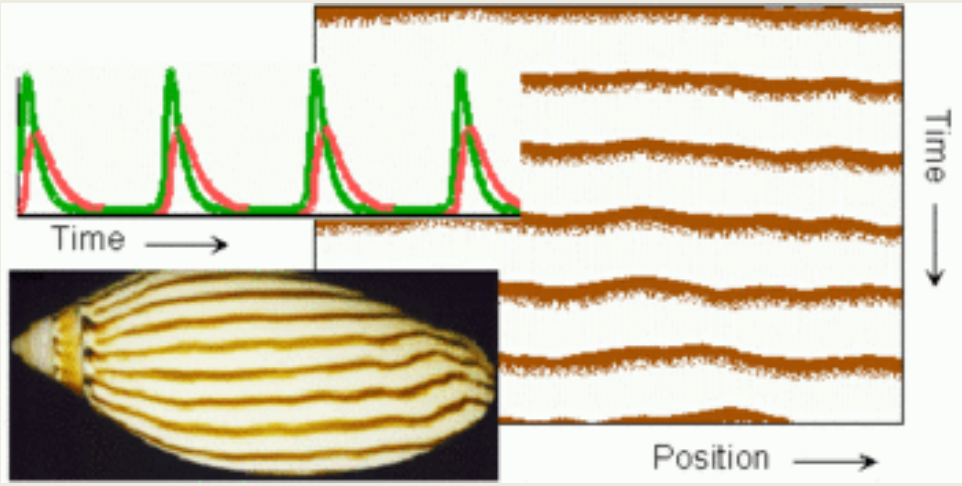
$$\frac{\partial a}{\partial t} = D_a \Delta a + \rho_a \frac{a^2}{(1+\kappa_a a^2)h} - \mu_a a + \sigma_a$$

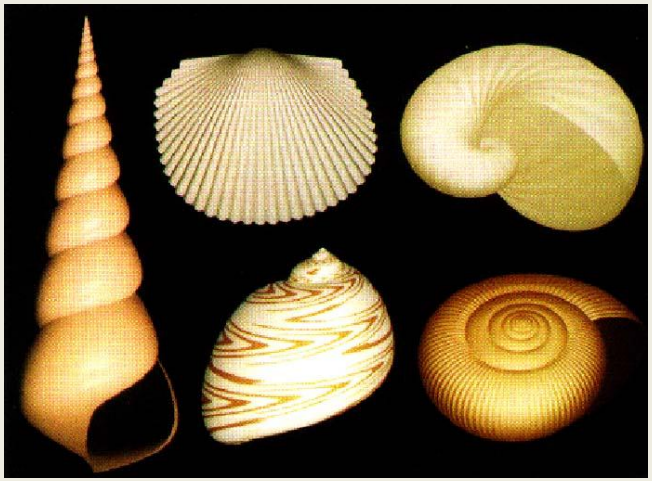
$$\frac{\partial h}{\partial t} = D_h \Delta h + \rho_h a^2 - \mu_h h + \sigma_h$$

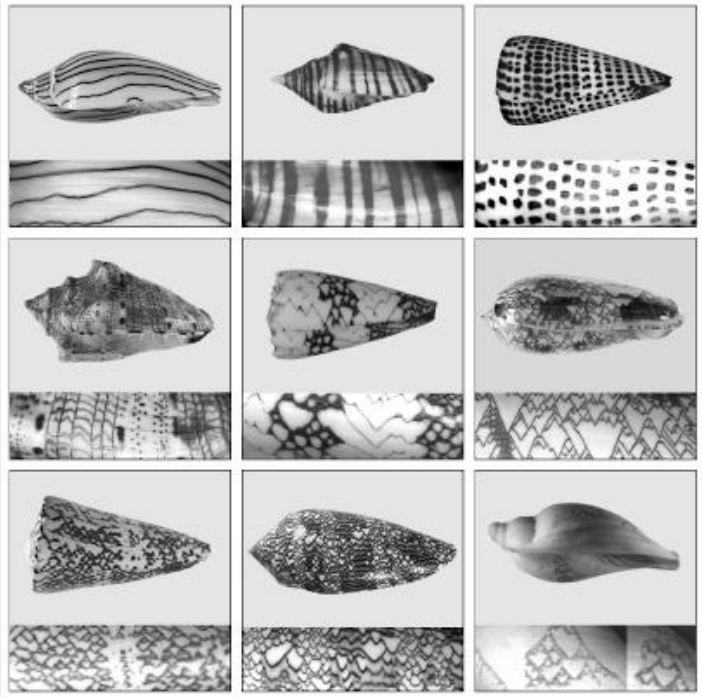


$$\kappa_a > 0$$

Results: Often Surprising!







Chemical Reactions

Chemical reactions depend usually on continuous densities.

However, sometimes, a "caricature" of a chemistry with chemicals either present or absent is already sufficient to demonstrate important properties (see lecture on self replication)

Note: On very short scales, chemistry is discrete. By using CA, we "undo" the averaging process of conventional kinetics.

Dynamical Systems - A Formal Way to Represent Complex Systems

Zurich University
of Applied Sciences
School of Engineering
ZAMP Center for Applied
Mathematics and Physics

Some Formal Definitions

Dynamical Systems

- A dynamical system is a 3-tuple (T,M,Φ), where
- T is the set of values the time parameter can take (for mathematical purists: an additively written monoid),
- M is a set containing the possible states (the state space) of the dynamical systems
- Φ is a function with

$$\Phi : T \times M \to M \begin{cases} \Phi(0, x) = x \\ \Phi(t_2, \Phi(t_1, x)) = \Phi(t_2 + t_1, x) \end{cases}$$

Time: Flows and Maps

Basically (T,M,Φ) produces a series of maps of M onto itself.

Given an initial condition $x_0 = x(0)$, a deterministic trajectory $x(t)$, $t \in T$ is produced by (T,M,Φ).

Note that for our purposes, T is either a interval in the real numbers $\mathbb{R}$ or the natural numbers $\mathbb{N}$. In the former case, time is continuous and one speaks of a **flow**, in the latter discrete and one calls it a **map**

Dynamical System = Time + States + Determinism

States: Finite States, Vectors, Functions, ….

What is M? Well, basically just a set, there are almost no restrictions. It can be:

- **A finite set**, e.g. leading to a finite state automaton.
- **A subset of $\mathbb{N}^n$, $\mathbb{R}^n$ or $\mathbb{C}^n$**, leading to a sequence of vectors, means a curve or trajectory in the according space.
- **A set of functions f(x),** leading to a parameterized function f(x,t). A prominent example is a probability distribution that evolves in time: **ρ(x,t).**
- ….

How Is the Dynamics Described?

Very many dynamical systems can be expressed as sets of ordinary **differential equations (ODE):**

$$\begin{aligned} \dot{x}_1 &= f_1(x_1,\ldots,x_n;\lambda_1,\ldots,\lambda_m) \\ \vdots &= \vdots \\ \dot{x}_n &= f_n(x_1,\ldots,x_n;\lambda_1,\ldots,\lambda_m) \end{aligned}$$

Discrete systems are given by **iterations:**

$$x_n = f(x_{n-1}, x_{n-2}, \ldots, x_{n-m})$$

The flows and maps are global, but here, their description is local. The big technical question is how to „integrate“ a local description in order to get a global flow.

Cool Notation

- The cool guys use the notation $\Phi_t x_0$.
- They then write $\Phi_s \Phi_t x_0 = \Phi_{s+t} x_0$.
- $\Phi_t x_0$ is an element of the state space. Φ_t transports x_0 through the state space M.
- Advantages:
  - It doesn't yield nested expressions.
  - "Group structure" becomes apparent.

More Variants (Examples, don‘t learn them!)

There are many more variants to describe dynamical systems:

- Delay equation (take into account „historic“ events)

$$\dot{x}(t) = f(x(t-\tau))$$

- Integro-differential equation (account for „history“)

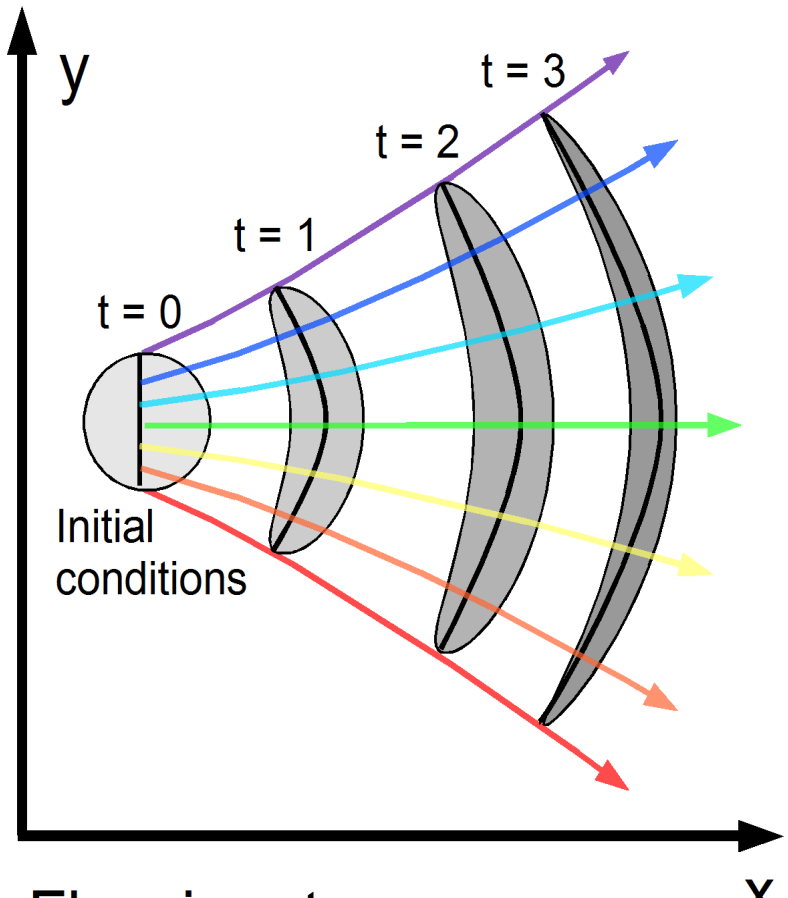
$$\dot{x}(t) = f(x(t)) + \int_{t'=t_0}^{t'=t} f(x(t'))dt'$$

- Partial Differential Equations (PDE) (= infinitely many ODE)

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi(x,t) = \frac{\partial^2}{\partial x^2}\varphi(x,t)$$

Flows and Maps

A Flow in Two Dimensions



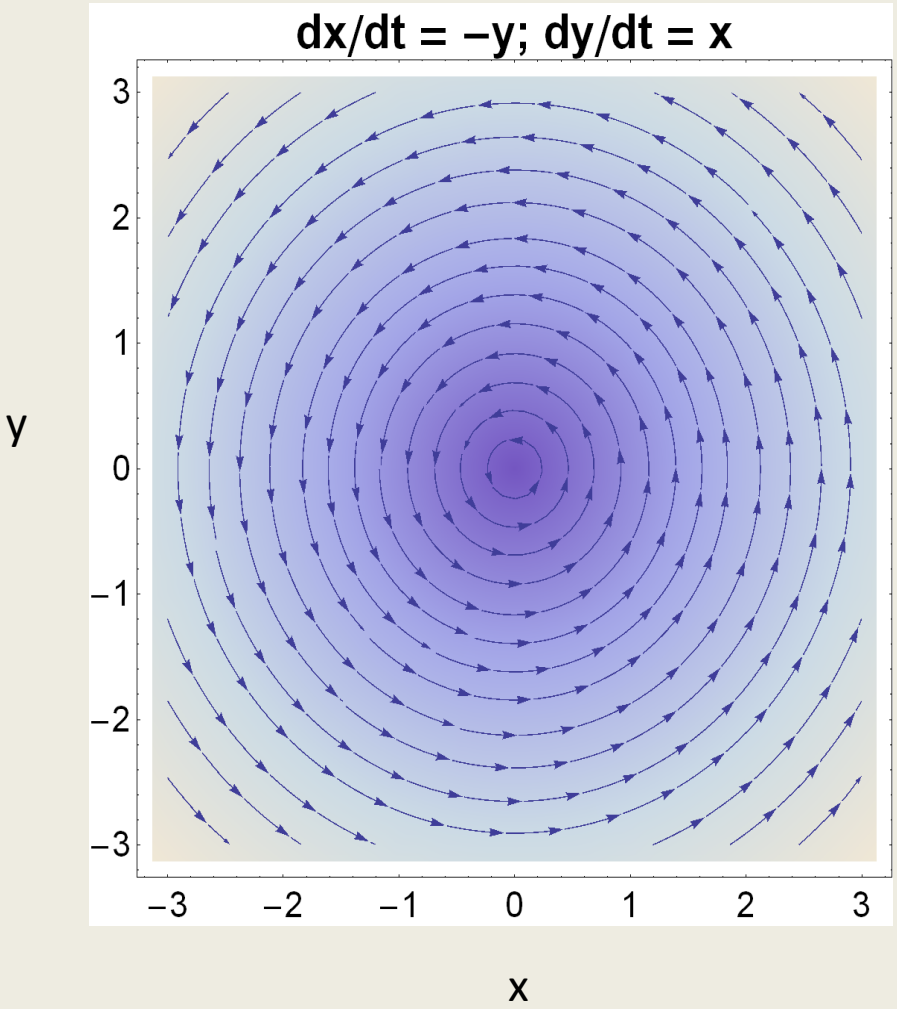
Flow in a two-dimensional system

A set of initial conditions, here a circle is transported through M by a flow.

The flow deforms the initial region, but for each and every initial condition we have a trajectory.

If the system dissipates energy, these areas will shrink.

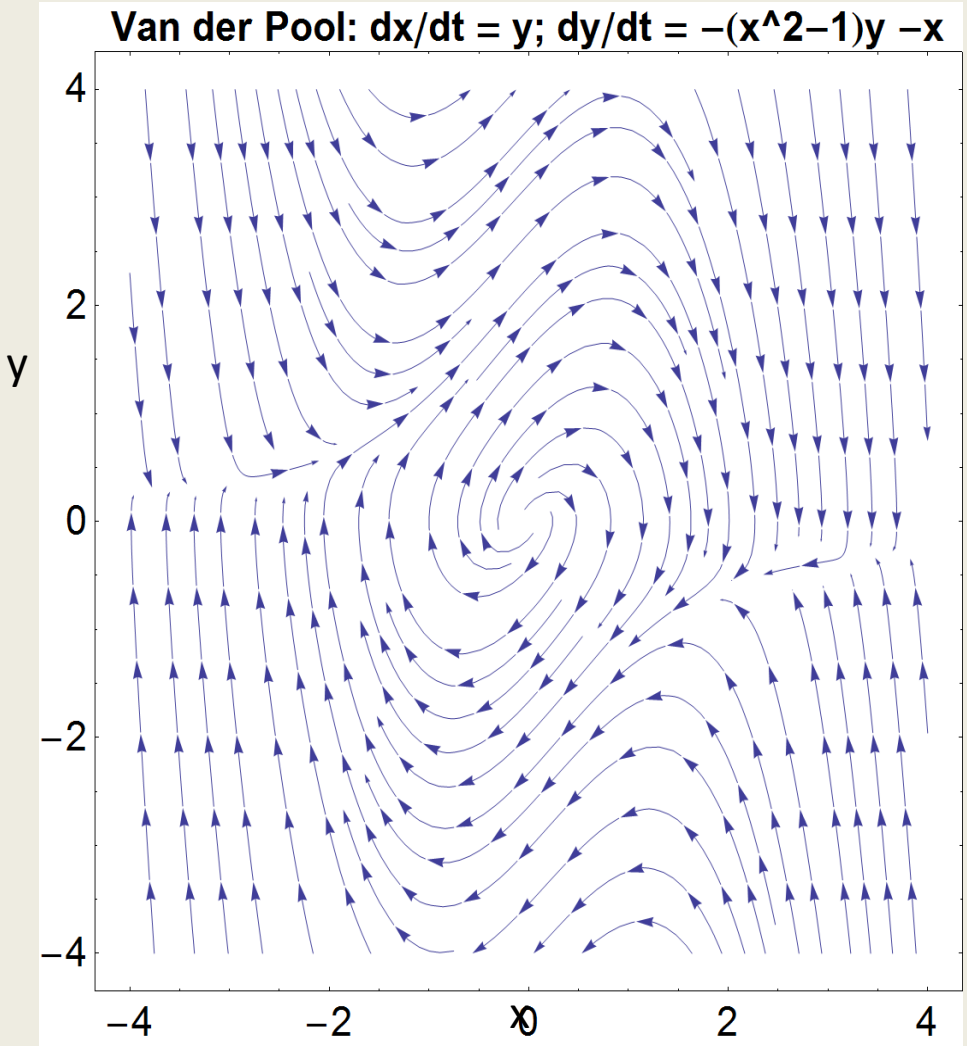
Visualization of a Two-Dimensional Flow 1



$$\dot{x} = -y$$

$$\dot{y} = x$$

Visualization of a Two-Dimensional Flow 2



$$\dot{x} = y$$

$$\dot{y} = -(x^2 - 1)y - x$$

Why Maps and Flows?

- Basic assumption: Complex behavior results from “simple” physics.
- Simple physics requires **that the future depends only on the present (the actual state**).
- Parameters may represent observable quantities (e.g. stimuli).
- **Iterations and systems of ODE doesn’t describe Φ in the sense of a blue print. They are rather developmental descriptions for the construction of Φ.**

Deterministic Micro – Stochastic Macro

- Physics requires that the future depends only on the present (the actual state).
- However, the actual state may depend on a large number of variables, a **microscopic state** (e.g. positions and velocities of all molecules). A dynamical system may work with averages, a **macroscopic state** (e.g. weather or cell models).
- Then, there are many microscopic states leading to the same macroscopic state. Dynamics depends on microscopic state: Macroscopic dynamics may become stochastic.

Quantitative and Qualitative Analysis of Flows

Zurich University
of Applied Sciences

School of
Engineering

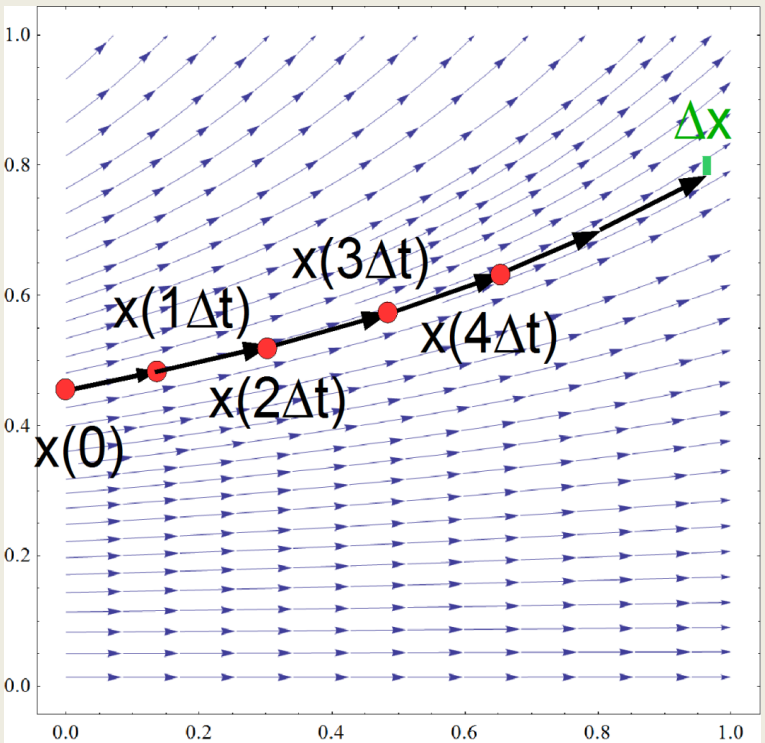
ZAMP Center for Applied
Mathematics and Physics

How To Get the Global Flow In General?

- Since the definition of a dynamical system is very open, there is no “general” receipt for the construction of the global flow/map Φ.
- Depending on
  - T (the “time”) and M (the states)
  - The way $\boldsymbol{\Phi}$ is described

  many integration schemes exist.
- Most often, Φ cannot easily be described globally. We must search for sensible, but less detailed information.

A Glimpse of Numerical Analysis



$$x((n+1)\Delta t) = x(n\Delta t) + f(x(n\Delta t))\Delta t$$

Illustration of **Euler method** for the integration of ODE: How to get a global flow from a local ODE?

The local information provided by the ODE tells you in what direction and with what speed you have to go on.

On a computer, you can't vary the speed/direction continuously, you have to dived your journey in little steps. The smaller the step size, the smaller Δx.

There are much better methods, your video game e.g. uses probably **Runge-Kutta 4th order**.

Important: Computers can only deal strictly with discrete systems. But many continuous systems can be approximated by discrete ones.

If We Can’t Get the Global Flow

- Instead of asking for the global flow, we may only be interested in the state(s) a system reaches after it has gone through an initial transient.
- Mathematically, we are interested in the attractors of the flow.
- Such attractors may be single pointsd

Attractors

Definition: An **attractor A** of a dynamical system (T,M,Φ) is a subset of $\mathbb{R}^n$ such that

1. **A** is an **invariant set**, i.e. any trajectory that starts on remains on **A**.
2. **A** attracts an open set of initial conditions ***U***, i.e. there is an open set ***U*** such that if $x(0) \in U$ the distance between $x(0)$ and **A** vanishes for $t \to \infty$. The maximal ***U*** with this property is called the **basin of attraction**.
3. **A** is **minimal**, i.e. there is no proper subset of satisfying conditions 1 and 2

Don't learn this! It is presented to state that the term "attractor" is well defined.

Properties of Attactors

- Simple attractors are just points.
- There are more interesting variants.
- The dimensionality of the system has an influence on what type of attractor can appear.
- Some systems with seveal attractor can exhibit chaotic behavior. Chaos is exactly the opposite of robustness and should be avoided.
- High sensibility to external stimuli is not chaotic dynamics.

Geometric Interpretation of ODE

Besides analytical and numerical solutions of differential equations, the geometric approach offers a further, very efficient possibility for the studying dynamical systems. With the help of a computer, it is very easy to draw vector fields. Using your personal biological visual system, it is very easy to "integratde" differential equations to and to get insight into the dynamical behavior of a system.

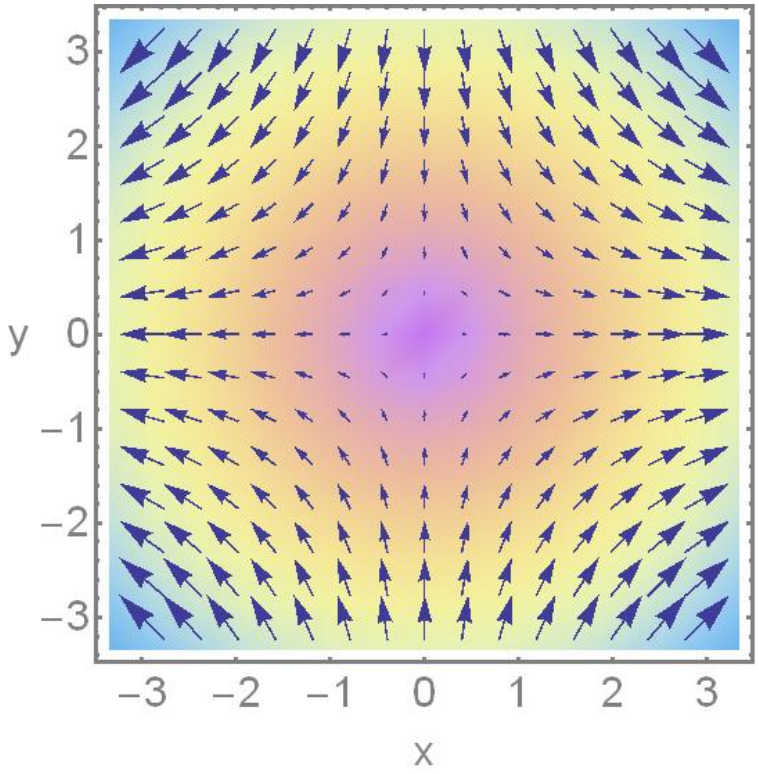
Vectorfields

- Gegeben sei ein System von Differentialgleichungen

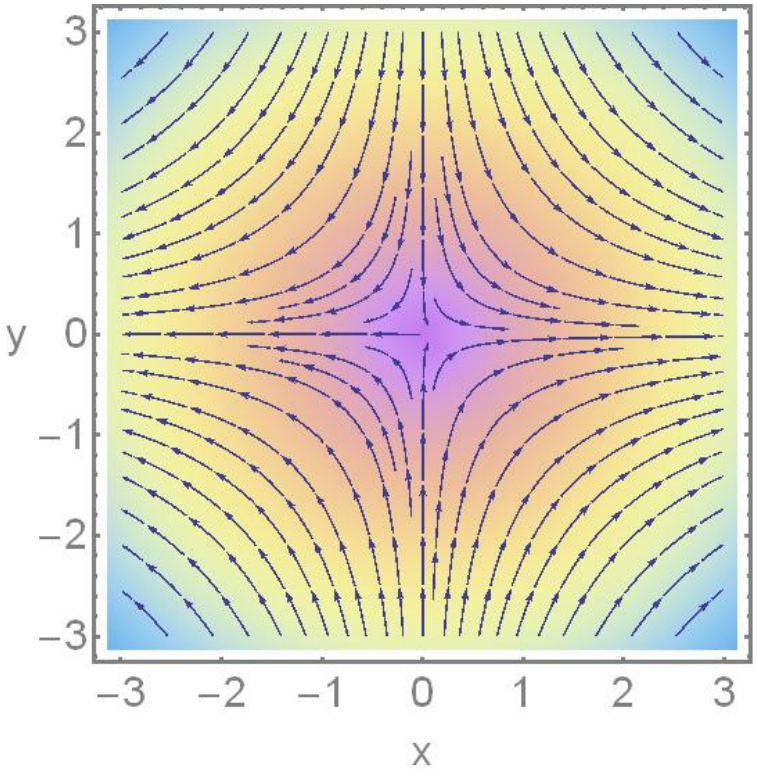
$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

- Ein Plot, in welchem die Geschwindigkeitsvektoren der Trajektorien angegeben werden, heisst das **Vektorfeld** oder das Phasenplot der **Differentialgleichung**.
- Ein Plot in welchem nur die Richtung, nicht aber die Grösse der Geschwindigkeitsvektoren einer Differentialgleichung gezeigt werden, heisst Richtungsfeld.

Richtungs- und Vektorfelder

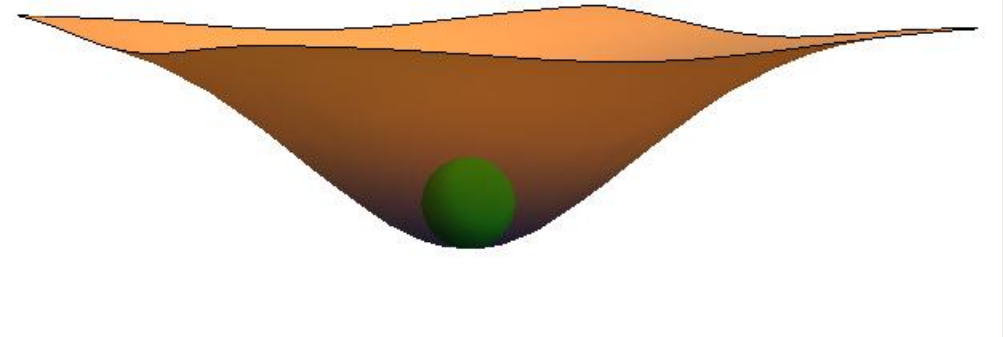


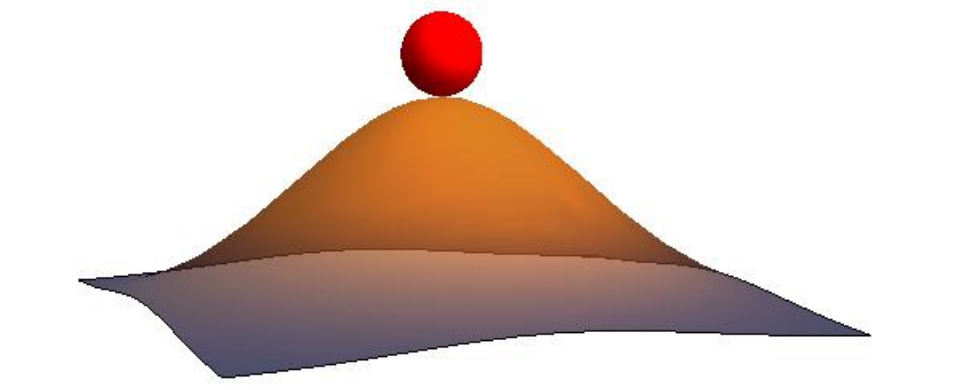
Phaseplot, vektorfield

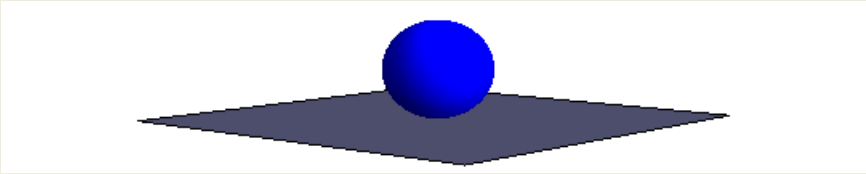


Directional field

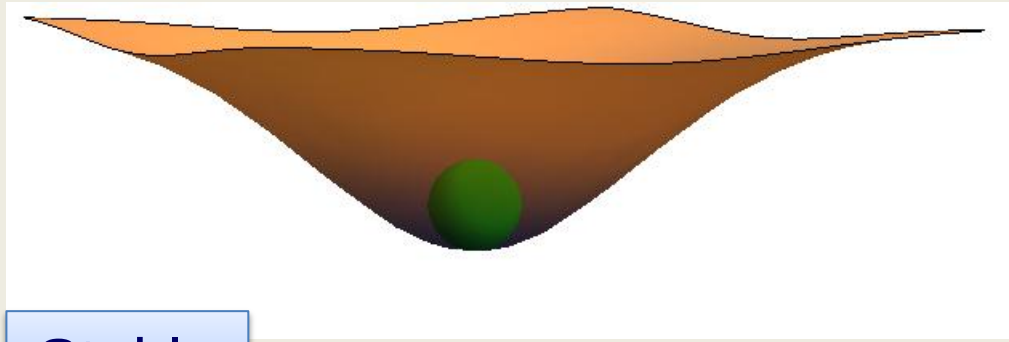
Stable – instable - Indifferent



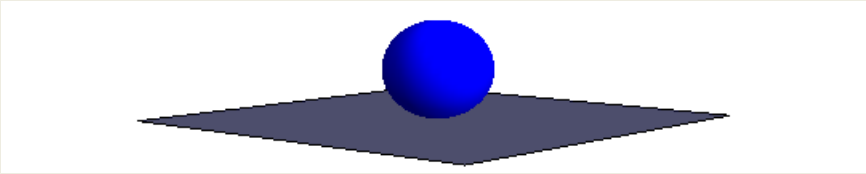




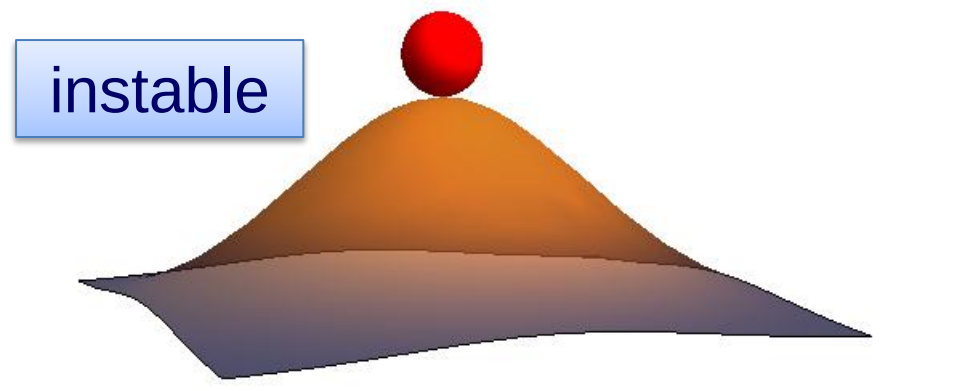
Stable – Instable - Indifferent



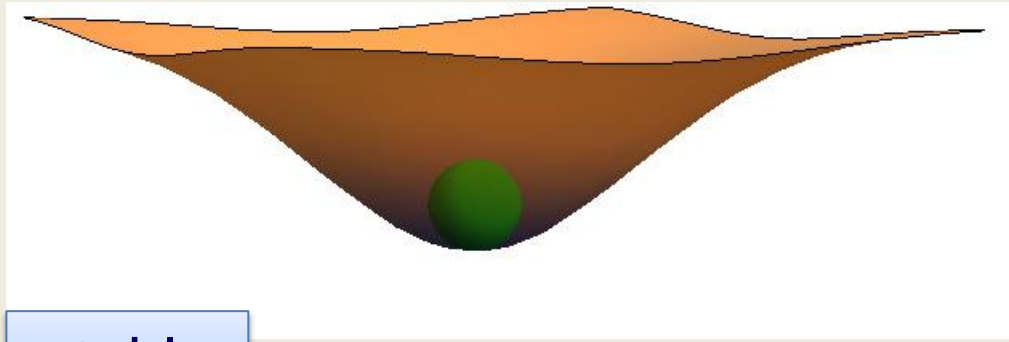
Stable



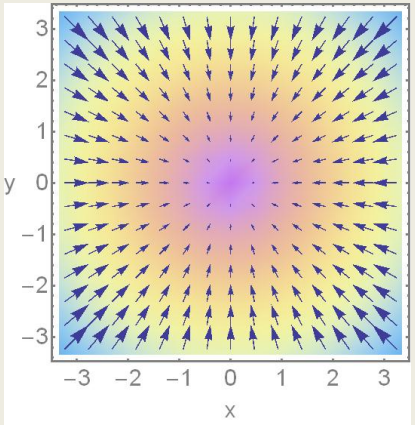
Indifferent



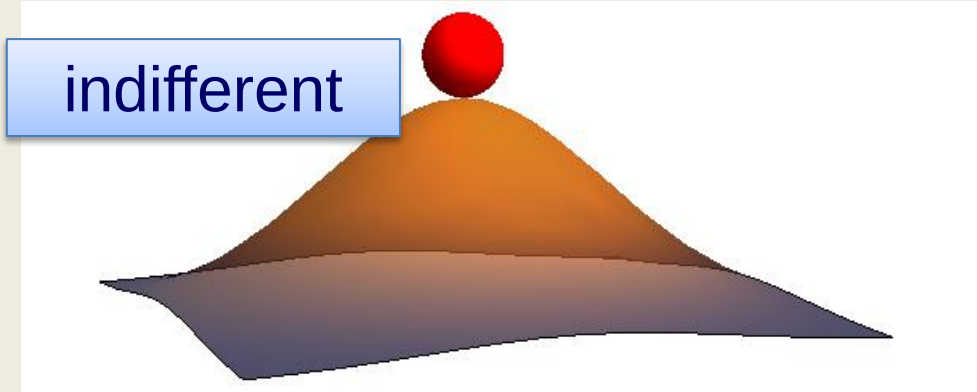
Stabil – Labil - Indifferent

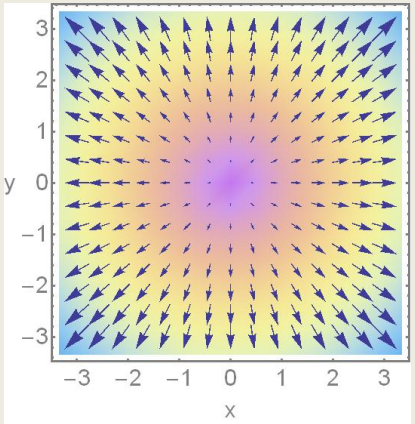


stable

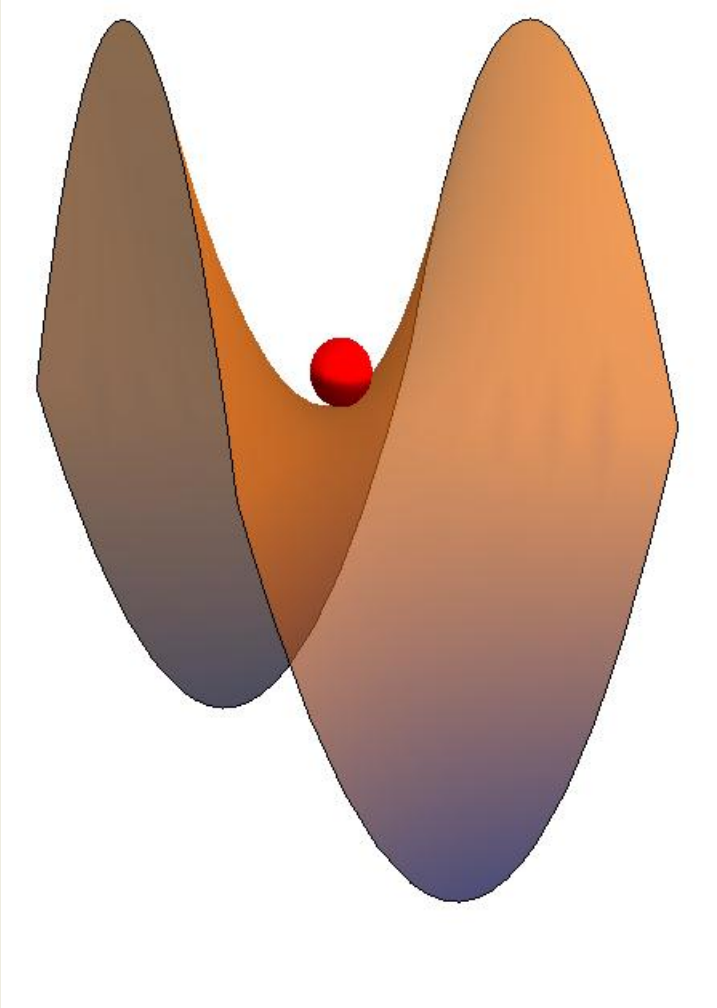


indifferent



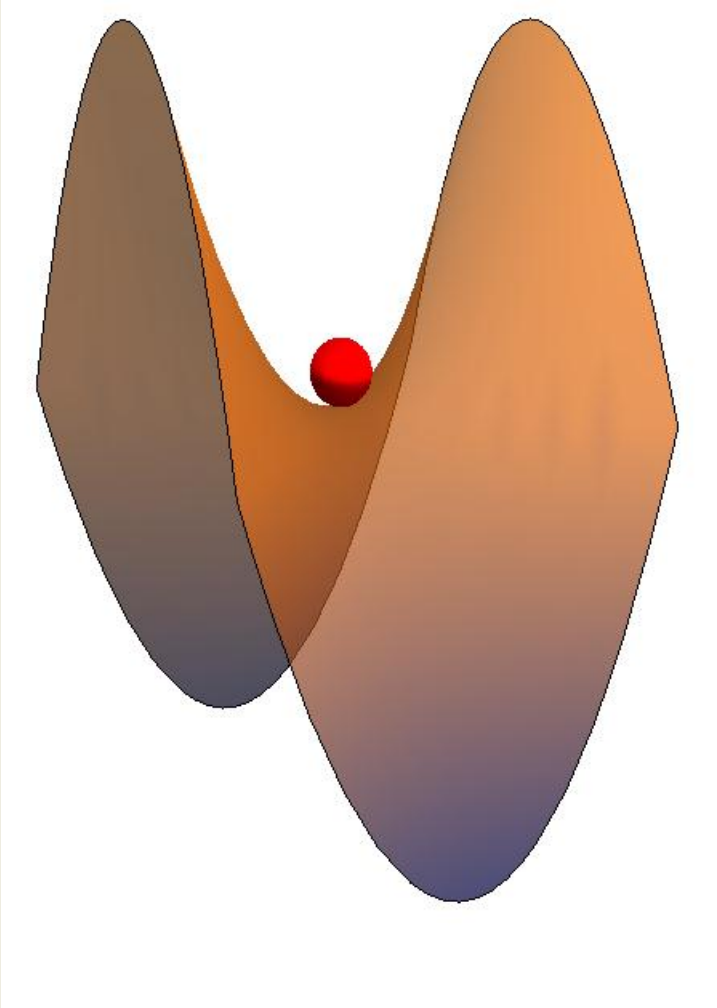


Saddle point: Stable or instable?

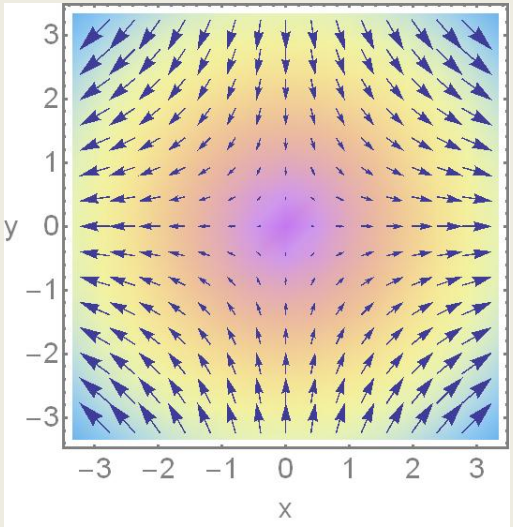


A "saddlepoint" is a point in which the surface goes upwards in one direction and downwards in another one.

Saddlepoint: stable or instable?



instable!

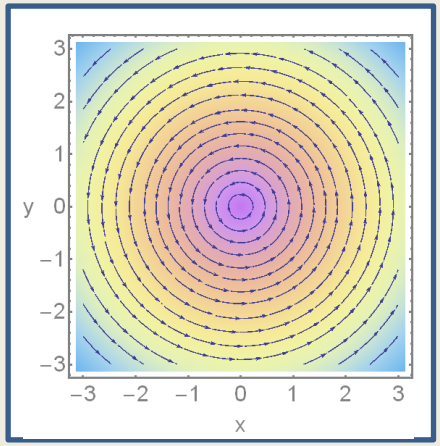


Sometimes useful, sometimes misleading

- The picture of the "hilly landscape" is sometimes very useful, especially if one deals with systems that can be described by some form of energy function.
- There are dynamical systems (many in fact) which cannot be described that way,

Ein gefährliches aber nützliches Bild

- Wenn wir davon ausgehen, dass ein Geschwindigkeitsfeld durch das "Hinabrollen" eines Balls entsteht, welches von uns bereits untersuchte System lässt sich dann nicht darstellen?

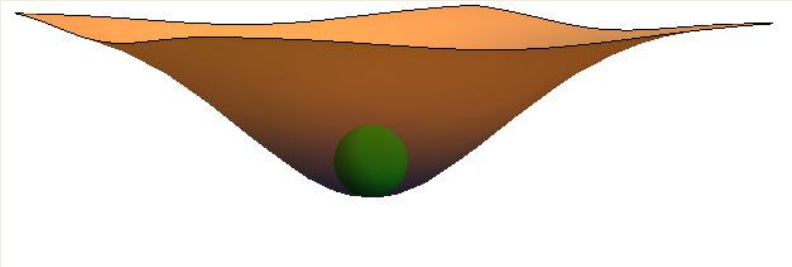


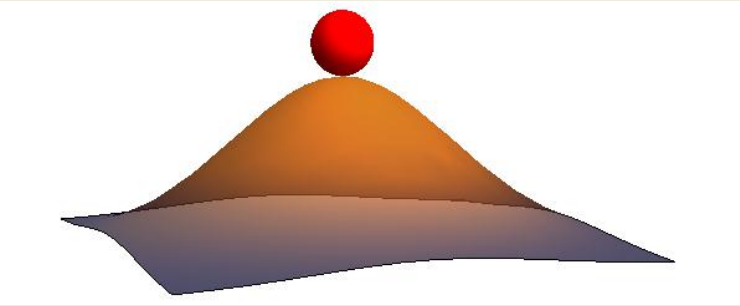
$$\dot{x} = -y$$
$$\dot{y} = x$$

Ein "Drehfeld" würde verlangen, dass es immer abwärts geht, wenn Sie im Gegenuhrzeigersinn um den Nullpunkt laufen. Das kann es nicht geben.

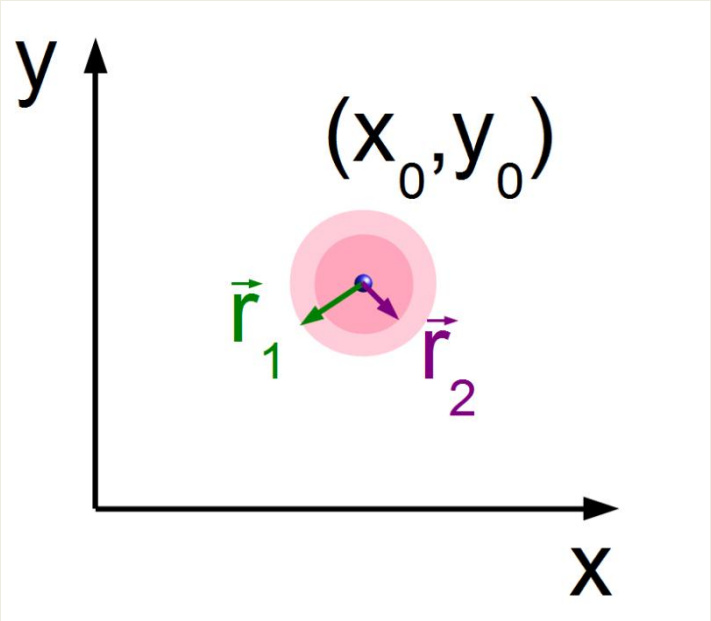
Ein gefährliches aber nützliches Bild

- Das irreführende an der "Hügel- und Tal"-Vorstellung ist, dass ein physikalischer Ball Position und Geschwindigkeit hat, währendem unsere Differentialgleichung jedem Punkt eine Geschwindigkeit zuordnet.
- Das passende am "Hügel- und Tal"-Bild ist, dass wir sofort ein Verständnis von labilen und stabilen Systemen erhalten.





Nochmals: Umgebungen



Gegeben sei ein Punkt $P = (x_0,y_0)$ in der Ebene. Eine **ε-Umgebung von P** bezeichnet die Menge aller Punkte Q, für die gilt

$$\text{Distanz}(P,Q) < \varepsilon$$

Beachten Sie das "<" – Zeichen! Es geht nicht um "≤"!
Wenn man von ε-Umgebungen spricht, geht es normalerweise um sehr (beliebig) kleine ε.
Stellen Sie sich eine ε-Umgebungen als kleine Kreisscheibe oder (in 3D) als Kugel vor.

Zur Information

Stabilität – Etwas mathematischer

- Sei P ein Gleichgewichtspunkt und U(P,ε) bezeichnet eine ε-Umgebung von P.
- Falls für alle ε-Umgebungen U(P,ε) einen Punkt Q gibt so, dass
  - $Q \neq P$
  - Q in U(P,ε)
  - eine Trajektorie mit Startpunkt Q verlässt die Umgebung U(P,ε)

  ist P **labil**.

Zur Information

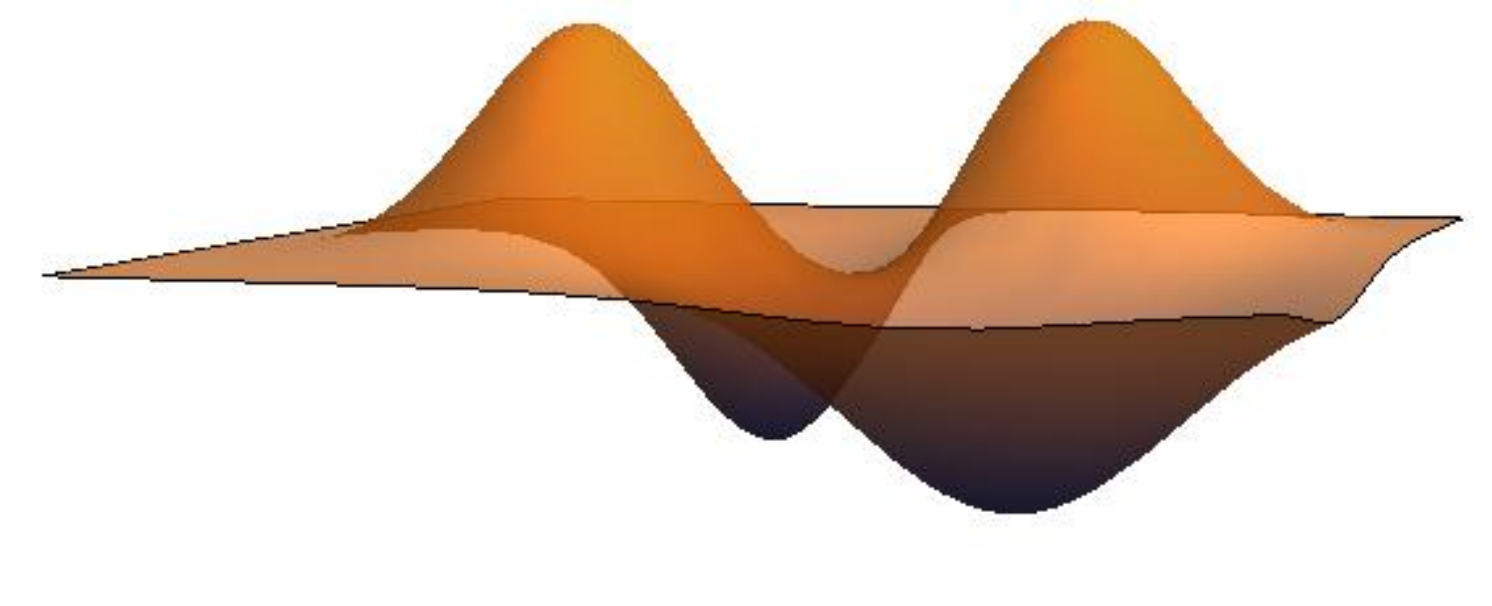
Stabilität – Etwas mathematischer

- Sei P ein Gleichgewichtspunkt und U(P,ε) bezeichnet eine ε-Umgebung von P.
- Falls für alle ε-Umgebungen U(P,ε) einen Punkt Q gibt so, dass
  - $Q \neq P$
  - Q in U(P,ε)
  - eine Trajektorie mit Startpunkt Q verlässt die Umgebung U(P,ε)

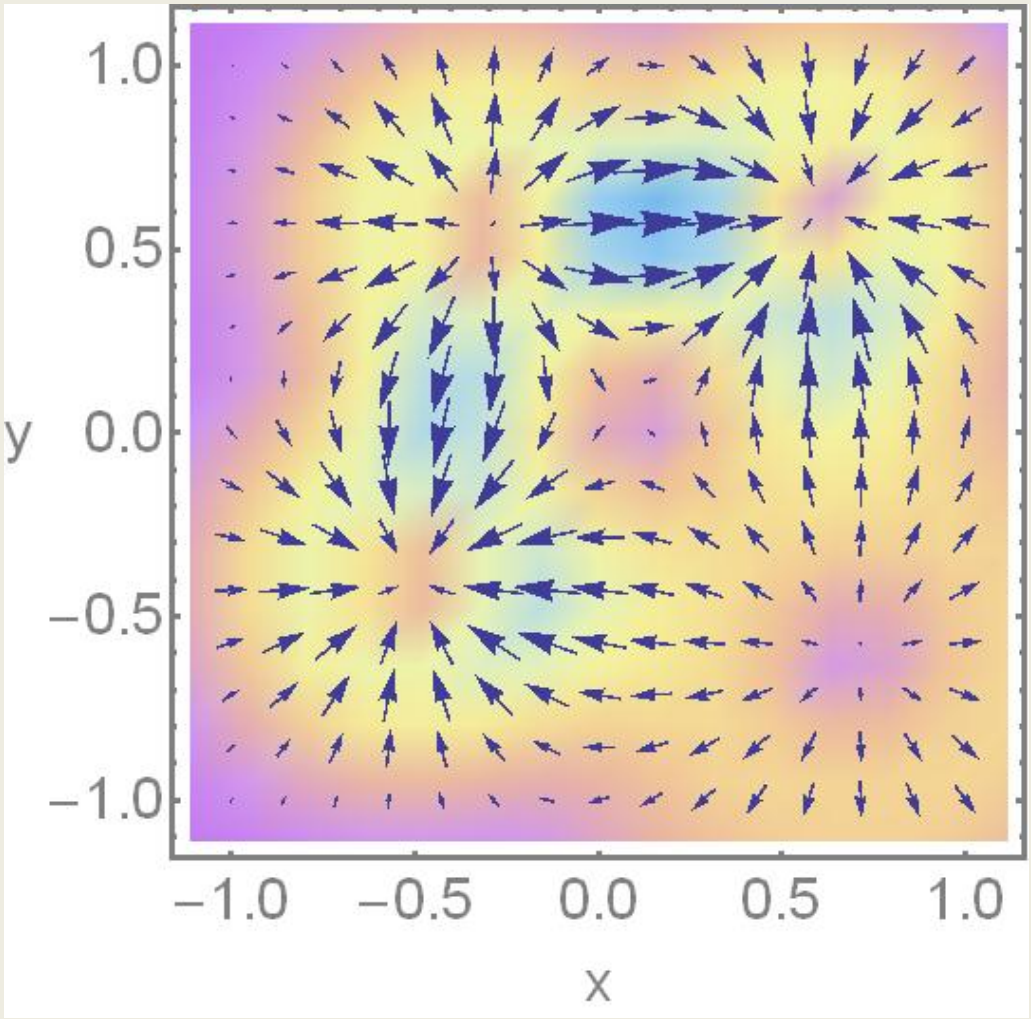
  ist P **labil**.
- Falls es ein ε_0 gibt, sodass für alle $\varepsilon < \varepsilon_0$ und alle Punkte Q in U(P,ε) eine Trajektorie mit Startpunkt Q immer näher zu P hin läuft, heisst P **stabil**.

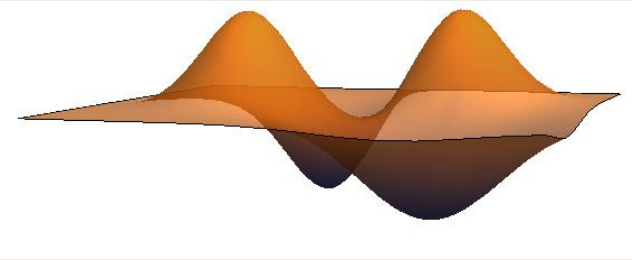
Zur Information

A Function

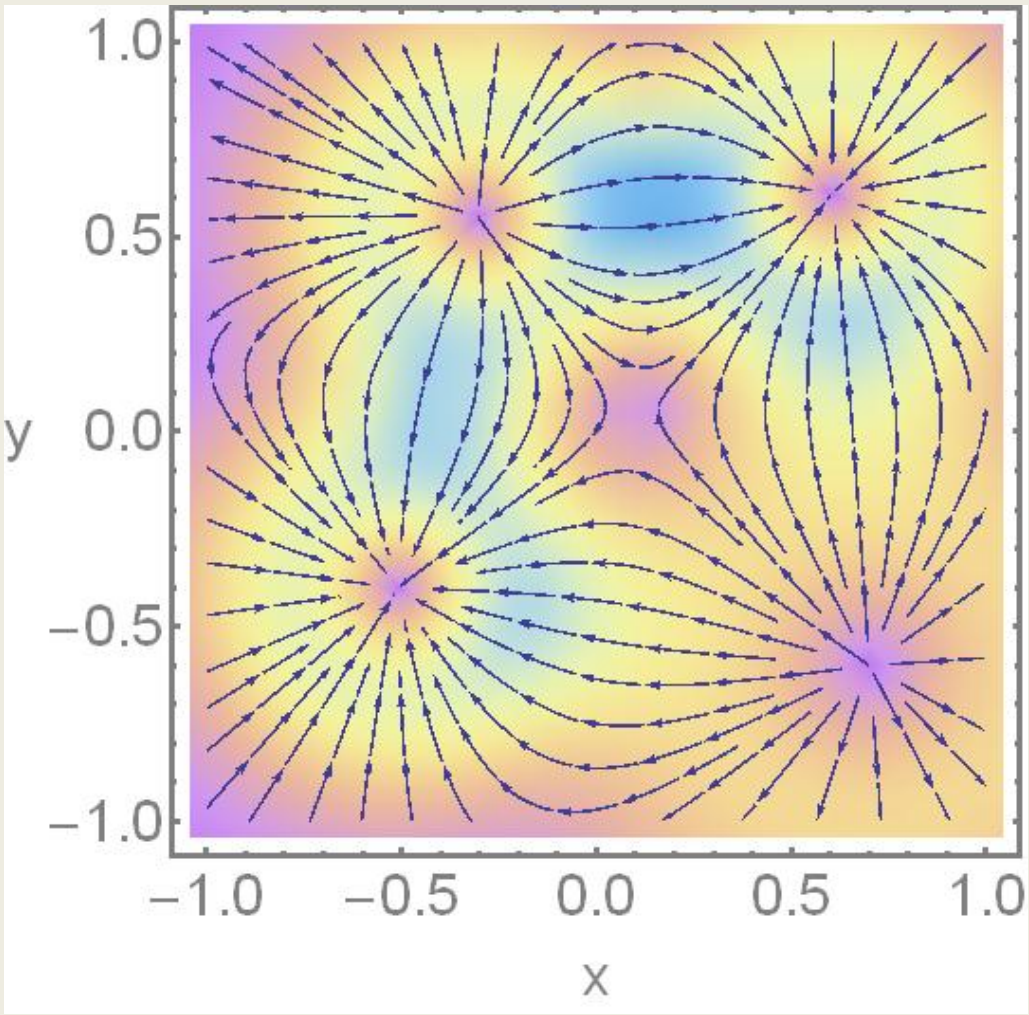


The According Vector Plot

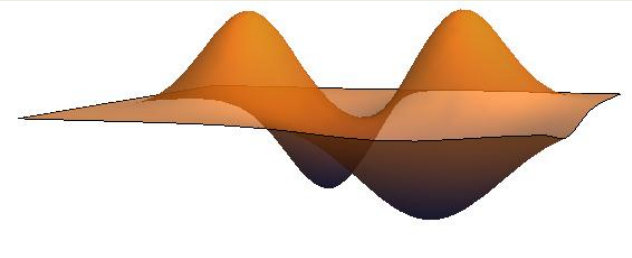




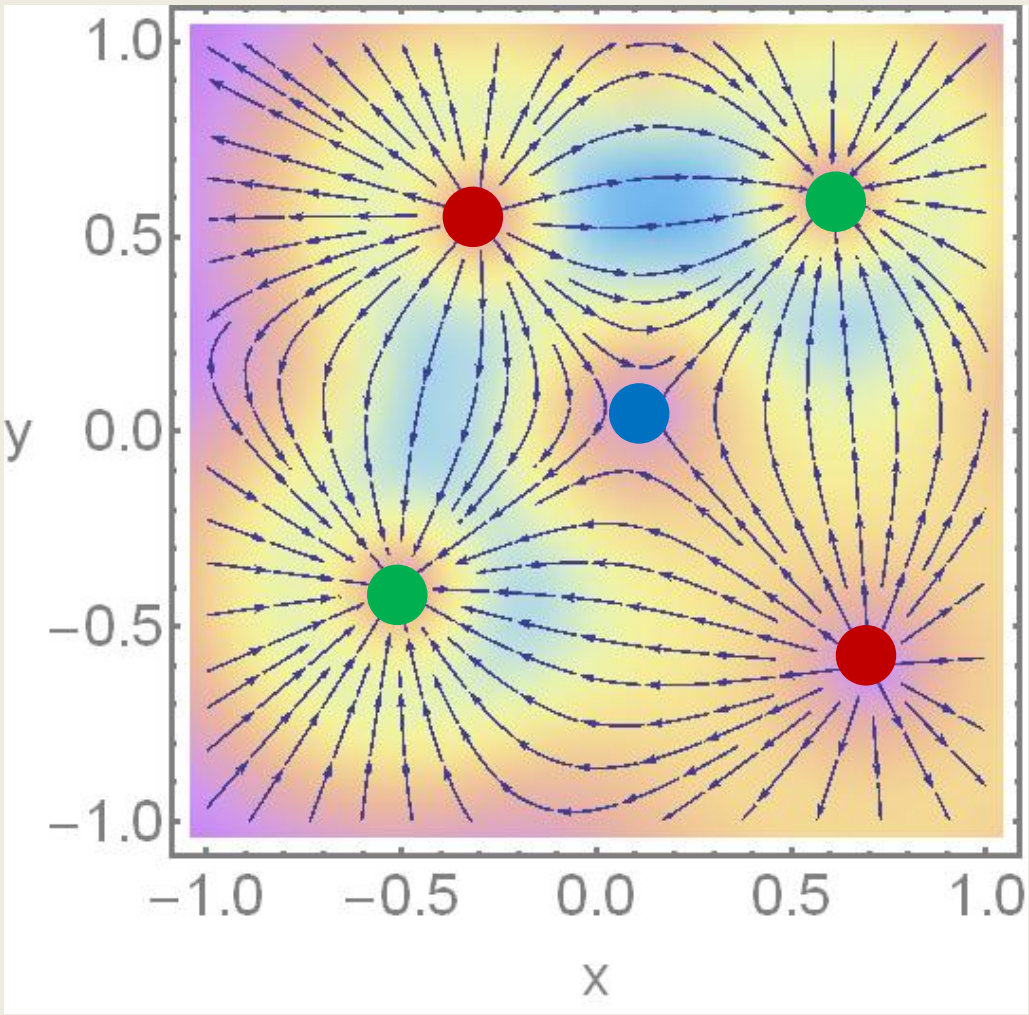
The According Streamplot



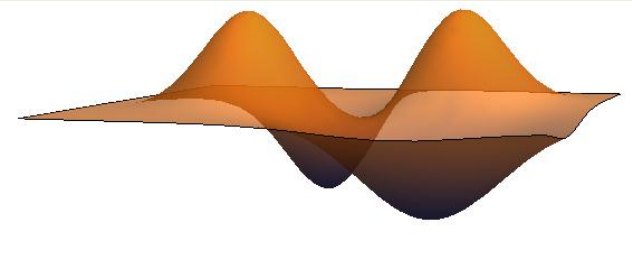
Where are the attractors, repellors and saddle points?



The According Streamplot

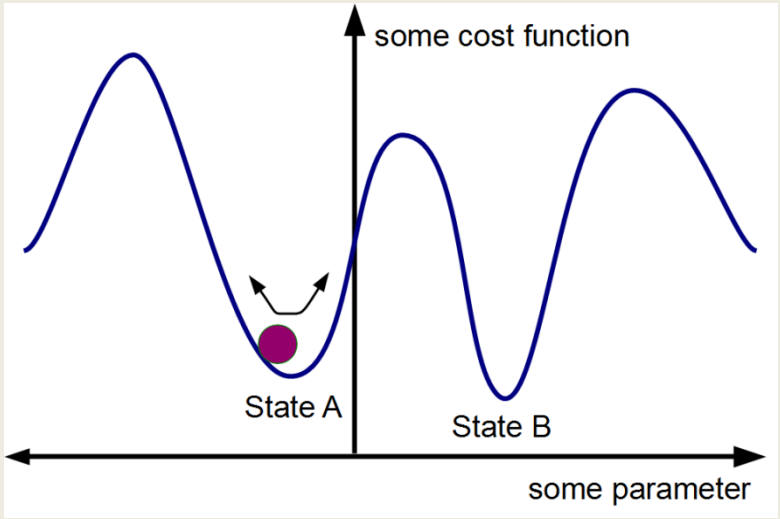


Attractor
Repellor
Saddlepoint

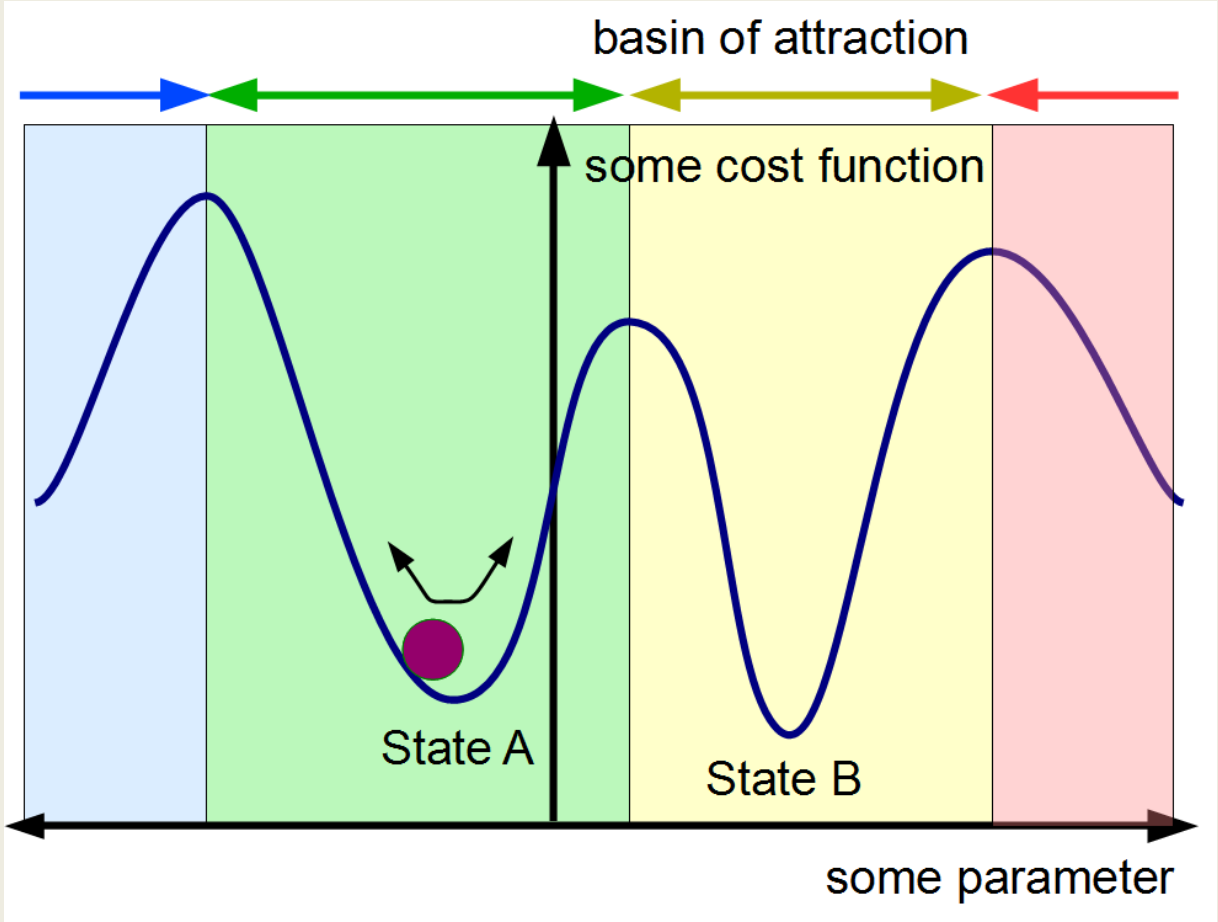


"Basin of Attraction"

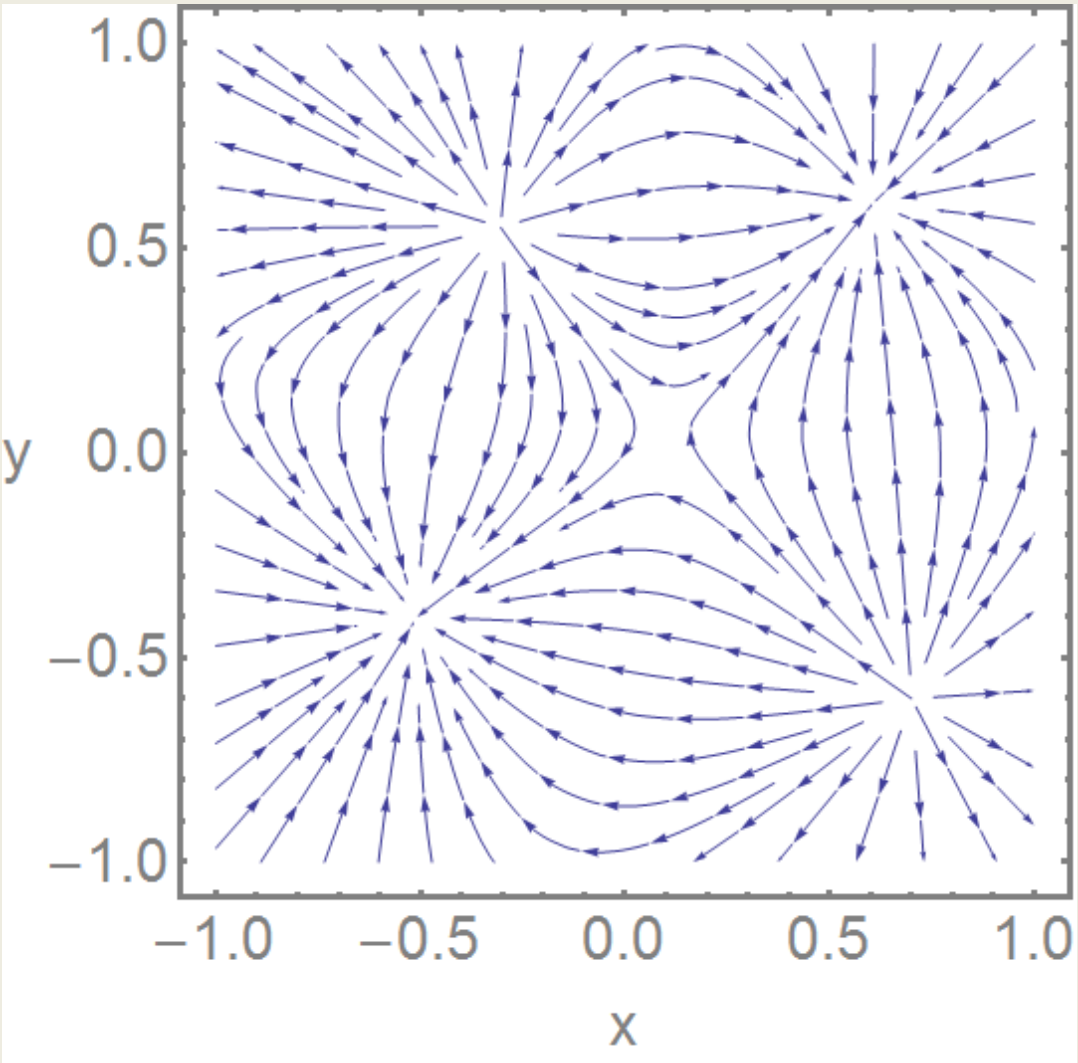
- In a hilly landscape, a ball will always roll into a valley
- There may be several valleys
- These points from which a ball will roll into one and the same valley are called the **"basin of attraction"** of that valley.



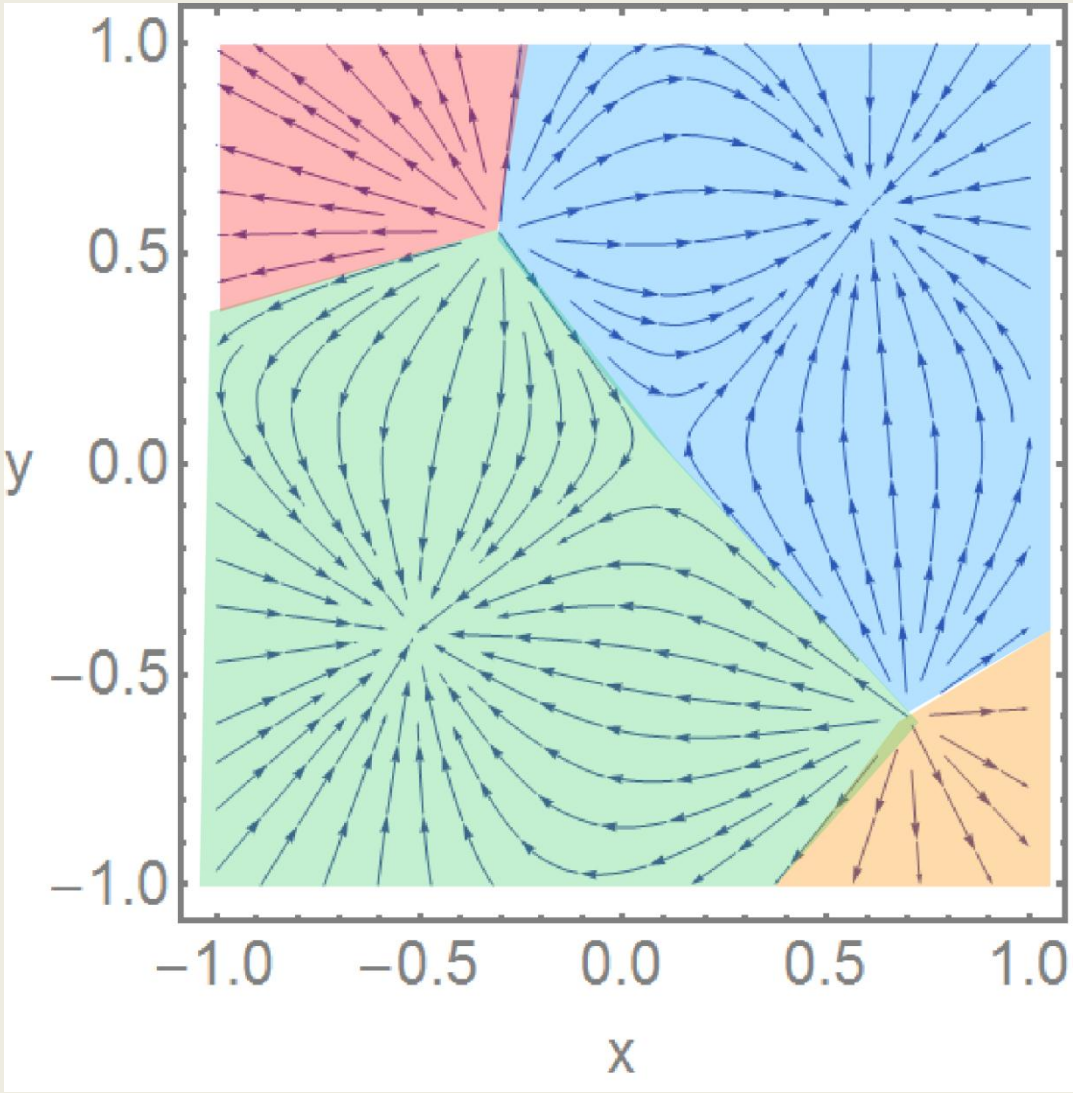
"Basin of Attraction"



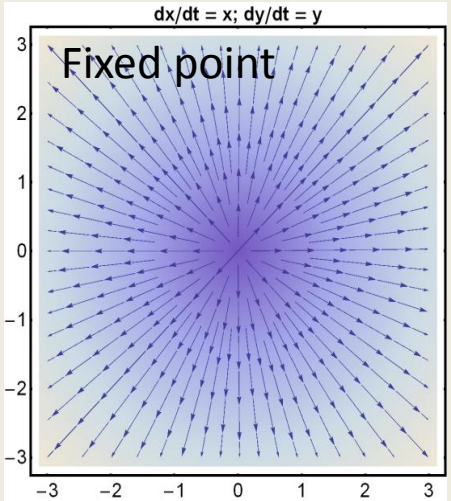
Basin of Attraction in two dimensions



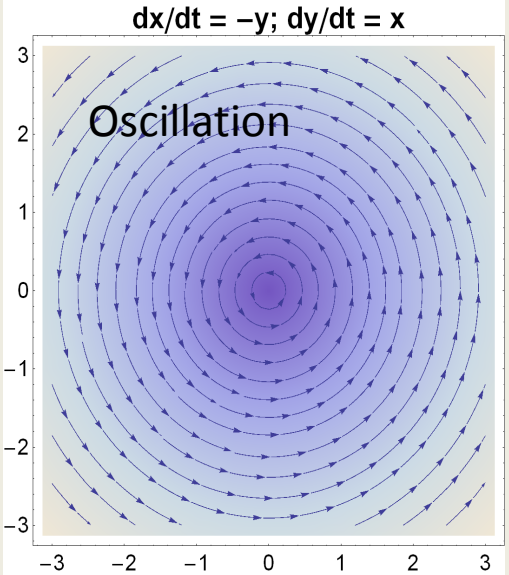
Basin of Attraction in zwei Dimensionen

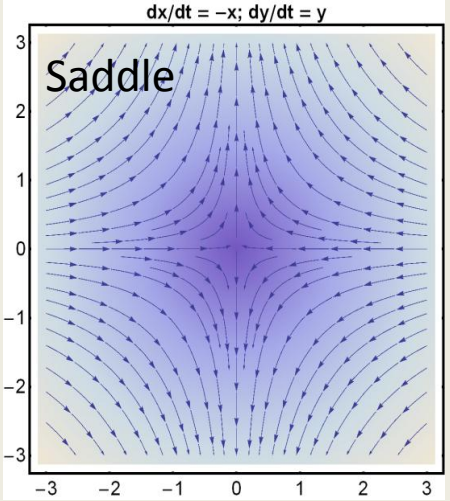


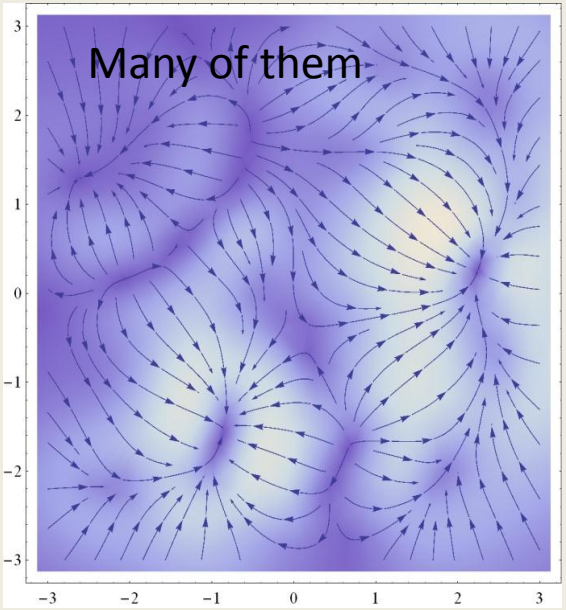
Fixed Points in Two Dimensions



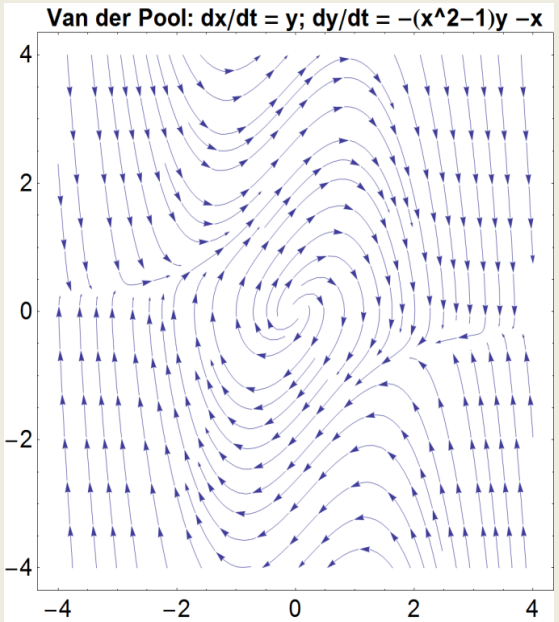
In two dimensions, we have a much richer phenomenology. Oscillations, saddle points. etc.

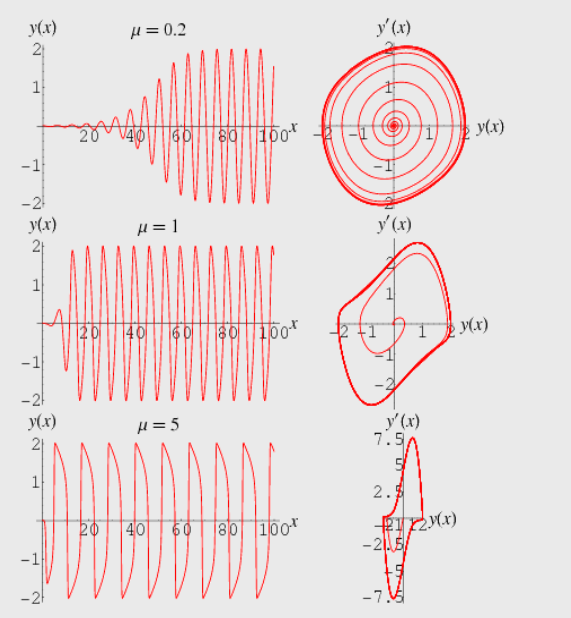






Fixed Points in Two Dimensions

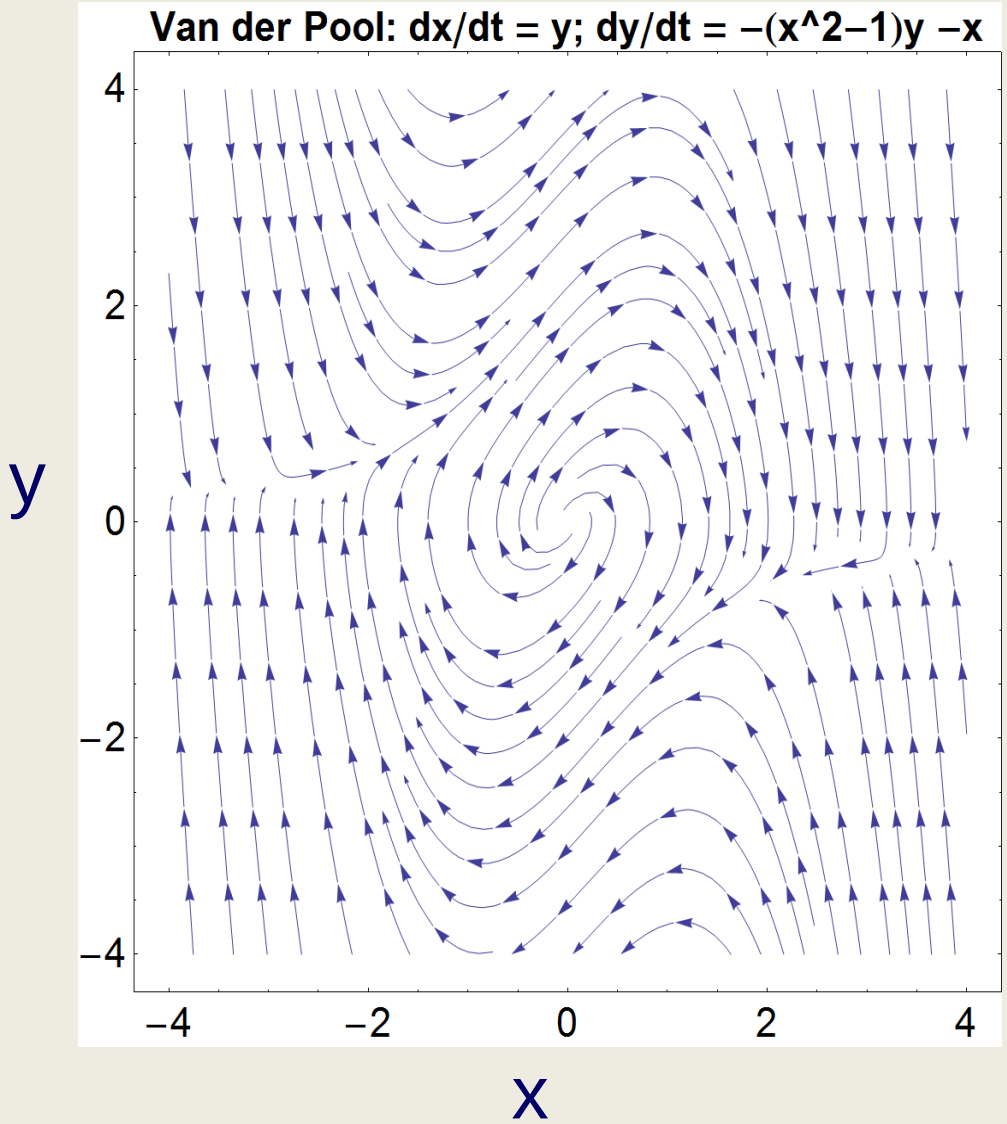




Besides oscillations and fixed points, two-dimensional systems can show **limit cycles.**

Theorem Poincare-Bendixon: In the plane, you can only have fixed points or closed orbits.

Visualization of a Two-Dimensional Flow



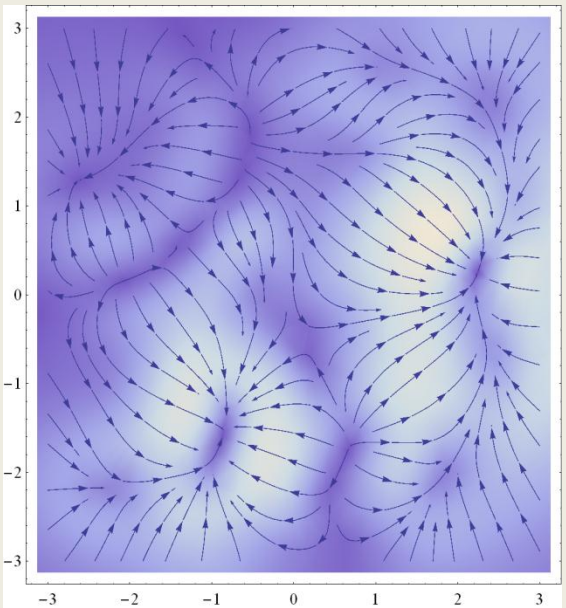
$$\dot{x} = y$$

$$\dot{y} = -(x^2 - 1)y - x$$

Attractors: Informal Definitions

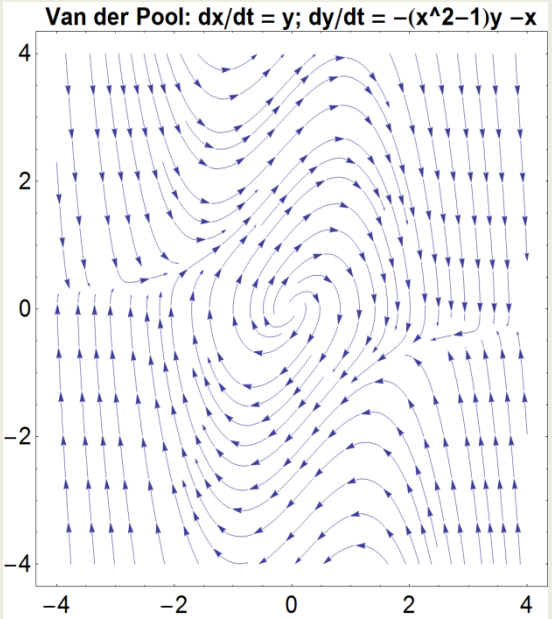
After sufficiently long time, many systems tend to attain only a very small subset of their potential states. This small subset **attracts** all other points.

x



y

x



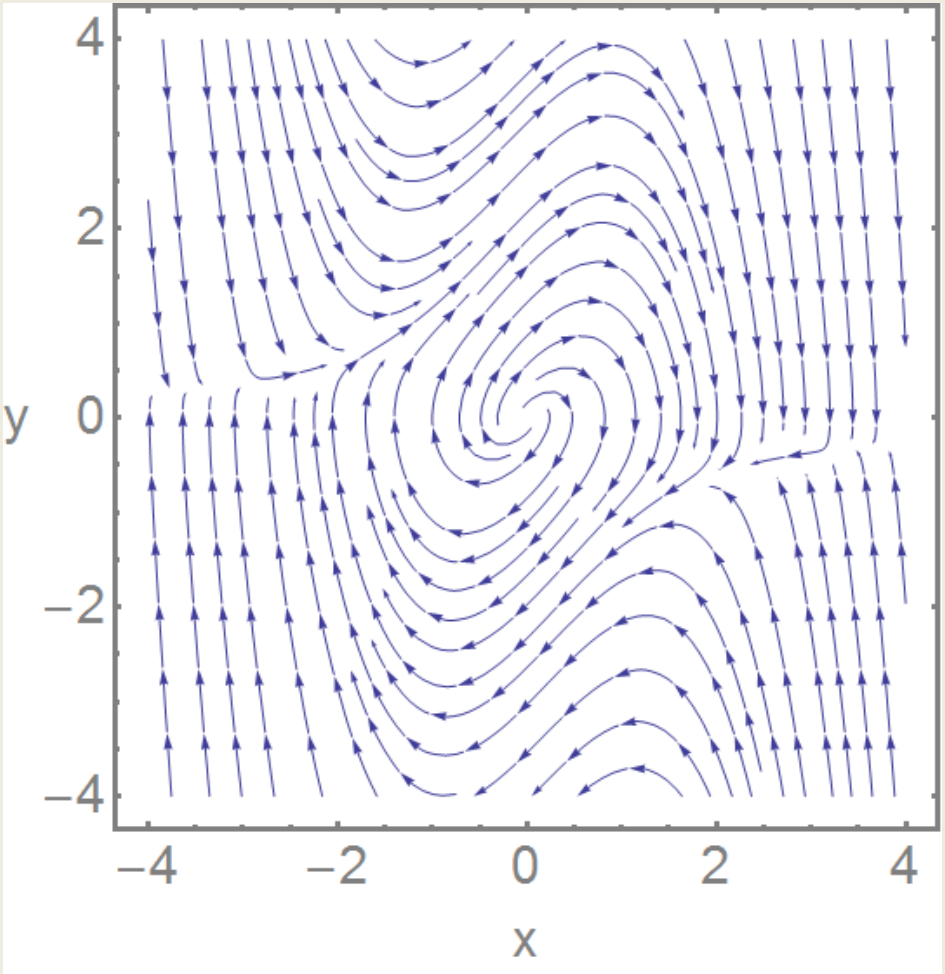
y

Limit Cycles

- Systems with two or morê dimensions need not to end up in an point attractor.
- They may eventually enter an orbit that goes on forever.
- Such an orbit is called a limit cycle.

Limit Cycles

- Do you see the limit cycle?

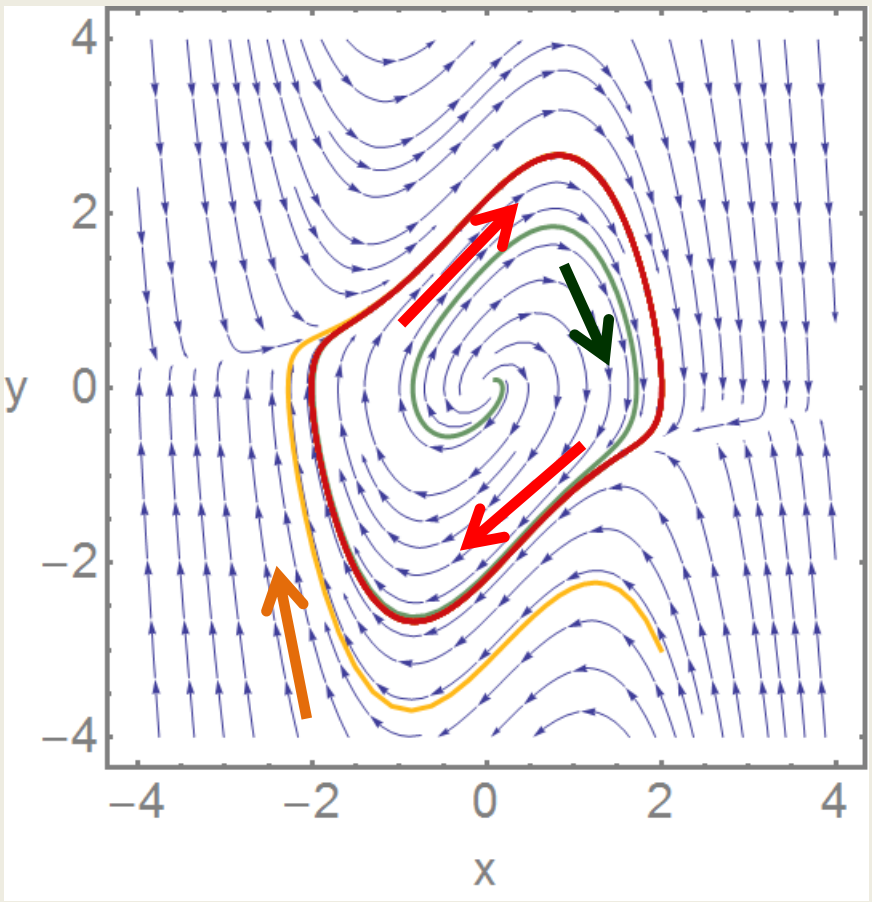


$$\dot{x} = y$$

$$\dot{y} = -y(x^2 - 1) - x$$

Van der Pol - Oszillator

Limit Cycles



$$\dot{x} = y$$

$$\dot{y} = -y(x^2 - 1) - x$$

Van der Pol - Oszillator

limit cycle
Transient from the interior
Transient from the exterior

Chaos

Strong and Weak Determinism

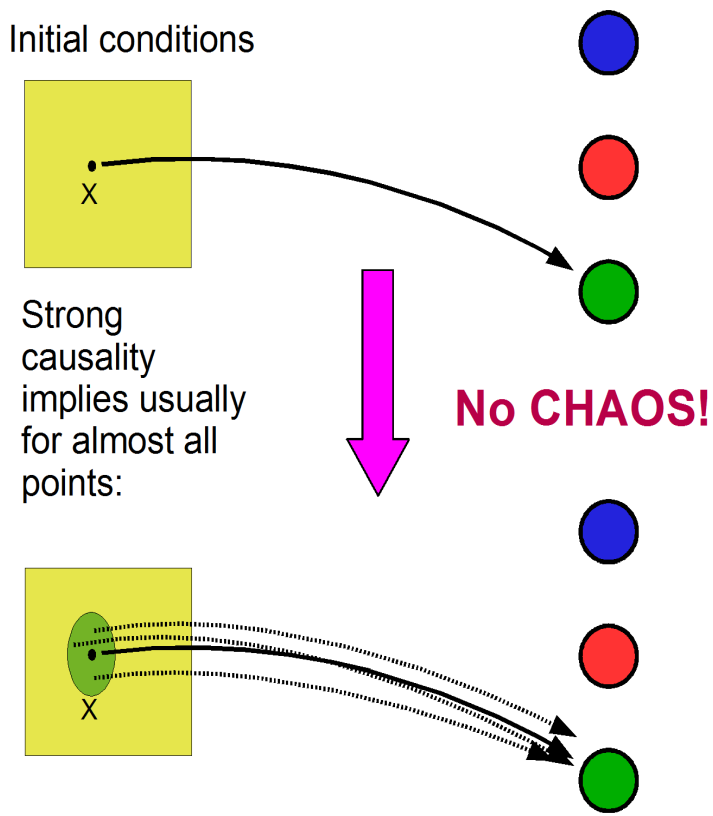
- **Weak determinism**: Identical initial conditions lead to identical results.
- **Strong determinism**: Weak determinism holds and additionally, similar conditions lead to similar results.
- Systems subject to strong determinism are easy to handle, e.g. bicycles, tin openers, computers.
- Systems subject to weak determinism are deterministic, but if one doesn't know exactly the initial conditions, the final outcome might not be predictable.

Chaos – Informal Definition

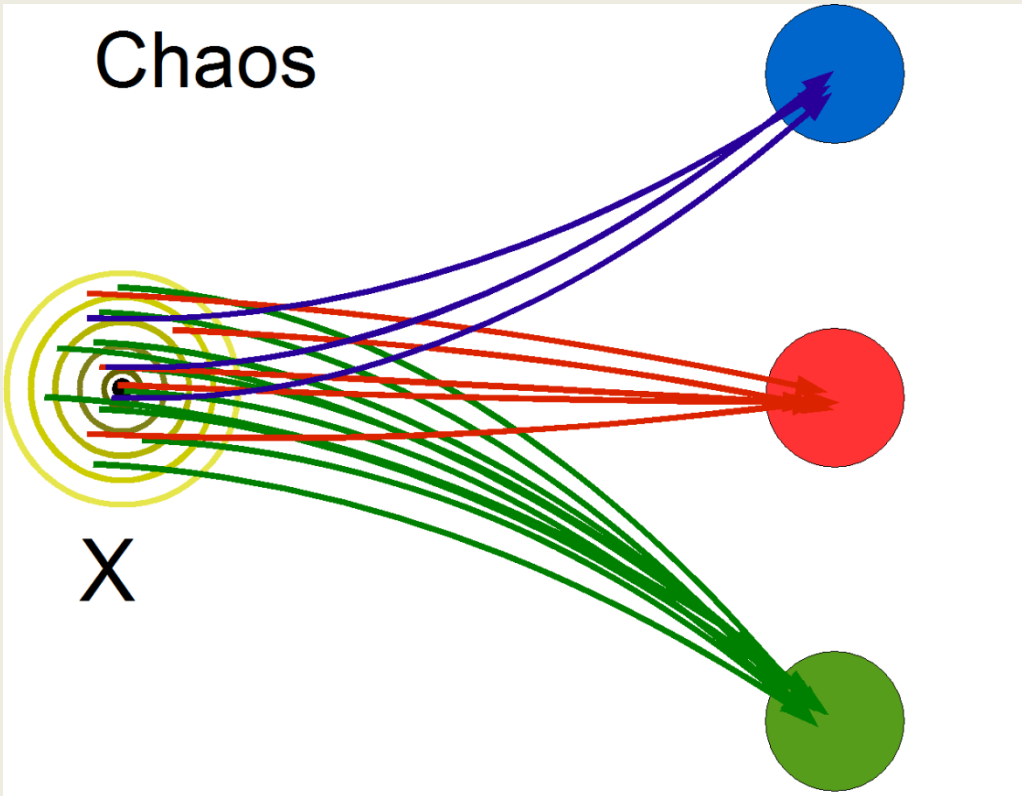
- Roughly said, systems that are subject to weak but not to strong determinism are called **chaotic**.
- Note that whether or not a system behaves in a chaotic manner may depend on some parameters. It may also be the case that the system is only chaotic for some portions of the space of potential initial conditions.

Chaos – Geometrical Perspective

System with three fixed points.



Means, if X goes green, so do his immediate buddies. This is true for almost all X's, except some borderliners. If X lifes in a n-dim space, the borderliners life on n-1 dim subsets.



Despite the fact that the fate of X is determined and leads to the „green“ fixed point, one finds in all, arbitrary small, neighborhoods of X points going to another fixed point.

Case Study: Logistic Map

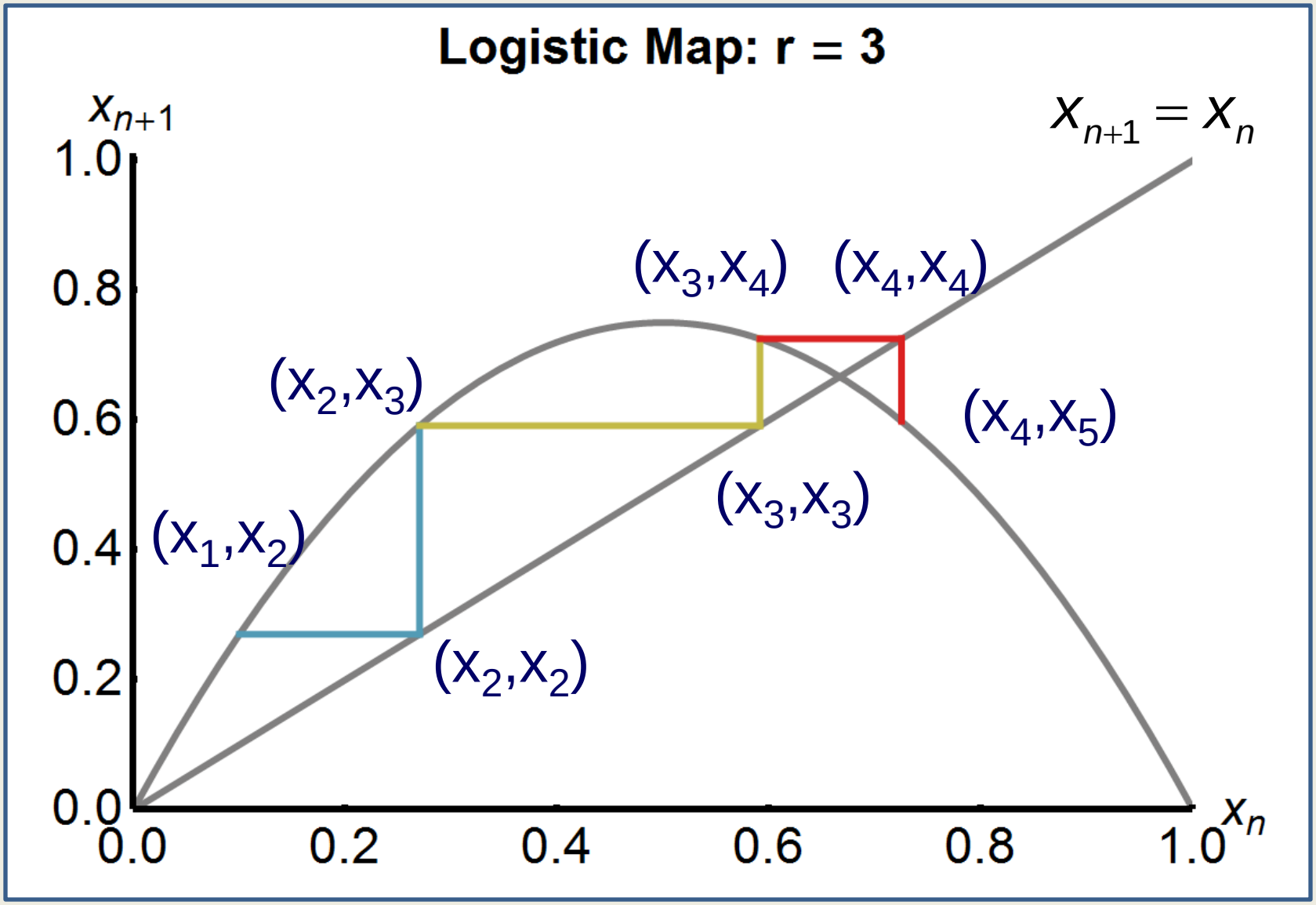
- The logistic map is a very simple ecological model.
- Two assumption:
  - Sexual reproduction ➔ $\sim$ population size
  - Competition for food ➔$\sim$ prob. of meeting a potential competitor ➔ $\sim$ (population size)^2.

$$x_{n+1} = r x_n (1 - x_n)$$

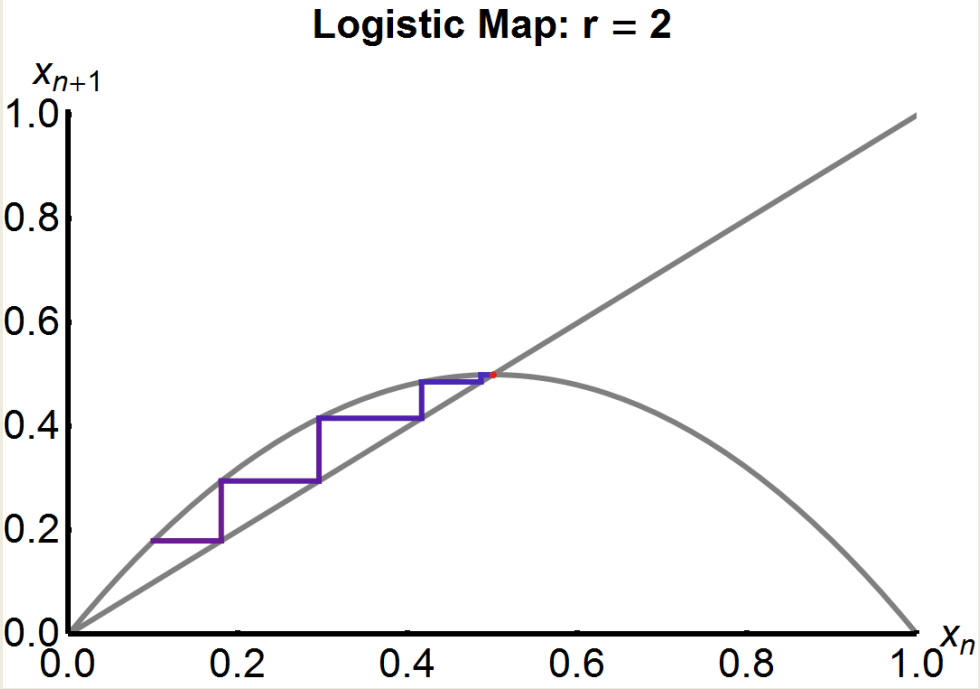
x_n: Population size at discrete time n

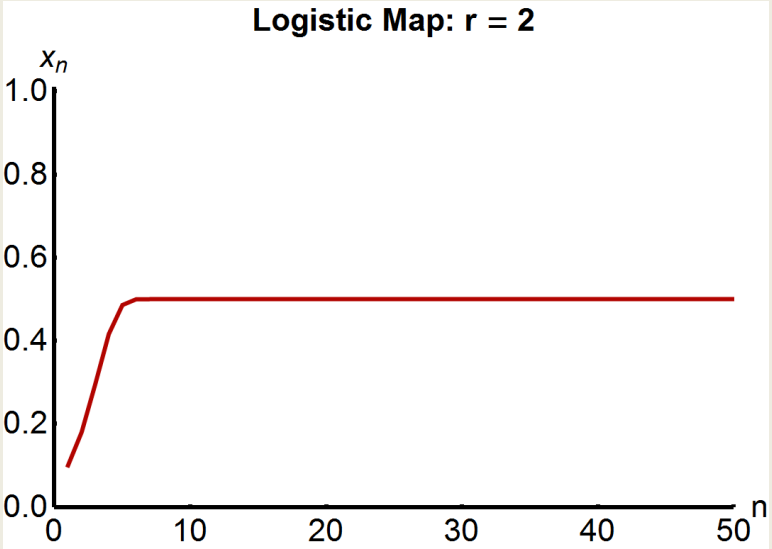
r : Parameter (after normalization)

Logistic Map Visualized



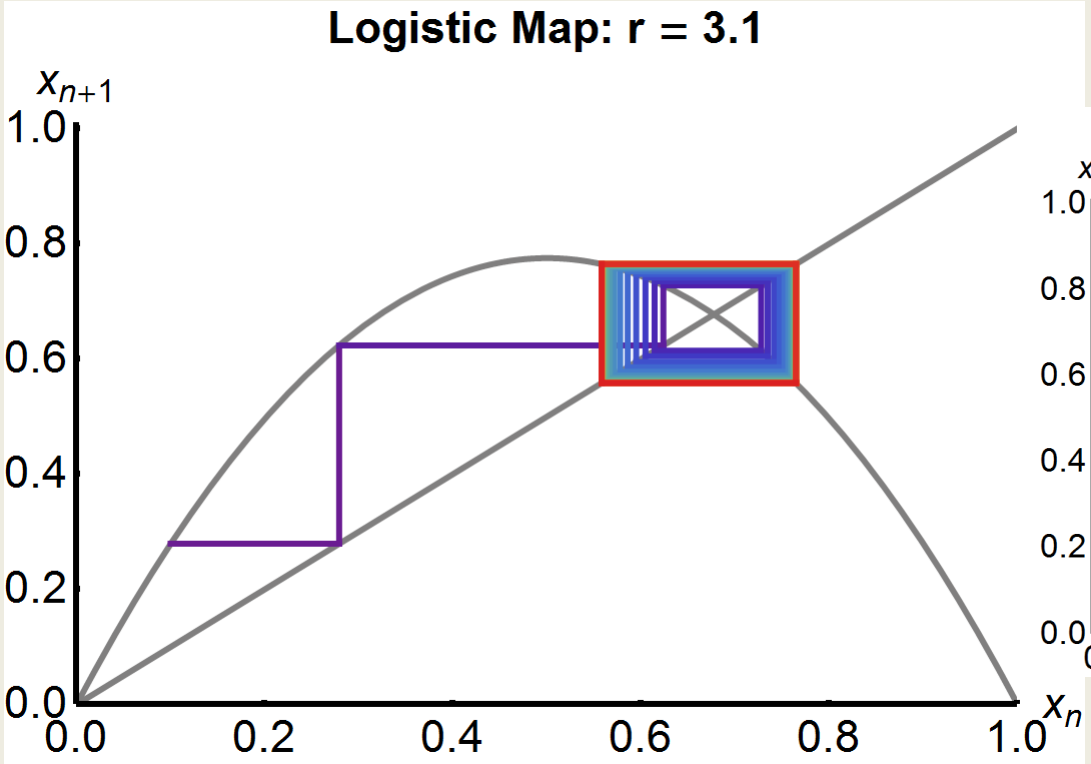
Behavior of the Logistic Map

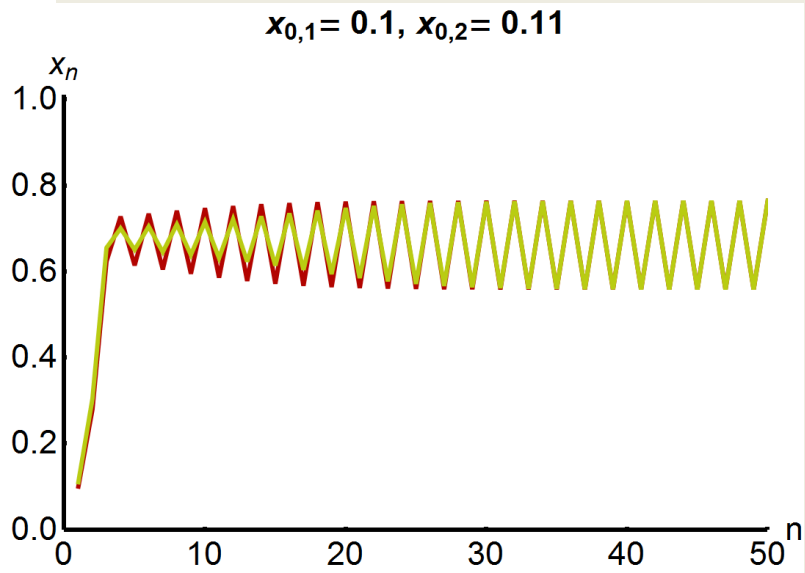




For small r: convergence to a single point.

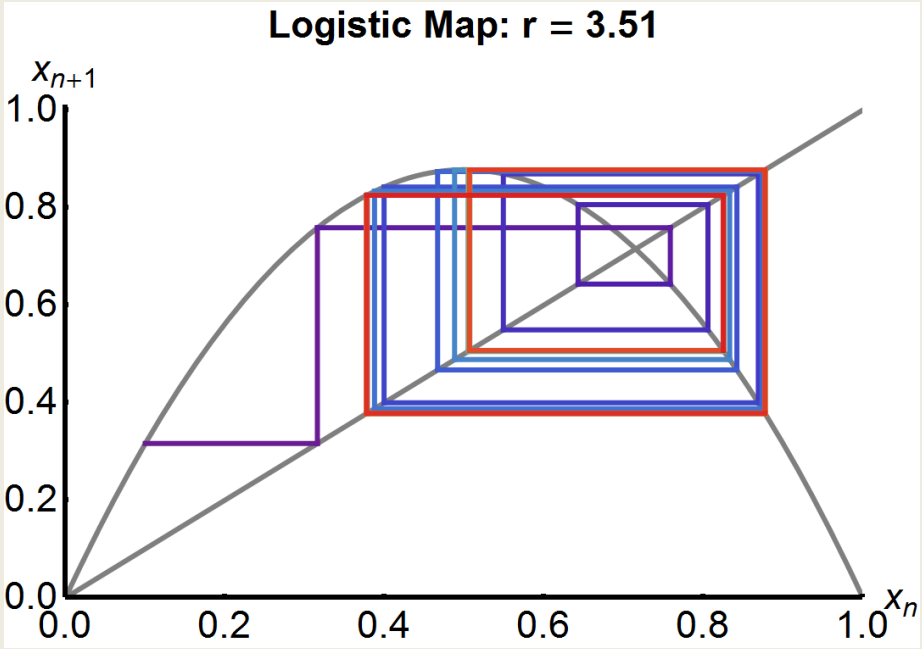
Behavior of the Logistic Map

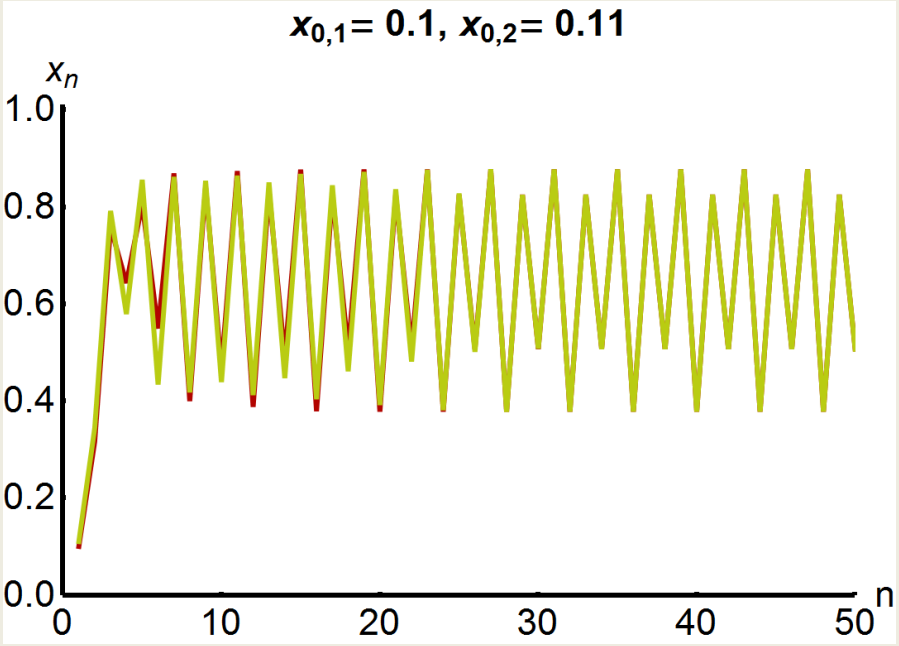




For larger r: convergence to oscillation between two points.
No dependence on initial conditions.

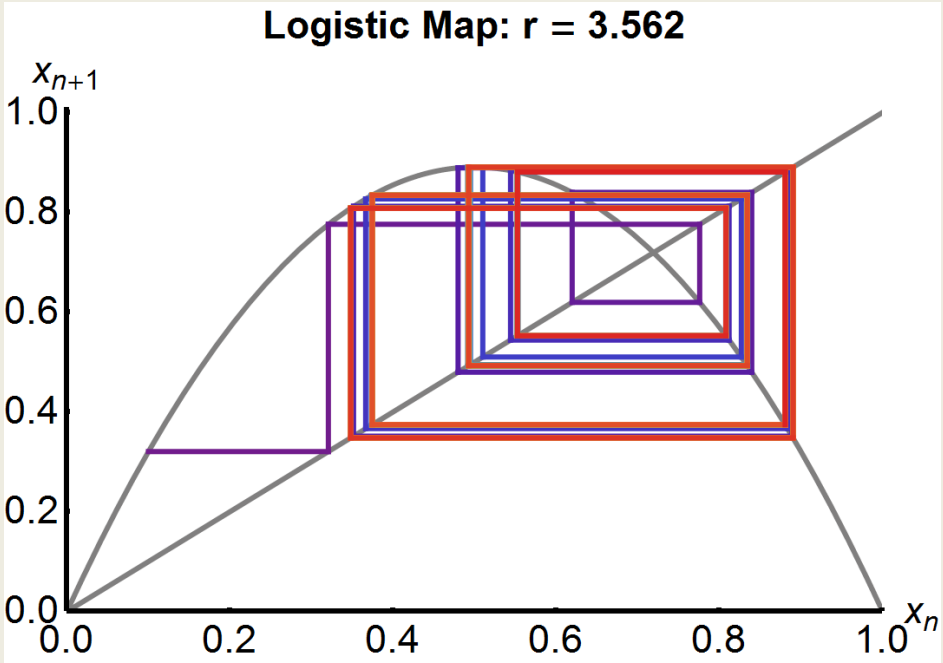
Behavior of the Logistic Map

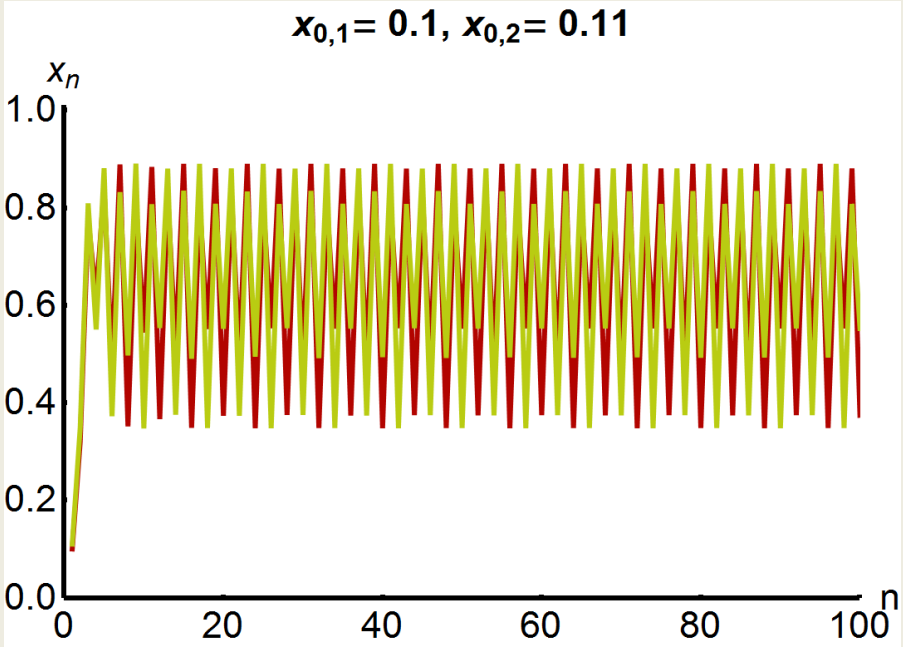




For even larger r: convergence to oscillation between four points.
No dependence on initial conditions.

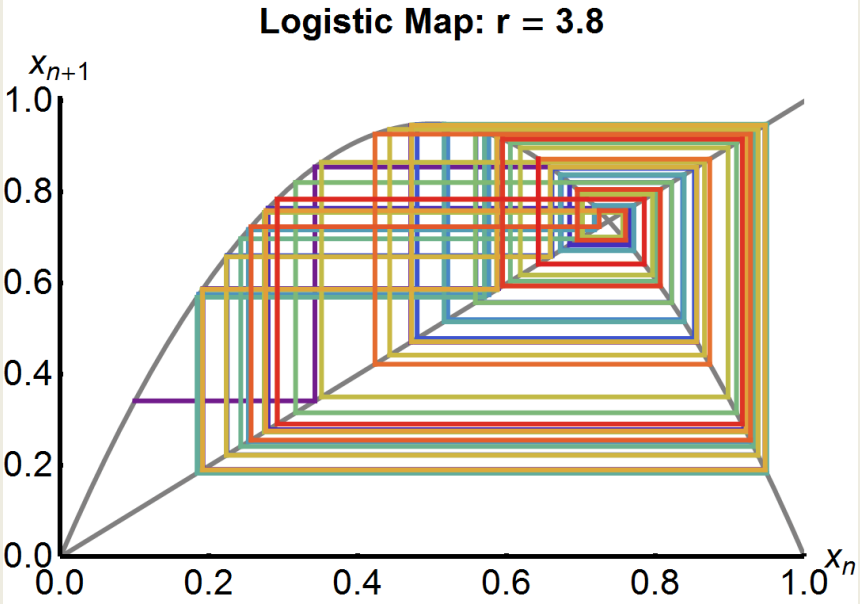
Behavior of the Logistic Map

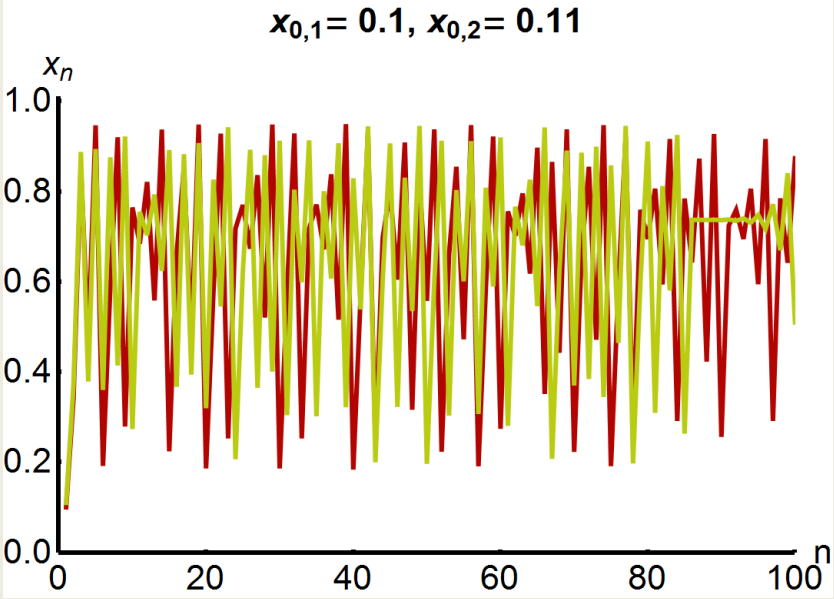




For even larger r: convergence to oscillation between eight points.
No important dependence on initial conditions.
Actually, we observe a phase shift.

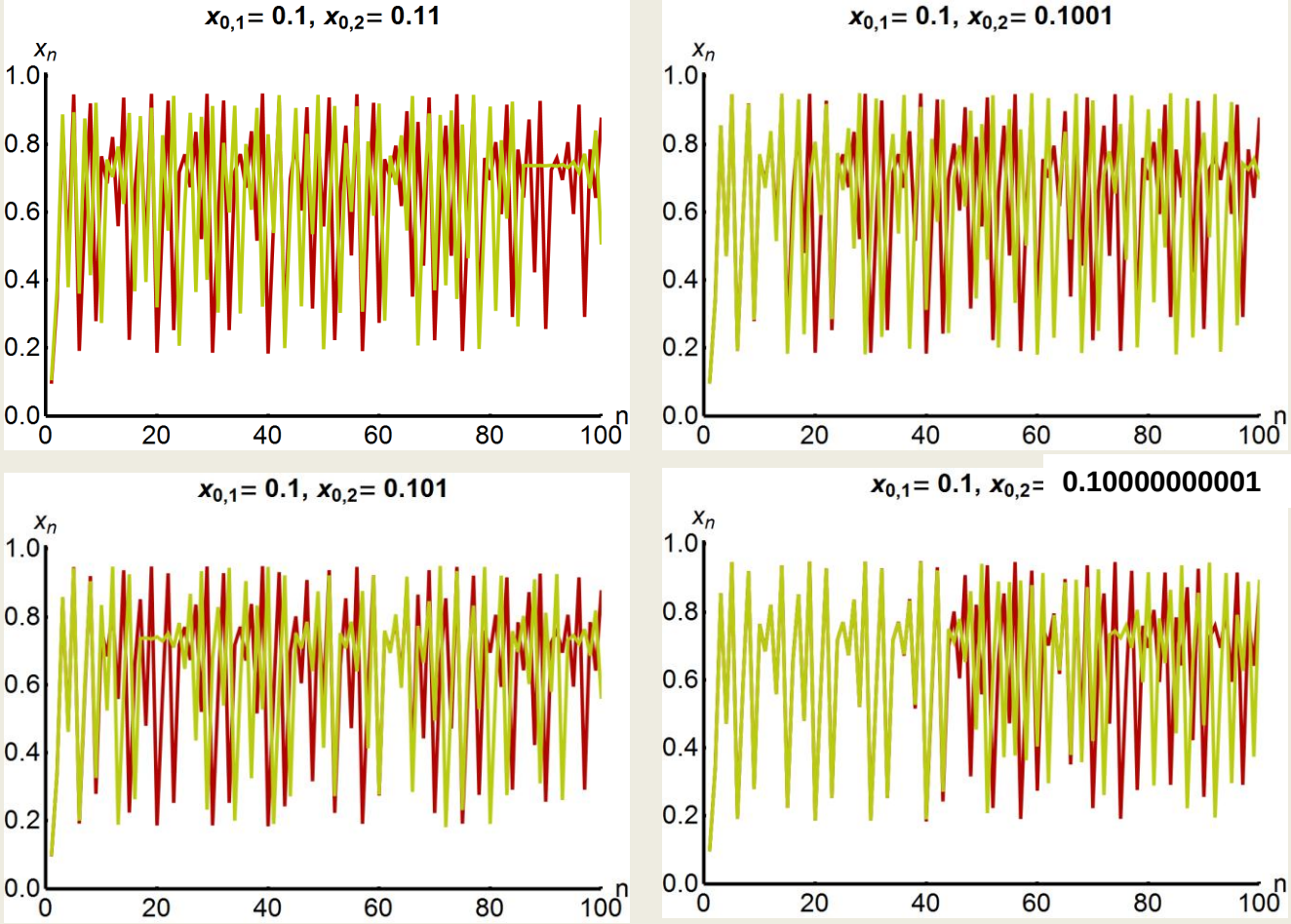
Behavior of the Logistic Map





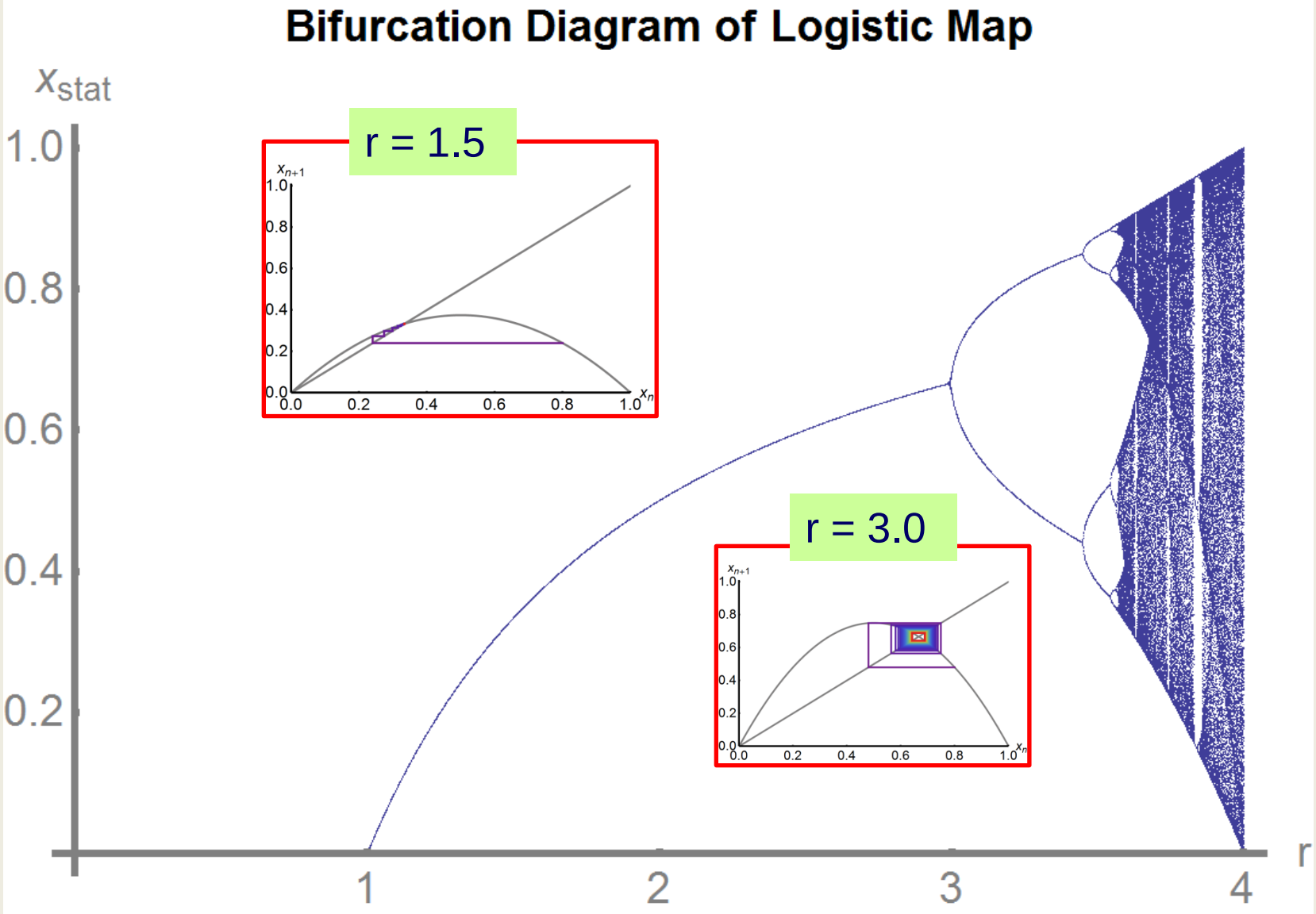
Above critical value of r: Transition to chaotic behavior.

Behavior of the Logistic Map

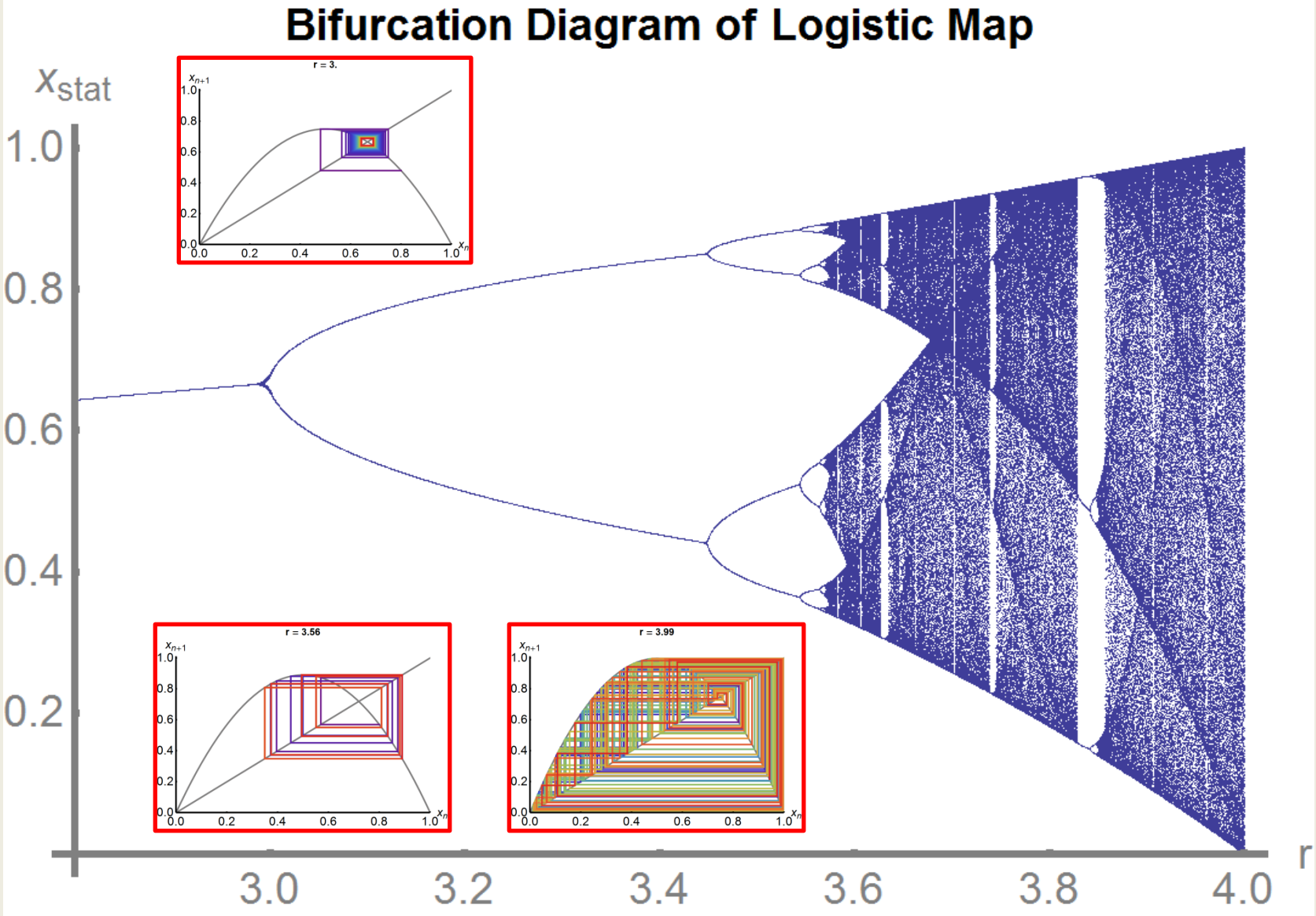


Initial conditions are very critical: Arbitrary small deviations eventually take large effect.

Logistic Map: Bifurcation Diagram



Logistic Map: Bifurcation Diagram



Case Study: Lorenz Attractor

- E. Lorenz aspired a model for some meteorological phenomena (convection).
- He observed critical dependence on initial condititions.
- First „practical“ observation of weak determinism.

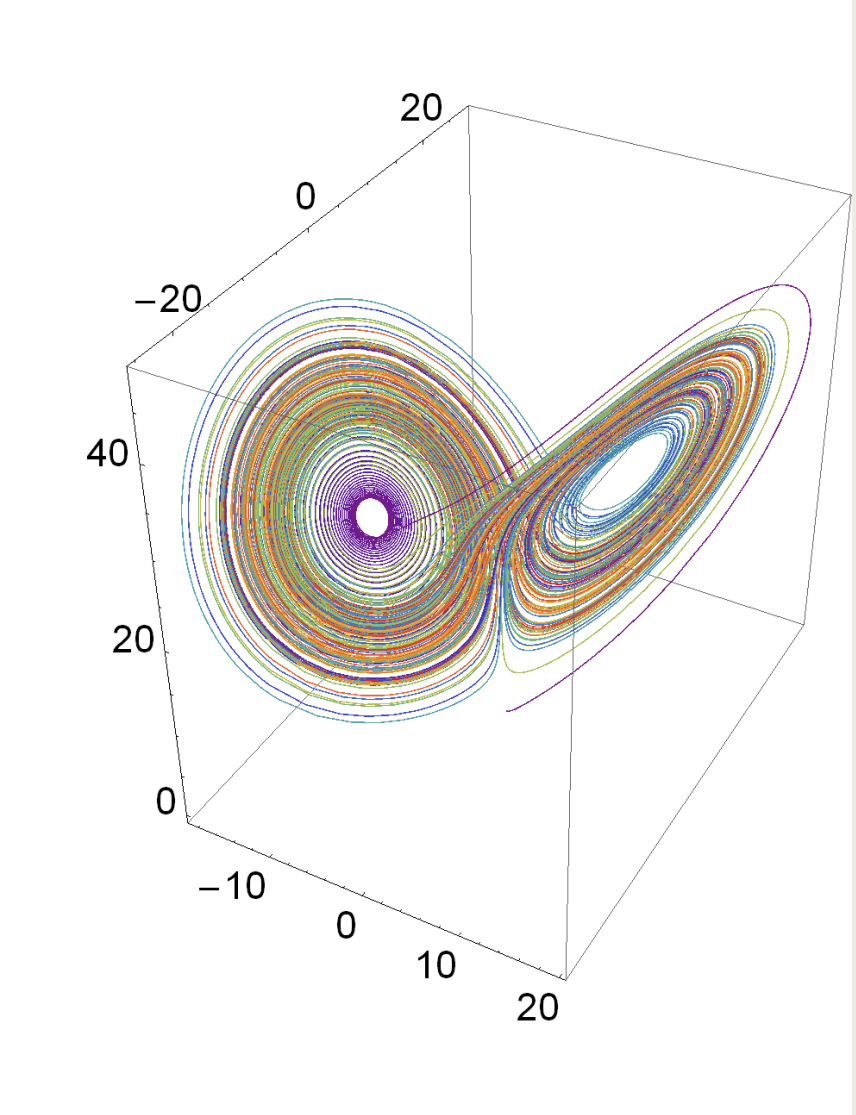
$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = x(\rho - z) - y$$

$$\dot{z} = xy - \beta z$$

σ: Prandtl number
ρ: Rayleigh number

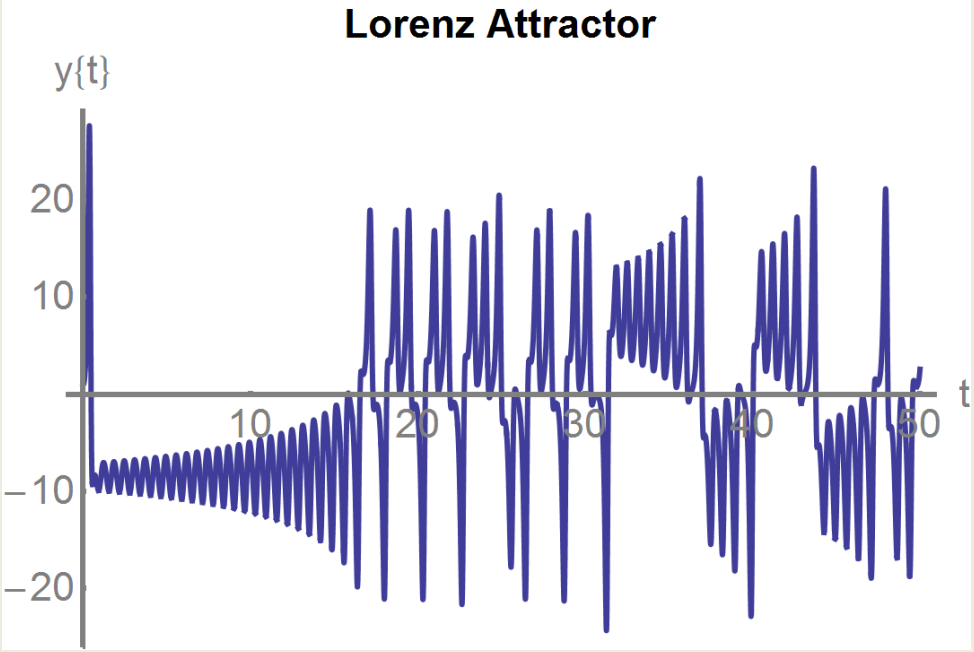
Lorenz Attractor



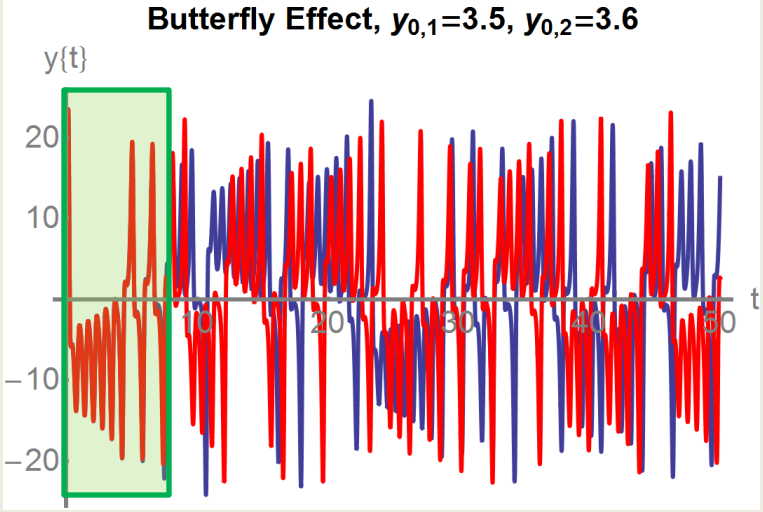
σ: 10, β: 8/3, ρ: 28

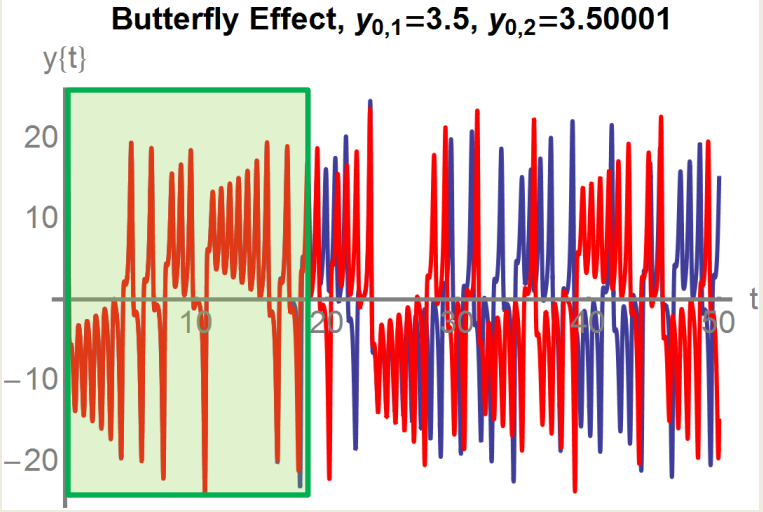
Fractal dimension: 2.06

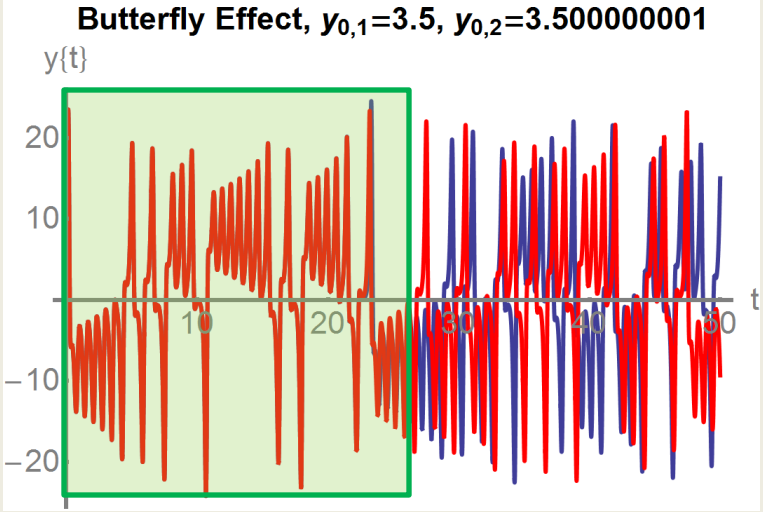
$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$



Lorenz Attractor: Butterfly Effect







Tiny variation have large effects: Orders of magnitude in precision lead only to linear increase of correctly predicted time!
The Lorenz system is strictly deterministic, but in various senses unpredictable.

Determinism and Non-Determinism

- Abstractly, the trajectories of the Lorenz – system are given in a unique way for each possible set of initial conditions ➔ the system is deterministic.
- The system may be chaotic: Given that we can't measure initial conditions to any degree of precision and given that numerical computations are subject to random errors, we can't calculate the long term behavior of the system. **But this is a problem of our computation, not of the system itself.** There is no sudden emergence of non-determinism.

Dynamical Systems – State of the Art

	Number of variables				
	n = 1 Growth, decay, equilibrium	n = 2 Oscillations	n = 3 Chaos	n >> 1 Collective phenomena	Continuum Waves and patterns
Linear	Populations RC – circuits Radioactive decay	Mass and spring RCL- circuit 2-body problem	Civil and electrical engineering	Solid state physics Molecular dynamics	Wave equation Elasticity Hydrodynamics
Non-linear	Fixed points Bifurcations Overdamped systems Logistic map	Pendulum Anharmonic oscillator Limit cycles Biological osc. Predator-prey Non-linear electronics	Strange attractors 3-body problem Chemical kinetics Fractals Iterated maps Practical uses of chaos	Non-equilibrium statistical mech. Lasers Heart cells Neural networks Immune system Eco-systems	QFT Earthquakes Reaction-diffusion systems. Fibrillation Epilepsy Turbulent flow Climate

Dynamical Systems
Connection to CA

Zurich University
of Applied Sciences
School of
Engineering
ZAMP Center for Applied
Mathematics and Physics

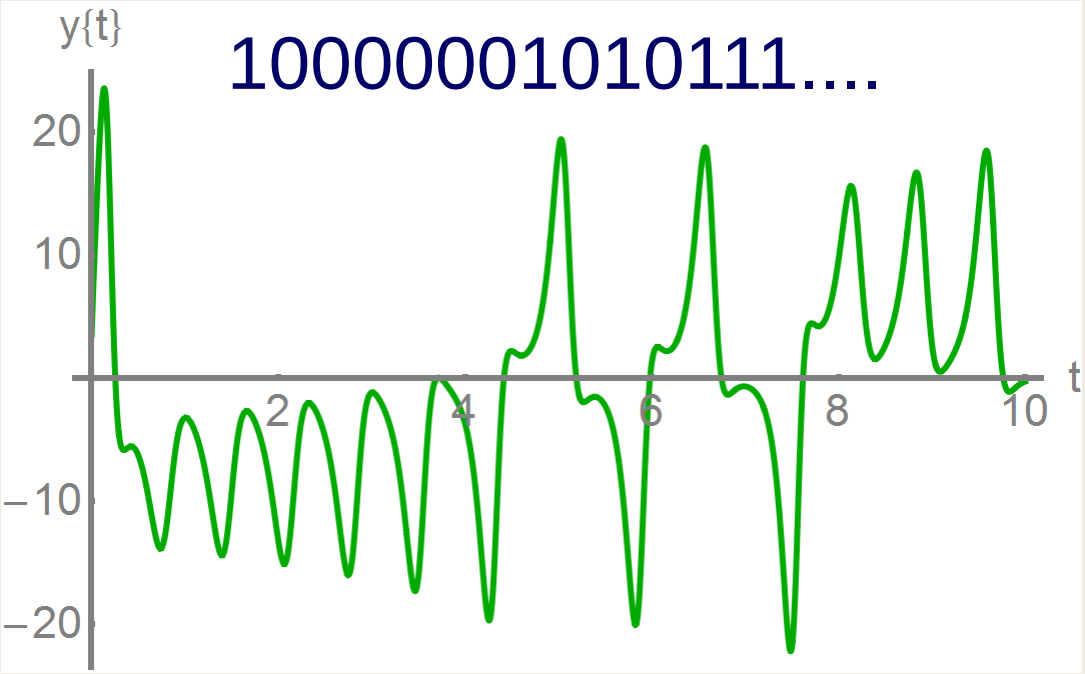
Relation to other IT Systems

	Fixed points	Limit cycles, Closed orbits, regular behavior	Chaos, strange attractors, complex behavior
Systems of differential equations			
Dim 1	Yes	No	No
Dim 2	Yes	Yes	No
Dim $\geq$ 3	Yes	Yes	Yes
Cellular Automata			
Class 1	Yes	No	No
Class 2	No	Yes	No
Class 3	No	No	Chaos
Class 4	Yes (?)	Yes (?)	Complex beh.
Automata			
FSA	Yes	Yes	No
TM	Yes (Halting)	Yes	Yes

Dynamical Systems and Information

Dynamical systems can switch between different states.

A dynamical system together with an initial condition can be understood as a (maybe compressed) representation of a sequence.



Dynamical Systems - Summary

- “Dynamical systems” are locally governed, deterministic structures that develop in time.
- They can be discrete or continuous.
- “Dynamical systems” also refers to an important body of mathematical knowledge and formalism.
- As mathematical objects, DS are tractable (sometimes) with computers.
- The dynamics is deterministic, but may be chaotic.
- Dynamical systems model/emulate/describe processes as they are observed in nature

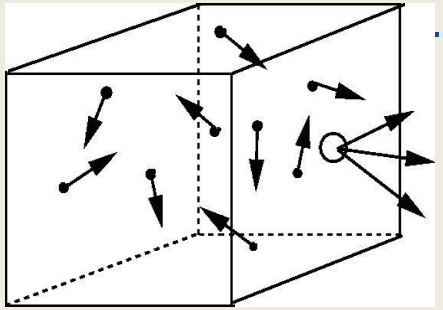
Complex Systems

Mechanisms for producing complex behavior using dynamical systems.

Zurich University
of Applied Sciences
zh aw School of Engineering
ZAMP Center for Applied Mathematics and Physics

What Is a Complex System?

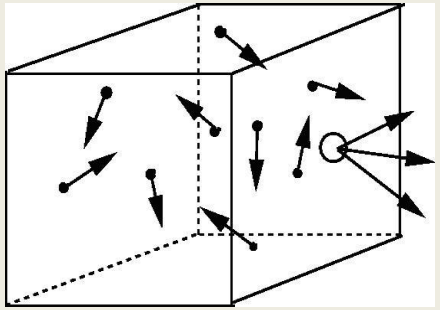
- There are various variants for defining and measuring complexity of a system and/or its respective dynamics. (see later chapter).
- General features:
  - Emergent dynamics more complex than dynamics of components
  - Self-organized non-random distribution of components and/or



simple → complex

Two Features of Complex Systems

- Multiplicity of states: The system can be in a multitude of states and these states are chosen with respect to environment.
- Emergence of behavior on different space and time scales: A complex system may well be hard to predict on short scales but exhibit prominent regularities on longer scales, despite the fact that all dynamics results from short-range interactions.

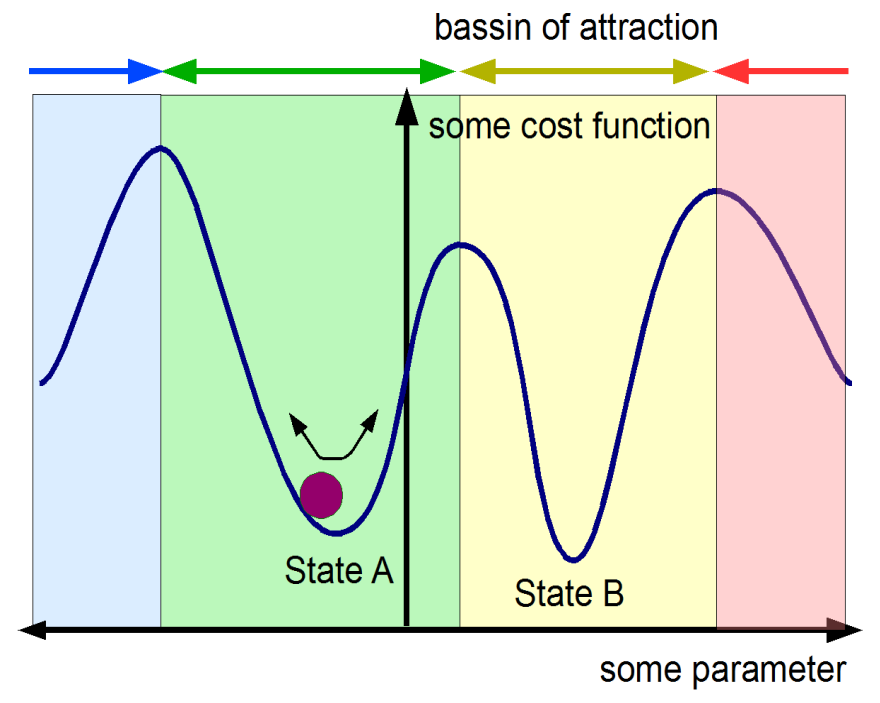


simple → complex

What Do We Want to Describe?

- **Spatial and temporal self-organized patterns** (see dedicated lecture). Local interaction lead to global structures.
- **Robustness:** A small perturbation gets lost after a short time in most of the cases.
- **Multiple states:** The systems we are interested in show a number of basic states.
  - A human being can switch between different global states: sleep, rest, attention, recovery, run, …..
  - Traffic states: Jam, flowing traffic, nose-to-tail traffic, …
- **Adaptivity:** State space can be explored sometimes.
- **Hysteresis:** Behavior of system depends on history.

Robustness – Equilibrium View



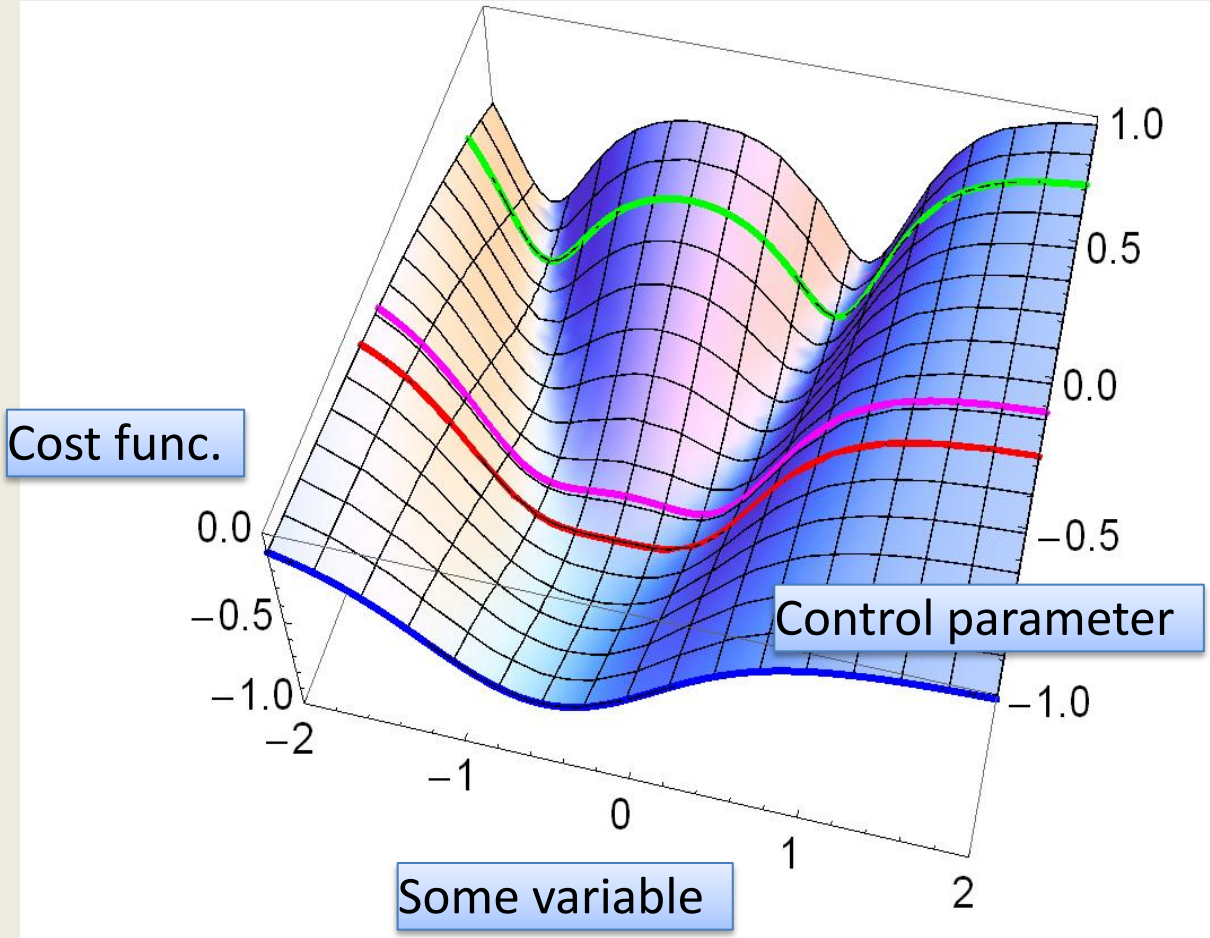
Envisage a ball in a landscape with hills and valleys. If the system **dissipates** energy (e.g. by friction), there can be **several macroscopically different states.**

This picture does not explain how a system can change its state. Such a system tends to an equilibrium and is not adaptive.

Decision Making

- How can a large, robust system react on a weak stimulus?
- The lecturer is a huge mass of cells, usually rather robust in his dynamics. But slight variations in the way his name is vocalized changes his course of actions.
- From an equilibrium point of view, this is hardly understandable.

Robustness and Adaptation

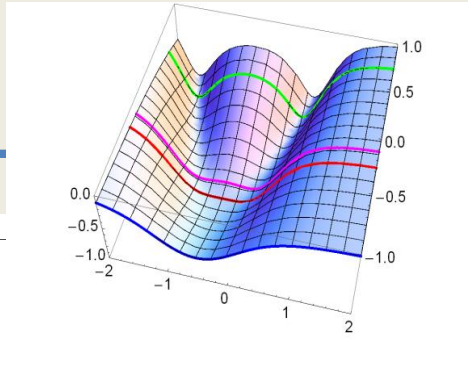


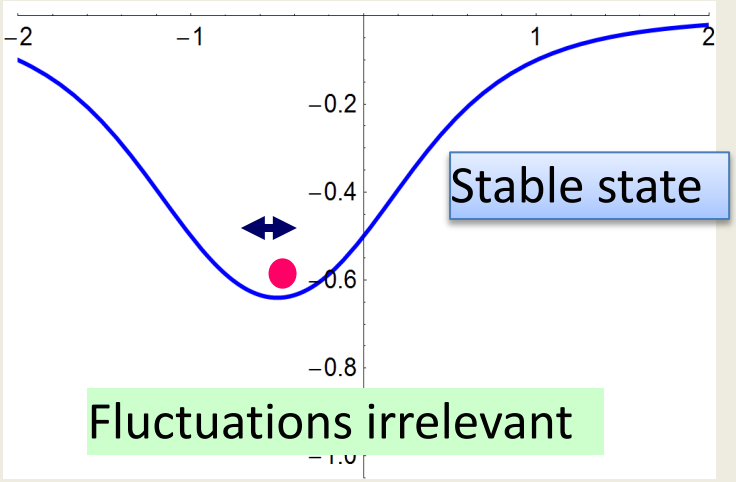
We have a function of some variable.

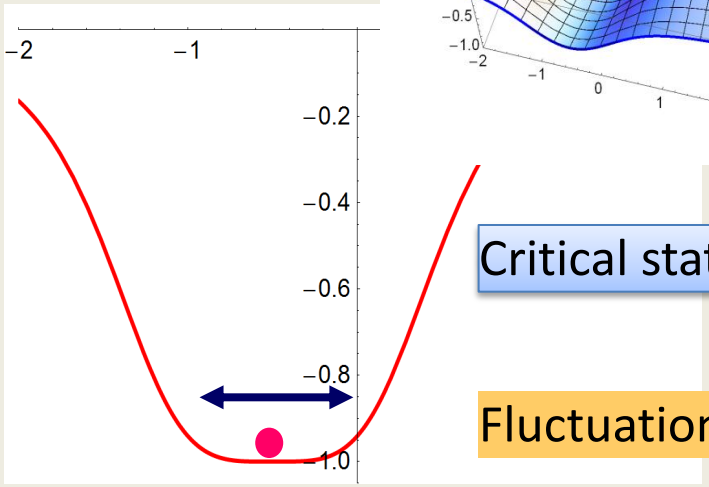
This function gives "costs", means it will be minimized.

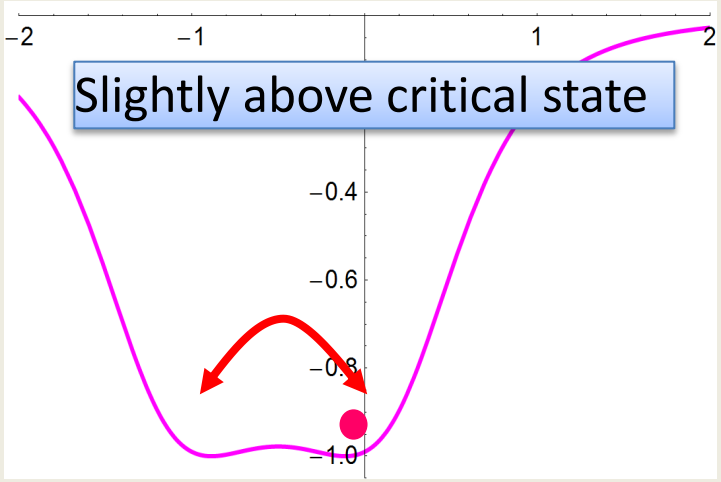
The shape of this function depends on a control parameter

Robustness and Adaptation

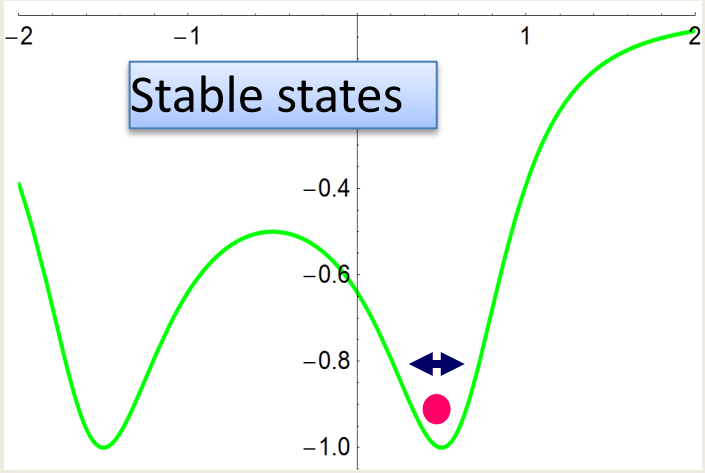








Fluctuations decide

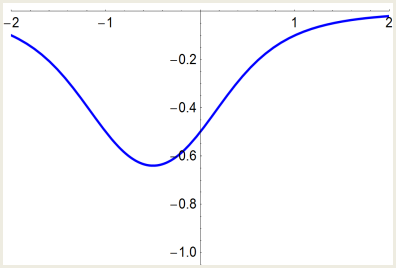


Fluctuations irrelevant

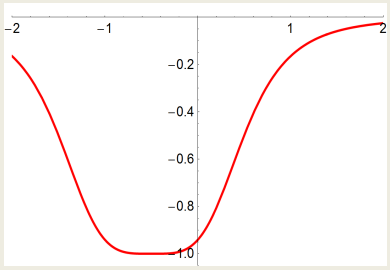
Decision Making with Complex Systems

Decision making algorithm with “attentiveness” parameter **a**:

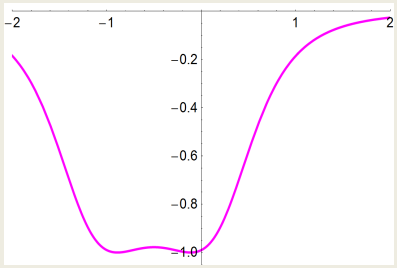
1. In the inattentive state, a has a low **a** value, the system is insensitive, the effect of external stimuli are readily damped.
2. As soon as something happens, **a** is raised towards $\mathbf{a_c}$.
3. **a** is slowly increased above $\mathbf{a_c}$. External stimuli can induce a decision.
4. **a** is further increased. Decision becomes robust state.



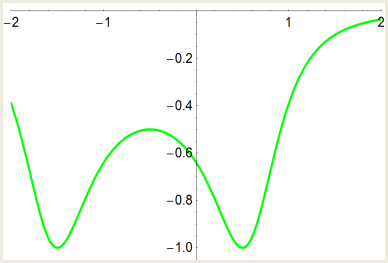
1. $a << a_c$



2. $a = a_c$

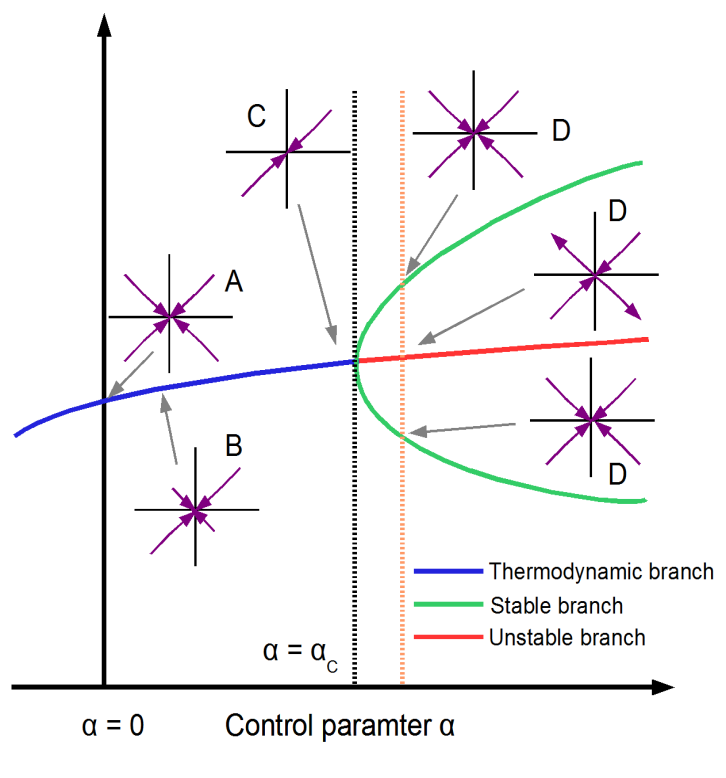


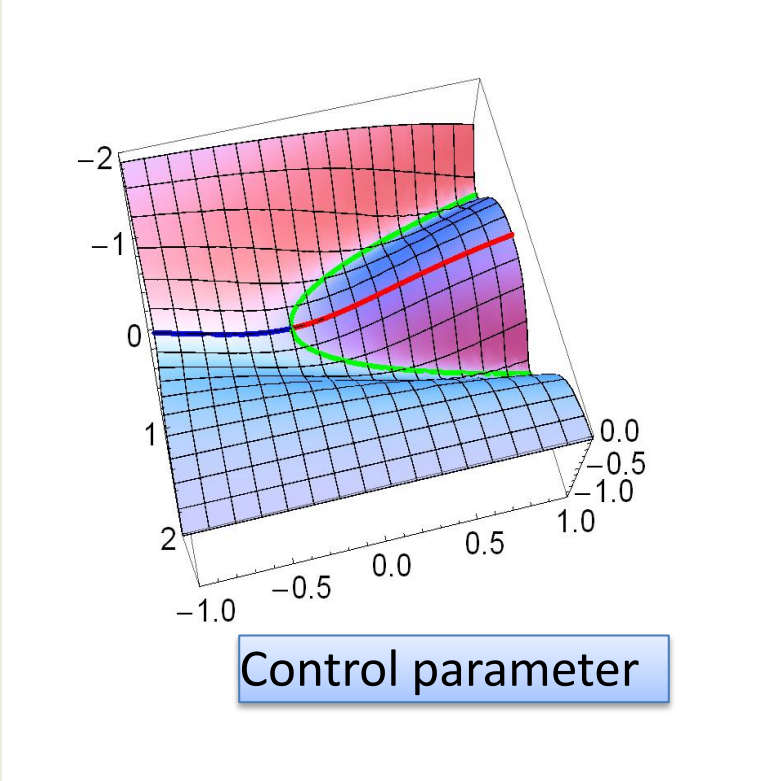
3. $a > a_c$



4. $a >> a_c$

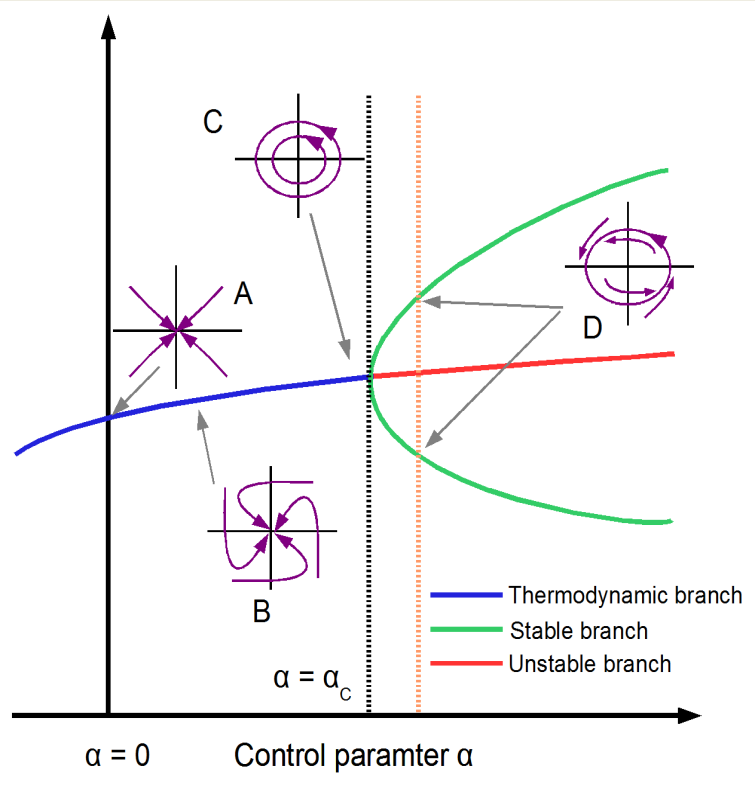
Bifurcations



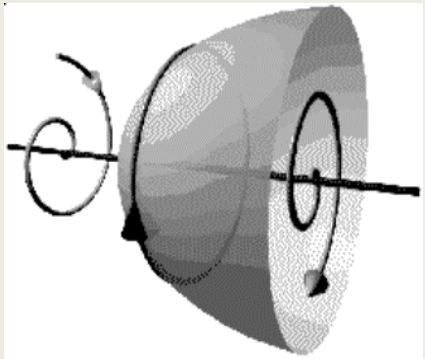


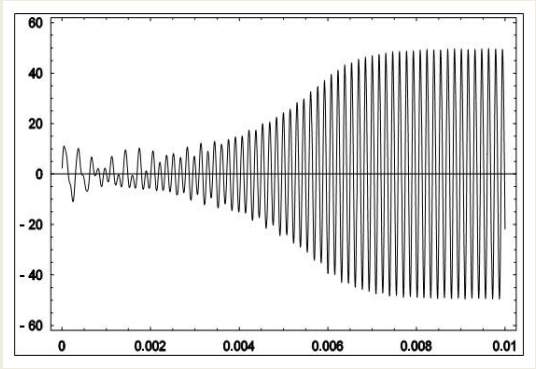
Varying the control parameter (an external stimulus) can change the system state ➔ **Adaption**. There is a subtle interplay between determinism and chance.

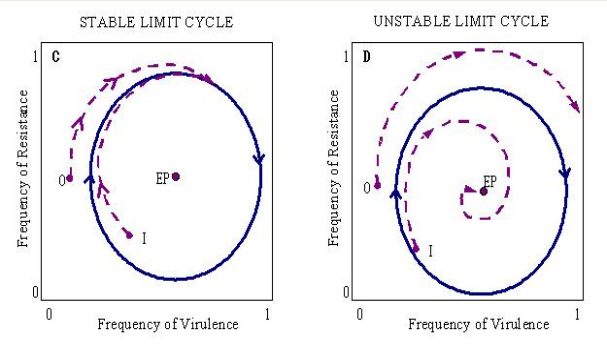
Bifurcations: Limit Cycles



Just for your information: A system can start to oscillate instead splitting its single stable state into two stable and one unstable state.







Hysteresis – Systems with History

- “Dependence on history” ➔ Given the (macroscopic) parameters describing the system, there are several stationary states the system can attain.
- Closely related question: How can it be that a system under constant external conditions can undergo an oscillatory change of states?
- A chemical reaction that does this is the **Belousov-Zhabotinski reaction**.

Hysteresis – Systems with History

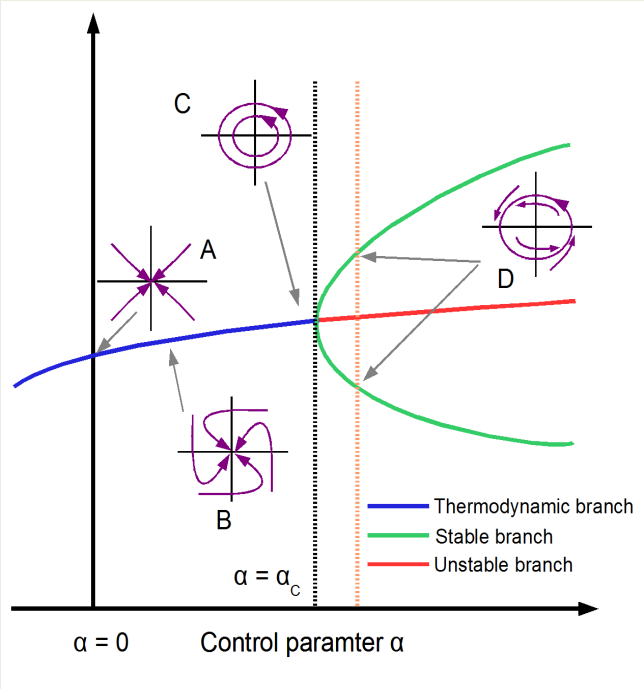
- How can it be that a system under constant external conditions can undergo an oscillatory change of states?
- The problem is: A famous theorem states that **the equilibrium state of a chemical reaction is unique → no oscillations in equilibrium chemistry.**
- **Direct consequence: Eventually, every isolated chemical system should reach its equilibrium.**
- Historical fact: People didn’t believe Belousov. However, people forgot that influx of matter or energy (e.g. because of a permanently uphold temperature gradient) breaks (The situation is in fact more involved, but for us, this is sufficient.)

Why Aren't We Dead?

Our heart beats also under constant external conditions. Note that also the electrical pulses driving the heart are the result of chemical processes. Also they should stop.

Under non-equilibrium conditions, the "only one stable state" theorem does not hold anymore.

Close to equilibrium, it does (see "thermodynamic" branch), but further away, a limit cycle may appear.



Little Intro to Chemical Dynamics

Zurich University
of Applied Sciences
School of
Engineering
ZAMP Center for Applied
Mathematics and Physics

Artificial Life

In artificial life, we need to set up physical or chemcial systems that behave in a designed way. Our goal is to get complex phenomena from simple dynamics. In what follows, some very basic facts about chemical kinetics are given. You don't have to know them, the serve as additional information. You may as well just believe what we will say about our simple model of the beating heart to be given after this intro.

Example: First order Reactions

- Assumption: A,B, C, .. substances, ρ_X: density of X, well-stirred reactors, k is a reaction rate.
- The transformation of A into B is described by an equation like:

$$A \xrightarrow{k} B$$

- k gives the probability per unit time that A is transformed into B.
- Given a well-stirred, constant volume, the densities of A and B will change according to (A is «eaten up», b is produced):

$$\frac{d\rho_A}{dt} = -k\rho_A, \frac{d\rho_B}{dt} = k\rho_A$$

Higher Order Reactions

- A reaction of nth order with **m** educts, **n** producs and respective stoechiometrich coefficients $\mathbf{a_i}$, und b_j is given by an equation of the form:

$$a_1 A_1 + \ldots + a_m A_m \xrightarrow{k} b_1 B_1 + \ldots + b_n B_n$$

- Densities of educts and products change according to

$$\frac{d\rho_{A_1}}{dt} = -k a_1 \rho_{A_1}^{a_1} \cdot \ldots \cdot \rho_{A_m}^{a_m}$$

$$\vdots$$

$$\frac{d\rho_{A_m}}{dt} = -k a_n \rho_{A_1}^{a_1} \cdot \ldots \cdot \rho_{A_m}^{a_m}$$

$$\frac{d\rho_{B_1}}{dt} = k b_1 \rho_{A_1}^{a_1} \cdot \ldots \cdot \rho_{A_m}^{a_m}$$

$$\vdots$$

$$\frac{d\rho_{B_n}}{dt} = k b_m \rho_{A_1}^{a_1} \cdot \ldots \cdot \rho_{A_m}^{a_m}$$

Detailed Balance: An Example

- Take the reaction

$$\left.\begin{array}{c} A+2X \underset{k_2}{\overset{k_1}{\rightleftharpoons}} 3X \\ X \underset{k_4}{\overset{k_3}{\rightleftharpoons}} B \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \dfrac{[A]}{[X]} = \dfrac{k_2}{k_1} \\ \dfrac{[X]}{[B]} = \dfrac{k_4}{k_3} \end{array}\right.$$

- When detailed balance holds, giving [A] determines the equilibrium. **Only one equilibrium can be observed.**

Non-Equilibrium

- Assume that [A] and [B] are kept constant, but NOT at the values on expects for the equilibrium state.
- Now, detailed balance hasn't to hold. The only statement we can made is that for a stationary state, as much X will be produced as will be consumed.

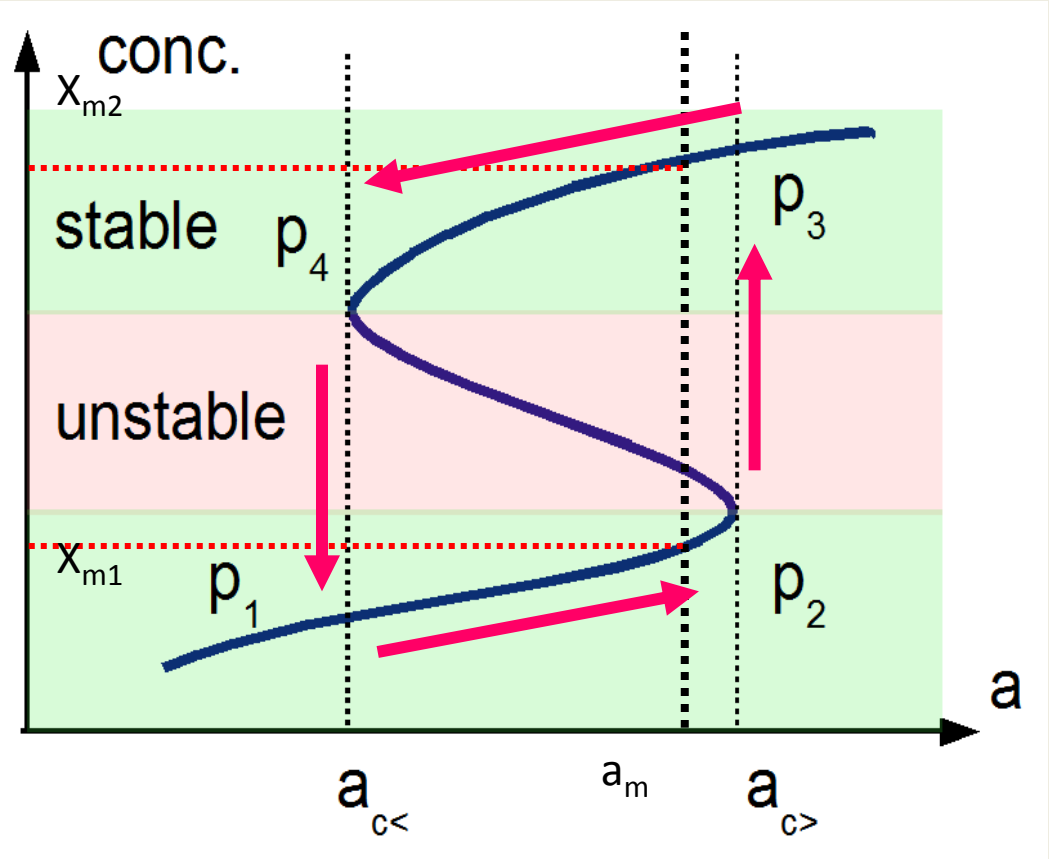
$$k_1[A][X]^2 - k_2[X]^3 - k_3[X] + k_4[B] = 0$$

- **This equation has three stationary solutions.**

$$A + 2X \underset{k_2}{\overset{k_1}{\rightleftharpoons}} 3X$$

$$X \underset{k_4}{\overset{k_3}{\rightleftharpoons}} B$$

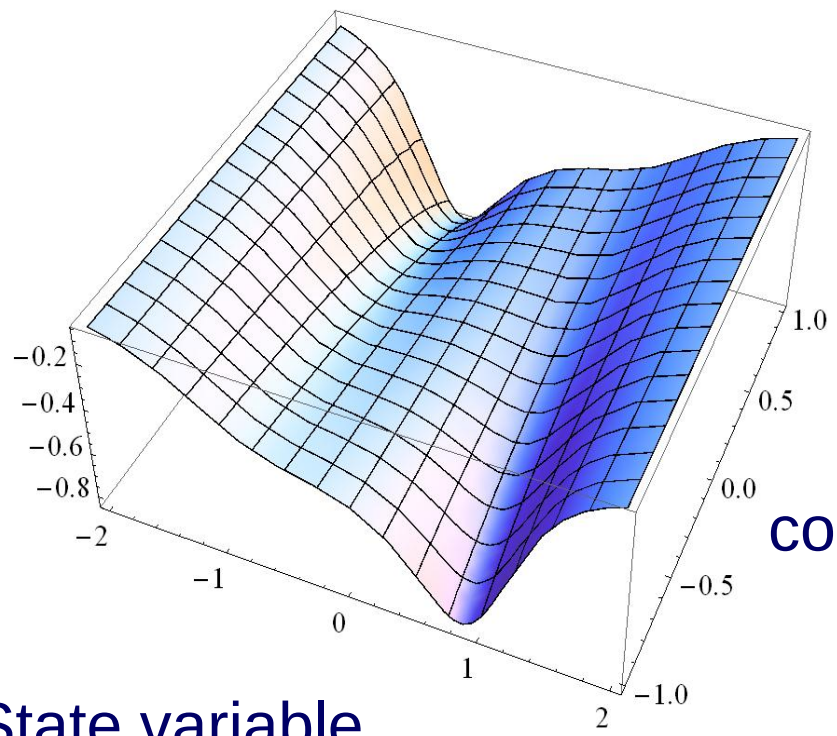
Hysteresis – Again and Real



The blue curve gives the stationary states. Start at p_1 and increase the control parameter a. You stay on the lower branch, until you reach p_2. There, because the lower stable states cease to exist, you switch to p_3. Now lower the control parameter. You stay on the higher level conc. until you reach p_4. There, you jump back again.

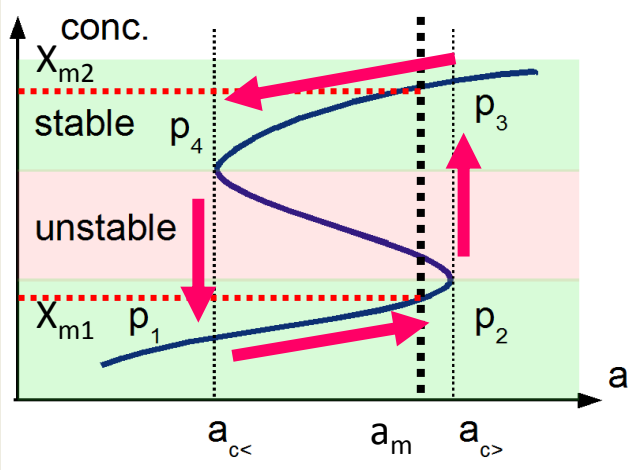
In non-equilibrium, it can be that one value a_m leads to two X_m, depending on the history.

Hysteresis – The Picture



State variable

control parameter

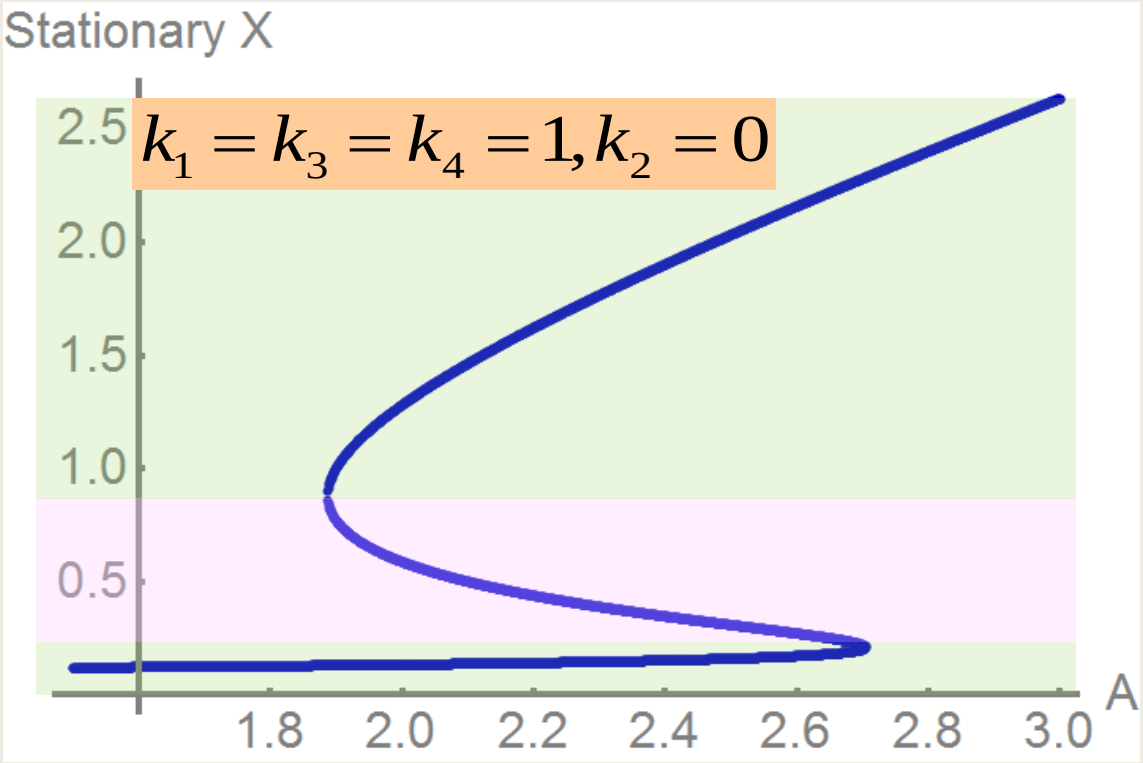


Let Us Build an Artificial Heart!

(Well, actually a chemical controller for a heart). We need:

- **A system that exhibits a multitude of stationary states**, such a change of a parameter allows one to drive the system through a hysteresis curve.
- **The system must alter its own control parameter**. Producing an oscillatory system by using an externally driven oscillating control parameter isn't a solution.
- Additional to the systems influence on the control parameter, we need a simple (in chemical terms), non-oscillatory external driving mechanism for the control parameter ➔ Otherwise, the system would just reach a stationary state and remain there.

A Multi-State System



A and B are "buffered", means kept at constant value. We chose B = 0.15.

$$A + 2X \underset{k_2}{\overset{k_1}{\rightleftharpoons}} 3X$$

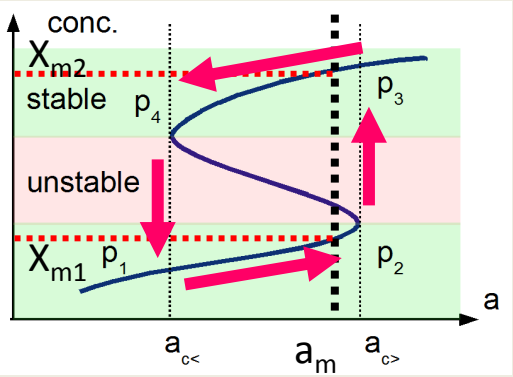
$$X \underset{k_4}{\overset{k_3}{\rightleftharpoons}} B$$

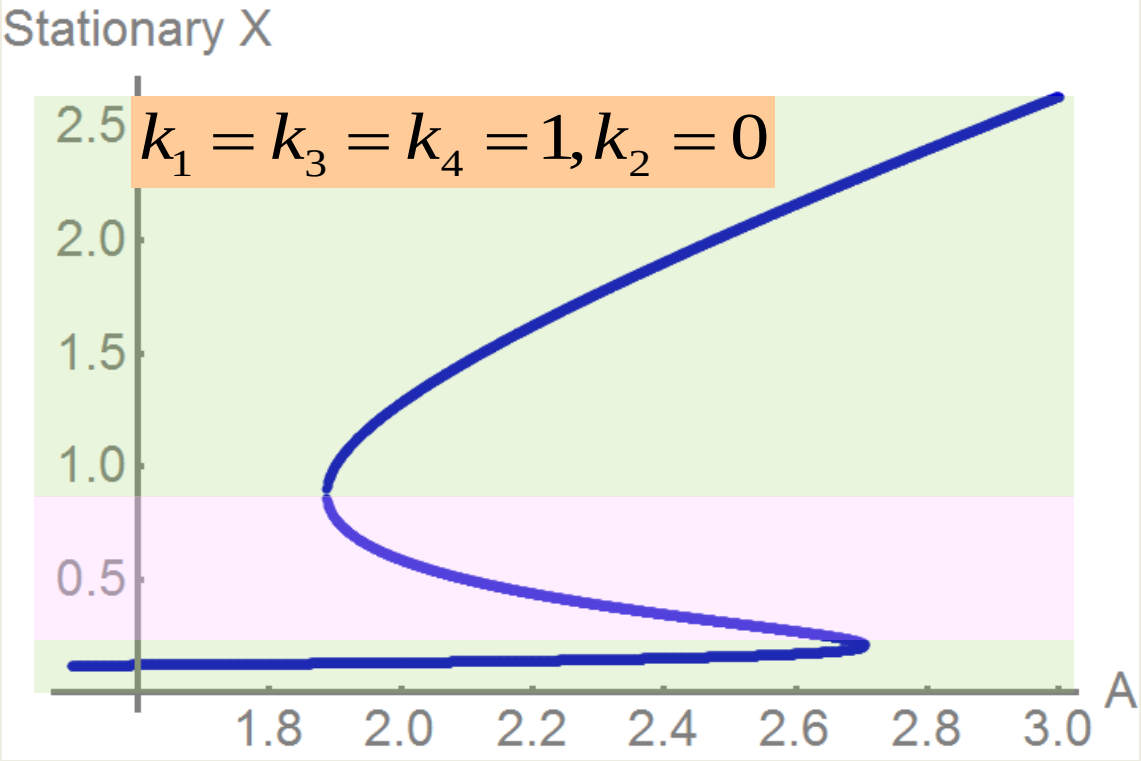
$$\frac{dA}{dt} = -k_1 A X^2 + k_3 X^3$$

$$\frac{dX}{dt} = k_1 A X^2 + k_4 B - k_2 X - k_3 X^3$$

Towards a Beating Heart

What reaction has to be added in order to drive the system through a hysteresis curve?





$$A + 2X \underset{k_2}{\overset{k_1}{\rightleftharpoons}} 3X$$

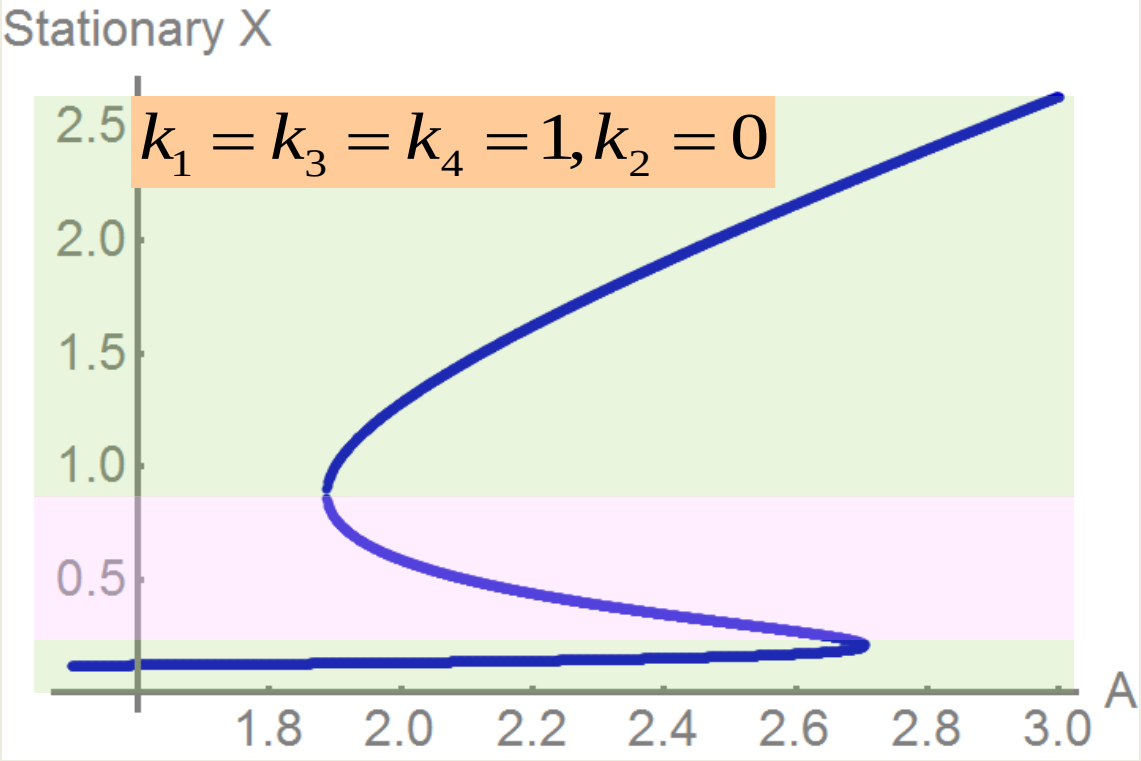
$$X \underset{k_4}{\overset{k_3}{\rightleftharpoons}} B$$

$$\frac{dA}{dt} = -k_1 A X^2 + k_3 X^3$$

$$\frac{dX}{dt} = k_1 A X^2 + k_4 B - k_2 X - k_3 X^3$$

Towards a Beating Heart

Observation: The term k_1 A X^2 consumes many A's on the upper branch of the hysteresis curve, but only few ones on the lower branch.



$$A + 2X \underset{k_2}{\overset{k_1}{\rightleftharpoons}} 3X$$

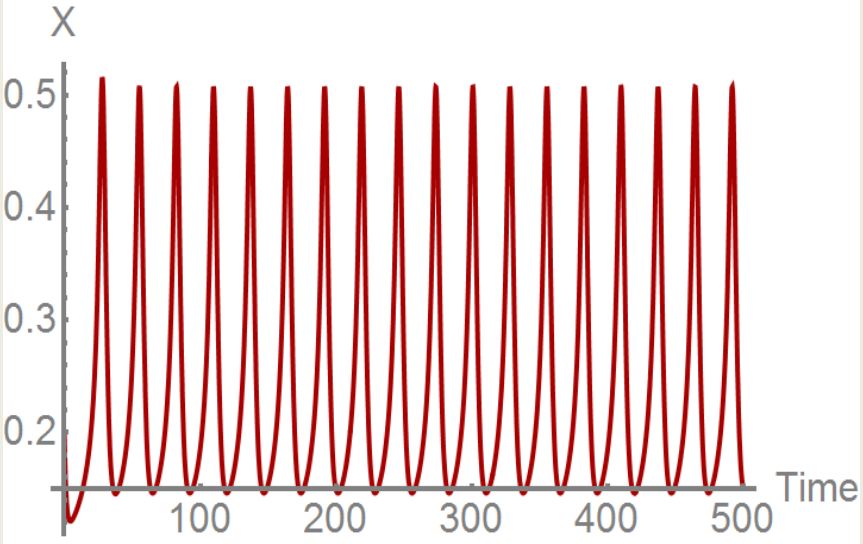
$$X \underset{k_4}{\overset{k_3}{\rightleftharpoons}} B$$

$$\frac{dA}{dt} = -k_1 A X^2 + k_3 X^3$$

$$\frac{dX}{dt} = k_1 A X^2 + k_4 B - k_2 X - k_3 X^3$$

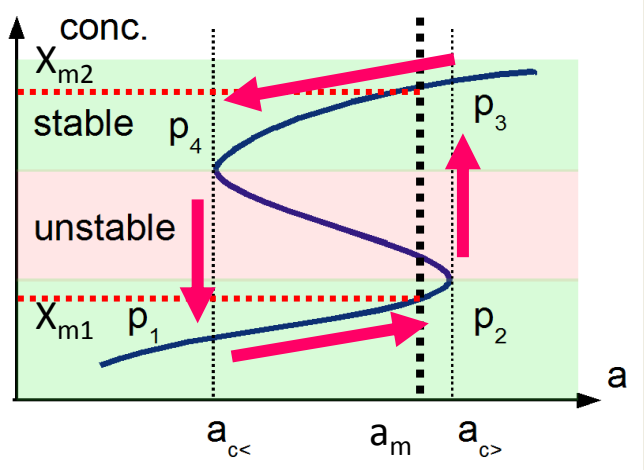
Towards a Beating Heart

A chemically simple constant influx with a rate $k_A = 0.15$ leads to a **self-driven oscillation!**



$$\frac{dA}{dt} = -k_1 A X^2 + k_3 X^3 \boxed{+ k_A}$$

$$\frac{dX}{dt} = k_1 A X^2 + k_4 B - k_2 X - k_3 X^3$$

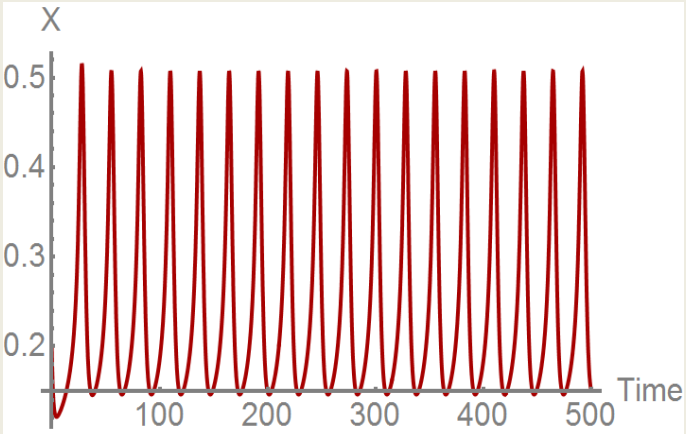


Towards a Beating Heart

Complex System: Constant influx ➔ more sophisticated dynamical oscillation.
Engineering artificial life means also to **combine simple processes such that qualitatively more complex dynamics results.**
Nature provides us with a limited set of “simple” chemical and mechanical processes. Identifying these **“primitives**” and study how to combine them is a major part in bio-inspired engineering.

$$\frac{dA}{dt} = -k_1 A X^2 + k_3 X^3 \boxed{+ k_A}$$

$$\frac{dX}{dt} = k_1 A X^2 + k_4 B - k_2 X - k_3 X^3$$

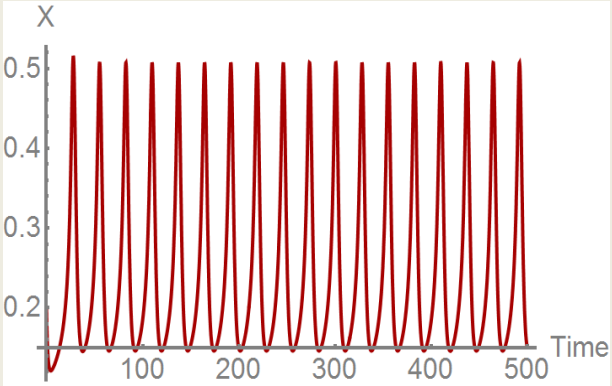


Towards a Beating Heart

Potential exercise/project: Expand the given system such that it react e.g. on the level of further chemical by increasing the frequency and couple the frequency to this chemical such that its level decreases under increase of frequency.

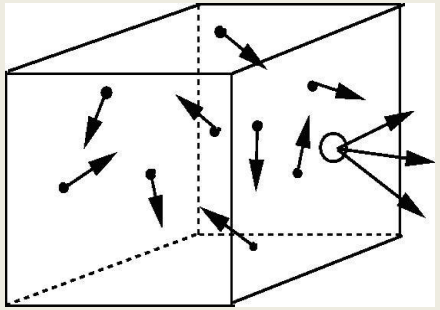
$$\frac{dA}{dt} = -k_1 A X^2 + k_3 X^3 \boxed{+ k_A}$$

$$\frac{dX}{dt} = k_1 A X^2 + k_4 B - k_2 X - k_3 X^3$$



Two Features of Complex Systems

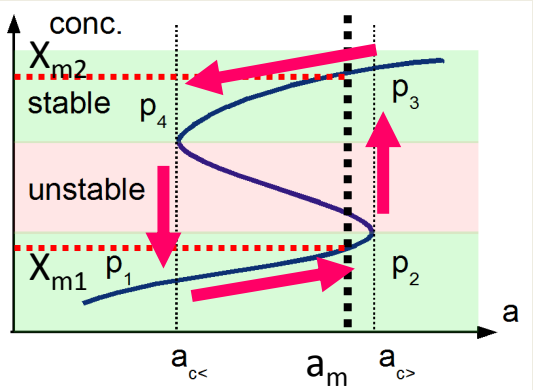
- Multiplicity of states: The system can be in a multitude of states and these states are chosen with respect to environment.
- Emergence of behavior on different space and time scales: A complex system may well be hard to predict on short scales but exhibit prominent regularities on longer scales, despite the fact that all dynamics results from short-range interactions.



simple complex

Two Features of Complex Systems

- Multiplicity of states:

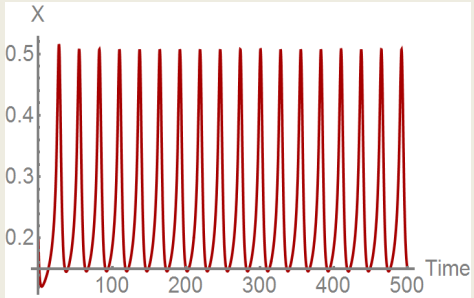


- Emergence of behavior on different space and time scales: The oscillatory dynamics results from the interplay of many molecules. To predict it from the (probably well known) properties of individual molecules is virtually impossible.

$$A + 2X \underset{k_2}{\overset{k_1}{\rightleftharpoons}} 3X$$

$$X \underset{k_4}{\overset{k_3}{\rightleftharpoons}} B$$

What you know



What you get

Complex Systems: Key Points

- “Complex systems” are dynamical systems consisting of many “simple” parts.
- The simple parts interact in a manner that yields phenomena not observed in the parts.
- Artificial life aims at combining simple processes such that complex behavior results.
- There is no real definition of “complex systems”. Look at it from an engineering point of view: If you understand the parts but not the machine, it is a complex system!