Combinatorial Auctions - Fall 2014
Assignment #7: Matching Markets

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Out Tuesday, November 18, 2014
Due **13:30** sharp: **Tuesday, November 25, 2014**
Submissions should be made to `dmitry.moor@ifi.uzh.ch`

[**Total: 50 Points**] This is a single person assignment. Points will be awarded for clarity, correctness and completeness of the answers. Reasoning must be provided with every answer, i.e., please show your work. You get most of the credit for showing the way in which you arrived at the solution, not for the final answer. You are free to discuss the assignment with other students. However, you are not allowed to share (even partial) answers and source code with each other, and **copying will be penalized**.

1. [**20 Points**] Two-sided matching markets

   Consider the following setting: three graduating medical students s_1, s_2, s_3 are going to be assigned to three mentors m_1, m_2, m_3 in a hospital.

   (a) [**10 Points**] Preferences of students and mentors are illustrated below:

   $$\begin{array}{llll} s_1: & m_1 \succ_{s_1} m_2 \succ_{s_1} m_3 & m_1: & s_3 \succ_{m_1} s_2 \succ_{m_1} s_1 \\ s_2: & m_2 \succ_{s_2} m_1 \succ_{s_2} m_3 & m_2: & s_3 \succ_{m_2} \emptyset \succ_{m_2} s_1 \succ_{m_2} s_2 \\ s_3: & m_1 \succ_{s_3} m_3 \succ_{s_3} m_2 & m_3: & s_1 \succ_{m_3} s_3 \succ_{m_3} s_2 \end{array}$$

   i. [**5 Points**] Using the Deferred Acceptance procedure, find a student-optimal and mentor-optimal allocations.

   ii. [**5 Points**] Provide all stable matchings. *HINT: you can use the result of the previous question to answer this one without doing an analysis of all possible matchings.*

   (b) [**10 Points**] Now assume the preferences of agents are specified as follows:

   $$\begin{array}{llll} s_1: & m_1 \succ_{s_1} m_2 \succ_{s_1} m_3 & m_1: & s_2 \succ_{m_1} s_1 \succ_{m_1} s_3 \\ s_2: & m_2 \succ_{s_2} m_1 \succ_{s_2} m_3 & m_2: & s_1 \succ_{m_2} s_2 \succ_{m_2} s_3 \\ s_3: & m_2 \succ_{s_3} m_3 \succ_{s_3} m_1 & m_3: & s_1 \succ_{m_3} s_2 \succ_{m_3} s_3 \end{array}$$

   Provide an example of how a mentor can manipulate the market to get a better student under the student-proposing Deferred Acceptance procedure.

2. [**30 Points**] Rank-value mechanism

   Let $N = \{1, ..., n\}$ denote a set of agents and $M = a, b, c, ...$ a set of items ($|M| = n$). All agents should receive exactly one item. Every agent $i \in N$ submits his preferences to the mechanism, for example, $a \succ_i c \succ_i b \succ_i ..., i \in N$. We say that for every agent $i \in N$ the *rank* of an item is equal to its position in the preference order of i. This is denoted by the *rank function* $r_i : M \to \{1, ..., n\}$ which maps an item to its rank. Using the previous example, $r_i(a) = 1$, $r_i(b) = 3$, $r_i(c) = 2$. We also specify a *rank valuation* $v =< v_1, ..., v_n >$: v_r is the "social value" derived from giving any agent its r-th choice item.

The Rank-value mechanism is given by the following integer program:

$$\begin{aligned} \max_x \sum_{i \in N} \sum_{j \in M} & x_{ij} v_{r_i(j)} \\ \texttt{s.t.} \sum_{i \in N} x_{ij} &\leq 1 \quad \forall j \in M \\ \sum_{j \in M} x_{ij} &\leq 1 \quad \forall i \in N \\ x_{ij} \in \{0, 1\} & \;\; \forall i, j \in N \times M \end{aligned} \tag{1}$$

Here, x_{ij} is a binary variable indicating that an item j is allocated to an agent i, i.e.,

$$x_{ij} = \begin{cases} 1, & \text{if an item } j \text{ is allocated to an agent } i, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the objective function is simply the total "social value" of allocation. The first constraint means that every item can be allocated to at most one agent. The second constraint means that every agent receives at most one item.

(a) **[10 Points]** Implement the Rank-value mechanism as an integer program. Use JOpt to formulate and solve the problem. Your algorithm should take an input file of the following format:

Line 1: `numberOfAgents,numberOfItems`
Line 2: $\texttt{value}_1,...,\texttt{value}_n$
Line 3: $\texttt{item}_{11},...,\texttt{item}_{1n}$
...
Line n+2: $\texttt{item}_{n1},...,\texttt{item}_{nn}$

Here, the first line contains the number of agents and items. The second line contains the rank valuation and all the following lines contain preference profiles of individual agents. For example, a line

$$\texttt{item}_{i1},...,\texttt{item}_{in}$$

means that a preference profile of an agent $i \in N$ looks like. $\texttt{item}_{11} \succ_i ... \succ_i \texttt{item}_{1n}$ (for convenience assume $\texttt{item}_{ij} \in 0, 1, ..., n-1 \;\; \forall i, j$). The output should look like:

Line 1: `x0_0 0`
...
Line $i \cdot n + j$: `xi_j 1`

...

Here, each line corresponds to a tuple of a binary optimization variable x_{ij} and its value.

(b) **[10 Points]** Assume there are three agents s_1, s_2, s_3 and three items $M = a, b, c$. The agent's preferences are

$$\begin{aligned} s_1 : &\quad a \succ_{s_1} c \succ_{s_1} b \\ s_2 : &\quad a \succ_{s_2} b \succ_{s_2} c \\ s_3 : &\quad b \succ_{s_3} a \succ_{s_3} c \end{aligned}$$

and the rank valuation is $v =< 10, 5, 1 >$.

i. For this preference profile determine the outcome of the Rank-value mechanism.

ii. For the same profile determine all possible outcomes of the Randomized Serial Dictatorship mechanism (see Definition 12.7). How often does the Randomized Serial Dictatorship produce the same outcome as the Rank-value matching?

(c) **[10 Points]** Now assume the value vector $v =< 100, 50, 10, 1 >$. Evaluate the Rank-value matching for the following preference profile:

$$\begin{aligned} s_1 : &\quad a \succ_{s_1} b \succ_{s_1} c \succ_{s_1} d \\ s_2 : &\quad a \succ_{s_2} c \succ_{s_2} b \succ_{s_2} d \\ s_3 : &\quad c \succ_{s_3} d \succ_{s_3} b \succ_{s_3} a \\ s_4 : &\quad d \succ_{s_4} a \succ_{s_4} b \succ_{s_4} c \end{aligned}$$

Which (false) report should s_1 submit to get a better allocation? What does this tell you about strategyproofness of the Rank-value matching mechanism?

3. **[Bonus Assignment]** Boston mechanism

The Random Serial Dictatorship mechanism is strategyproof and Pareto efficient (Theorem 13.6). In the same setting the Boston mechanism works as follows: in the first round it tries to give every agent its first choice. If more than one agent desire the same object as their first choice, the one with the lowest number gets it. Agents who have received an object are removed from the mechanism, together with their assigned object. Second choices of the remaining agents are considered in the same way, then third choices in the third round, etc.

Formally, this is described be the following steps.

- Each agent i makes a claim $\widehat{\succ}_i$ about its preference ordering.
- Step 1: For each item, the mechanism counts how many agents rank the item first. Let d_j^1 denote the number of agents who rank item j first.
  - If $d_j^1 = 0$, the item remains unallocated in this round.
  - If $d_j^1 = 1$, the (only) agent who ranks item j first gets it.
  - If $d_j^1 \geq 2$, the agent with the lowest number gets it.
- Step k: For each unallocated item and each agent who has not received an item yet, the mechanism counts how many agents rank the item kth favorite. Again d_j^k is the number of agents who rank item j in kth position and have not yet received another item. The items are allocated as in step 1.

(a) Implement the Boston mechanism. Show that Boston mechanism provides a different allocation from the Rank-value mechanism.

(b) Using the examples with three and four agent from the previous exercise, compare the total value of allocations of the Boston and the Rank-value mechanisms.

(c) Give an example that shows that the Boston mechanism is not strategyproof.

(d) Give two reasons why this mechanism might be popular in practice even though it is not strategyproof.