Computation and Economics - Fall 2014
Assignment #1: Game Theory

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Out Tuesday, September 23, 2013
Due **12:15** sharp: **Tuesday, October 7, 2013**
Submissions should be made in writing before the beginning of the lecture.

[**Total: 100 Points**] This is a single-person assignment. Points will be awarded for clarity, correctness and completeness of the answers. Reasoning must be provided with every answer, i.e., please show your work. You get most of the credit for showing the way in which you arrived at the solution, not for the final answer. You are free to discuss the assignment with other students. However, you are not allowed to share (even partial) answers with each other, and **copying will be penalized**.

1. [**11 Points**] Iterated elimination.

   (a) [**3 Points**] What strategies survive iterated elimination of strictly dominated strategies in the following normal-form game?

	L	C	R
T	0,6	3,4	1,5
M	2,1	3,3	2,4
B	2,3	5,1	3,4

   (b) [**2 Points**] Find a second sequence of eliminations for part 1a.

   (c) [**3 Points**] What strategies survive iterated elimination of strictly dominated strategies in the following normal-form game?

	L	C	R
T	0,0	-1,1	1,1
M	0,1	1,0	1,0
B	-1,0	0,1	0,1

   (d) [**3 Points**] Give an example of a game where no action can be eliminated.

2. [**16 Points**] Computational complexity.

   The following algorithm is called the *naive* (or *brute-force*) algorithm for finding a pure-strategy Nash equilibrium of a normal-form game:

   - Input: Players $N = \{1, \ldots, n\}$, action sets A_i and utility functions u_i for $i \in N$.
   - For each action profile $a \in A_1 \times \ldots \times A_n$ check whether it is a Nash equilibrium:
     - For each player i and each action $a'_i \in A_i$ check whether $u_i(a'_i, a_{-i}) \leq u_i(a_i, a_{-i})$.

  - If all of these checks succeeded, a is a Nash equilibrium, otherwise it is not.

- Output the first action profile which is a Nash equilibrium, or "NONE" if the check never succeeded.

For the following runtime analysis, let n be the number of agents like above and assume that any player has the same number m of actions, i.e., $m = |A_i|$ for any i.

(a) [**5 Points**] For $n = 2$, what is the worst-case run time of the naive algorithm, expressed in O-notation? (*Hint: Count how many times the algorithm compares terms of the form* $u_i(a_i, a_{-i})$ *and* $u_i(a'_i, a_{-i})$. *Assume that evaluation of* u_i*, comparisons of numbers, etc., all take constant time, so they are considered a single "step" of computation.*)

(b) [**6 Points**] What is the run time for general n?

(c) [**5 Points**] Does this run-time result give you reason for concern regarding the usefulness of the Nash equilibrium concept? Explain!

3. [**38 Points**] The Section Game.

Consider the following *Section Game*: Each week, an Economics & Computation assistant has to choose whether to put a lot of effort or little effort into preparing the section for the lecture. A student, on the other hand, must decide whether to attend section or not.[1]

- Preparing section is costly for the assistant, but if he is well-prepared and the student comes, the assistant feels rewarded (utility 5), while if the student does not attend, his effort was in vain (utility -5). If he is poorly prepared and the student attends section, he will feel embarrassed (utility -10).
- The student's utility depends on how well the assistant is prepared: If the assistant is well prepared, he receives utility 10 from an enlightening class. Otherwise, he is wasting his time (utility -5).
- If the student does not attend section, he gets 0 utility, as well as the assistant if he is being lazy and this remains unnoticed.

(a) [**6 Points**] Provide a formal description including the payoff matrix for this game.

(b) [**4 Points**]

  i. [**2 Points**] Which actions remain after iterated elimination of strictly dominated actions?

  ii. [**2 Points**] Is there a dominant strategy for either player? If yes, name it, if not, explain why.

(c) [**6 Points**] Find all pure strategy Nash equilibria of the game.

(d) [**8 Points**] Draw the best response graph for the two players and find a non-trivial mixed-strategy Nash equilibrium, i.e., each agent's strategy has a support of two actions.

(e) [**8 Points**]

  i. [**2 Points**] Which of the pure strategy Nash equilibria from part 3c are Pareto optimal amongst pure strategy profiles?

[1] We make the simplifying assumption here that there is only a single student. This could also model a situation where there actually are several students in the class, but they coordinate their actions.

ii. **[6 Points]** Is the mixed-strategy Nash equilibrium from part 3d Pareto optimal amongst mixed strategies? If yes, give a proof, if not, give another mixed strategy which Pareto dominates it.

(f) **[6 Points]** Consider a variation of the Section Game where there are two students instead of only one (and one assistant). Suppose that the students' payoffs are unchanged, except if both students attend and the assistant is well prepared: then the students distract each other and each gets payoff 8 (instead of 10). If $m \in \{0, 1, 2\}$ students attend, the payoff of the assistant changes as follows:

$$u_{\text{Assistant}}(\text{Effort}, \ldots) = 10m - 5, \quad u_{\text{Assistant}}(\text{Slack}, \ldots) = -10m.$$

Give a normal-form representation of this game.

4. **[20 Points]** Preservation of Nash equilibria.

Let G be a game and let α and β be real numbers such that $\alpha > 0$. Define the *positive affine transformation* of G with respect to α and β to be the game G' that arises from G by changing the utility functions u_i of all agents i to

$$u'_i(a) := \alpha \cdot u_i(a) + \beta.$$

(a) **[10 Points]** Show that pure strategy Nash equilibria are preserved under positive affine transformations, i.e. that the PSNE of the game G' are the same as those of G.

(b) **[10 Points]** Are the mixed strategy Nash equilibria preserved as well? If yes, give a proof, if no, give a counterexample.

5. **[15 Points]** P2P File Sharing.

(a) **[8 Points]** The strategic-piece-revelation strategy in the BitTorrent protocol uses "under-reporting" of pieces. Consider instead a strategy based on "over-reporting" pieces, i.e., a client reporting to have pieces that it doesn't actually have. Explain why such a strategy might make sense. Be explicit about whether the strategy makes sense against the default BitTorrent client, or whether you are assuming some kind of other behavior. Also describe how a client using this strategy could be detected.

(b) **[7 Points]** The strategies in file-sharing games are provided by software, with new clients (= strategies) such as BitThief released over time. Suppose that a client is universally adopted and even proved to be a Nash equilibrium with itself. Why might you still worry this is insufficient to provide stability of the ecosystem?

6. **[Bonus Assignment]** Repeated Prisoners' Dilemma.

Consider the Prisoners' Dilemma game from the lecture. Suppose, two players play the game repeatedly. Their payoff from the first round is the payoff of the single game. Payoffs in later rounds are discounted by a factor $0 < \delta < 1$, for example, if $\delta = 0.9$, and player 1 cooperates and player 2 defects in the 5th round, the payoff for player 1 from round 5 is $\delta^{5-1} u_1(D, C) = 0.9^4 \cdot (-5) \approx -3.28$.

Figure 1: Possible outcome in the game from exercise 6

If the players' actions are $a^k = (a_1^k, a_2^k)$ in the kth round, the total payoff to player i after m rounds is

$$u_i(a^1, a^2, \ldots, a^m) = \sum_{k=1}^{m} \delta^{k-1} u_i(a_1^k, a_2^k).$$

In an infinitely repeated game, the payoff is

$$u_i(a^1, a^2, \ldots) = \sum_{k=1}^{\infty} \delta^{k-1} u_i(a_1^k, a_2^k).$$

If $0 < \delta < 1$, this is well-defined through the geometric series: for $0 < \delta < 1$ we have

$$\sum_{k=1}^{\infty} \delta^{k-1} = \frac{1}{1-\delta}.$$

In a repeated game, an agent can observe the actions from the previous m rounds before he has to decide on the next action (in round $m+1$). The *GrimTrigger* strategy works as follows:

- Play C in round 1.
- If the other agent has never played D in the previous m rounds, play C in the $(m+1)$st round, otherwise play D.

Show that GrimTrigger is not a Nash equilibrium if the game is repeated finitely many times, but it is a Nash equilibrium of the infinitely repeated game for some discount factor $0 < \delta < 1$. Proceed as follows:

(a) If the game is repeated m times, prove that GrimTrigger is not a Nash equilibrium of this game for any discount factor δ. *Hint: What happens in the last round?*

(b) Prove that GrimTrigger is a best response to itself in the infinitely repeated game. *Hint: Determine the utility of player 1 if both agents play GrimTrigger forever. Then determine the maximum utility player 1 can attain if he deviates from GrimTrigger at some point. Compare these values.*