11 Equivalence of Swap Monotonicity and Strategyproofness for Deterministic Mechanisms

Proposition 5. A deterministic mechanism $f$ is strategyproof if and only if it is swap monotonic.

Proof. Since deterministic mechanisms are just special cases of random mechanisms, Theorem 1 applies: A deterministic mechanism $f$ is strategyproof if and only if it is swap monotonic, upper invariant, and lower invariant. Thus, strategyproofness implies swap monotonicity (i.e., sufficiency in Proposition 5). For necessity, observe that swap monotonicity implies upper and lower invariance for deterministic mechanisms: If a swap (say from $P_i: a > b$ to $P'_i: b > a$) affects an agent’s assignment, then the assignment must change strictly for the two objects $a$ and $b$ that are swapped. But under a deterministic mechanism, this change can only be from 0 to 1 or from 1 to 0. The only possible changes are therefore the ones where an agent receives $a$ with certainty if it reports $P_i: a > b$ and receives $b$ with certainty if she reports $P'_i: b > a$. □
Comparing Mechanisms by Vulnerability to Manipulation and Degree of Strategyproofness

The next proposition shows that the comparison of mechanisms by their vulnerability to manipulation and by their degrees of strategyproofness are consistent but not equivalent.

**Proposition 6.** For any setting \((N, M, q)\) and mechanisms \(f, g\), the following hold:

1. If \(g\) is strongly as manipulable as \(f\), then \(\rho_{(N,M,q)}(f) \geq \rho_{(N,M,q)}(g)\).

2. If \(\rho_{(N,M,q)}(f) > \rho_{(N,M,q)}(g)\), and if \(f\) and \(g\) are comparable by the strongly as manipulable as relation, then \(g\) is strongly as manipulable as \(f\).

In Proposition 6, the strongly as manipulable as relation is extended to random assignment mechanisms as follows:

**Definition 10.** For a given setting \((N, M, q)\) and two mechanisms \(f, g\), we say that \(g\) is strongly as manipulable as \(f\) if, for all agents \(i \in N\), all preference profiles \((P_i, P_{-i}) \in \mathcal{P}^N\), and all utility functions \(u_i \in U_{P_i}\), the following holds: If there exists a misreport \(P'_i \in \mathcal{P}\) such that

\[
E_{f_i(P_i, P_{-i})}[u_i] < E_{f_i(P'_i, P_{-i})}[u_i],
\]

then there exists a (possibly different) misreport \(P''_i \in \mathcal{P}\) such that

\[
E_{g_i(P_i, P_{-i})}[u_i] < E_{g_i(P''_i, P_{-i})}[u_i].
\]

In words, \(g\) is strongly as manipulable as \(f\) if any agent who can manipulate \(f\) in a given situation can also manipulate \(g\) in the same situation.

**Proof of Proposition 6.** Statement 1. Observe that, if \(f\) is strongly as manipulable as \(g\), then any agent who can manipulate \(g\) also finds a manipulation to \(f\). Thus, the set of utility functions on which \(g\) makes truthful reporting a dominant strategy cannot be larger than the set of utilities on which \(f\) does the same. This in turn implies \(\rho_{(N,M,q)}(f) \geq \rho_{(N,M,q)}(g)\).

Statement 2. Observe that, if \(\rho_{(N,M,q)}(f) > \rho_{(N,M,q)}(g)\), then there exists a utility function \(\tilde{u}\) in \(\text{URBI}\left(\rho_{(N,M,q)}(f)\right)\), which is not in \(\text{URBI}\left(\rho_{(N,M,q)}(g)\right)\), and for which \(g\) is manipulable, but \(f\) is not. Thus, \(f\) cannot be strongly as manipulable as \(g\), but the converse is possible.
13 Discounted Dominance and Partial Strategyproofness

In this section, we present \( r \)-discounted dominance, a new dominance notion that generalizes stochastic dominance but includes \( r \) as a discount factor. We then provide an alternative definition of partial strategyproofness by showing that it is equivalent to the incentive concept induced by \( r \)-discounted dominance.

Definition 11. For a bound \( r \in [0, 1] \), a preference order \( P_i \in \mathcal{P} \) with \( P_i : j_1 > \ldots > j_m \), and assignment vectors \( x_i, y_i \), we say that \( x_i \) \( r \)-discounted dominates \( y_i \) at \( P_i \) if, for all ranks \( K \in \{1, \ldots, m\} \), we have

\[
\sum_{k=1}^{K} r^k \cdot x_{i,j_k} \geq \sum_{k=1}^{K} r^k \cdot y_{i,j_k}.
\]  

Observe that, for \( r = 1 \), this is precisely the same as stochastic dominance. However, for \( r < 1 \), the difference in the agent’s assignment for the \( k \)th choice is discounted by the factor \( r^k \). Analogous to stochastic dominance for SD-strategyproofness, we can use \( r \)-discounted dominance (\( r \)-DD) to define the corresponding incentive concept.

Definition 12. Given a setting \((N, M, q)\) and a bound \( r \in (0, 1] \), a mechanism \( f \) is \( r \)-DD-strategyproof if, for all agents \( i \in N \), all preference profiles \((P_i, P_{-i}) \in \mathcal{P}^N \), and all misreports \( P'_i \in \mathcal{P} \), \( f_i(P_i, P_{-i}) \) \( r \)-discounted dominates \( f_i(P'_i, P_{-i}) \) at \( P_i \).

The next theorem yields the equivalence to \( r \)-partial strategyproofness.

Proposition 7. Given a setting \((N, M, q)\) and a bound \( r \in [0, 1] \), a mechanism \( f \) is \( r \)-partially strategyproof if and only if it is \( r \)-DD-strategyproof.

Proof. Given the setting \((N, M, q)\), we fix an agent \( i \in N \), a preference profile \((P_i, P_{-i}) \in \mathcal{P}^N \), and a misreports \( P'_i \in \mathcal{P} \). The following claim establishes equivalence of the \( r \)-partial strategyproofness constraints and the \( r \)-DD-strategyproofness constraints for any such combination \((i, (P_i, P_{-i}), P'_i)\) with \( x = f_i(P_i, P_{-i}) \) and \( y = f_i(P'_i, P_{-i}) \).
Claim 1. Given a setting \( (N, M, q) \), a preference order \( P \in \mathcal{P} \), assignment vectors \( x, y \), and a bound \( r \in [0, 1] \), the following are equivalent:

A. For all utility functions \( u_i \in U_{P_i} \cap \text{URBI}(r) \) we have \( \sum_{j \in M} u_i(j) \cdot x_j \geq \sum_{j \in M} u_i(j) \cdot y_j \).

B. \( x \) \( r \)-discounted dominates \( y \) at \( P \).

Proof of Claim 1. Sufficiency (B \( \Rightarrow \) A). Let \( P_{i_1} : j_1 > \ldots > j_m \). Assume towards contradiction that Statement B holds, but that for some utility function \( u_i \in U_{P_i} \cap \text{URBI}(r) \), we have

\[
\sum_{j=1}^{m} u_i(j) \cdot (x_{j} - y_{j}) < 0. \tag{34}
\]

Without loss of generality, we can assume \( \min_{j \in M} u_i(j) = 0 \). Let \( \delta_k = x_{j_k} - y_{j_k} \) and let

\[
S(K) = \sum_{k=1}^{K} u_i(j_k) \cdot (x_{j_k} - y_{j_k}) = \sum_{k=1}^{K} u_i(j_k) \cdot \delta_k. \tag{35}
\]

By assumption, \( S(m) = S(m - 1) < 0 \) (see (34)), so there exists a smallest value \( K' \in \{1, \ldots, m - 1\} \) such that \( S(K') < 0 \), but \( S(k) \geq 0 \) for all \( k < K' \). Using Horner’s method, we rewrite the partial sum and get

\[
S(K') = \sum_{k=1}^{K'} u_i(j_k) \cdot \delta_k = \left( \frac{S(K' - 1)}{u_i(j_{K'})} + \delta_{K'} \right) \cdot u_i(j_{K'}) \tag{36}
\]

\[
= \left( \left( \frac{S(K' - 2)}{u_i(j_{K' - 1})} + \delta_{K' - 1} \right) \cdot \frac{u_i(j_{K' - 1})}{u_i(j_{K'})} + \delta_{K'} \right) \cdot u_i(j_{K'}) \tag{37}
\]

\[
= \left( \left( \ldots \left( \delta_1 \cdot \frac{u_i(j_1)}{u_i(j_2)} + \delta_2 \right) \cdot \frac{u_i(j_2)}{u_i(j_3)} + \ldots \right) \cdot \frac{u_i(j_{K' - 1})}{u_i(j_{K'})} + \delta_{K'} \right) \cdot u_i(j_{K'}). \tag{38}
\]

Since \( u_i \) satisfies \( \text{URBI}(r) \), the fraction \( \frac{u_i(j_{K' - 1})}{u_i(j_{K'})} \) is bounded from below by \( 1/r \). But since \( S(K' - 1) \geq 0 \) and \( u_i(j_{K' - 1}) > 0 \), we must have that

\[
\left( \frac{S(K' - 2)}{u_i(j_{K' - 1})} + \delta_{K' - 1} \right) = \frac{S(K' - 1)}{u_i(j_{K' - 1})} \geq 0. \tag{39}
\]

Therefore, when replacing \( \frac{u_i(j_{K' - 1})}{u_i(j_{K'})} \) by \( 1/r \) in (38) we only make the term smaller. By the
same argument, we can successively replace all the terms \( \frac{u_i(j_{k-1})}{u_i(j_k)} \) and obtain
\[
0 > S(K') \geq \left( \left( \cdots \left( \frac{\delta_1}{r} + \frac{\delta_2}{r} \right) \cdot \frac{1}{r} + \cdots \right) \cdot \frac{1}{r} + \delta_{K'} \right) \cdot u_i(j_{K'}) = \frac{u_i(j_{K'})}{r^{K'}} \cdot \sum_{k=1}^{K'} r^k \cdot \delta_k.
\] (40)

This contradicts \( r \)-discounted dominance of \( x \) over \( y \) at \( P_i \), since
\[
\sum_{k=1}^{K'} r^k \cdot \delta_k = \sum_{k=1}^{K'} r^k \cdot (x_{jk} - y_{jk}) \geq 0.
\] (41)

**Necessity** (\( A \Rightarrow B \)). Let \( P_i : j_1 > \ldots > j_m \). Assume towards contradiction that Statement A holds, but \( x \) does not \( r \)-discounted dominate \( y \) at \( P_i \), i.e., for some \( K \in \{1, \ldots, m\} \), we have
\[
\sum_{k=1}^{K} r^k \cdot x_{jk} < \sum_{k=1}^{K} r^k \cdot y_{jk},
\] (42)
and let \( K \) be the smallest rank for which inequality (42) is strict. Then the value
\[
\Delta = \sum_{k=1}^{K} r^k \cdot (y_{jk} - x_{jk}),
\] (43)
is strictly positive. Let \( D \geq d > 0 \), and let \( u_i \) be the utility function defined by
\[
u_i(j_k) = \begin{cases} Dr^k, & \text{if } k \leq K, \\ dr^k, & \text{if } K + 1 \leq k \leq m - 1, \\ 0, & k = m. \end{cases}
\] (44)

This utility function satisfies URBI(\( r \)). Furthermore, we have
\[
\sum_{j \in M} u_i(j) \cdot x_j - \sum_{j \in M} u_i(j) \cdot y_j = \sum_{l=1}^{m} u(j_l) \cdot (x_{i,j_l} - y_{i,j_l})
\] (45)
\[
= D \cdot \sum_{k=1}^{K} r^k \cdot (x_{jk} - y_{jk}) + d \cdot \sum_{k=K+1}^{m-1} r^k \cdot (x_{jk} - y_{jk})
\leq -D \cdot \Delta + d.
\] (46)

For \( d < D \cdot \Delta \), \( \sum_{j \in M} u_i(j) \cdot x_j - \sum_{j \in M} u_i(j) \cdot y_j \) is strictly negative, a contradiction. \( \Box \)

This concludes the proof of Proposition 7. \( \Box \)
Proposition 7 generalizes the equivalence between EU-strategyproofness and SD-strategyproofness (Erdil, 2014). Moreover, it yields an alternative definition of $r$-partial strategyproofness in terms of discounted dominance. This shows that the partial strategyproofness concept integrates nicely into the landscape of existing incentive concepts, many of which are defined using dominance notions (e.g., SD-, weak SD-, LD-, and ST$^1$-strategyproofness).

The dominance interpretation also unlocks the partial strategyproofness concept to algorithmic analysis: Recall that the definition of $r$-partial strategyproofness imposes inequalities that have to hold for all utility functions within the set $\text{URBI}(r)$. This set is infinite, which makes algorithmic verification of $r$-partial strategyproofness infeasible via its original definition. However, by the equivalence from Proposition 7, it suffices to verify that all (finitely many) constraints for $r$-discounted dominance are satisfied (i.e., the inequalities (33) from Definition 11). These inequalities can also be used to encode $r$-partial strategyproofness as linear constraints to an optimization problem. This enables an automated search in the set of $r$-partially strategyproof mechanisms while optimizing for some other design objective under the automated mechanism design paradigm (Sandholm, 2003).

14 URBI($r$) and the Probabilistic Serial Mechanism

3-by-3 Settings: Consider the same setting as in the motivating example in the introduction with three agents $N = \{1, 2, 3\}$ and three objects $M = \{a, b, c\}$ with unit capacity. Recall that

\[
\text{PS}_1(P_1, P_{-1}) = (3/4, 0, 1/4), \quad \text{(47)} \\
\text{PS}_1(P'_1, P_{-1}) = (1/2, 1/3, 1/6). \quad \text{(48)}
\]

The gain in expected utility for agent 1 from misreporting is therefore

\[
- \frac{1}{4} \cdot u_1(a) + \frac{1}{3} \cdot u_1(b) - \frac{1}{12} \cdot u_1(c). \quad \text{(49)}
\]

1Sure thing dominance; see (Aziz, Brandt and Brill, 2013)
This is weakly negative if and only if
\[
\frac{u_1(b) - u_1(c)}{u_1(a) - u_1(c)} \leq 3/4, \quad (50)
\]
which is the condition for \( u_1 \) to satisfy URBI(3/4). Next, recall that we have computed the degree of strategyproofness of PS in this setting to be \( \rho_{(N,M,q)}(PS) = 3/4 \) (Figure 2 in Section 5.1). Any agent therefore has a dominant strategy to report truthfully under PS in this setting \( \text{if and only if} \) that agent’s utility function satisfies URBI(3/4). In other words, the set of utility functions for which PS induces good incentives is precisely URBI(3/4).

4-by-4 Settings (adapted from (Abächerli, 2017)): Next, we show a similar insight for a setting with four agents \( N = \{1, 2, 3, 4\} \) and four objects \( M = \{a, b, c, d\} \) with unit capacity. We have computed the degree of strategyproofness of PS in this setting to be \( \rho_{(N,M,q)}(PS) = 1/2 \) (Figure 2 in Section 5.1). Consider the preferences
\[
P_1 : a > b > c > d, \quad (51)
\]
\[
P_2 : b > c > d > a, \quad (52)
\]
\[
P_3 : c > a > b > d, \quad (53)
\]
\[
P_4 : c > b > d > a, \quad (54)
\]
and the misreport
\[
P'_1 : b > a > c > d \quad (55)
\]
by agent 1. The resulting assignments are
\[
\begin{align*}
\text{PS}_1(P_1, P_{-1}) &= (3/4, 0, 0, 1/4), \quad (56) \\
\text{PS}_1(P'_1, P_{-1}) &= (1/2, 1/2, 0, 0). \quad (57)
\end{align*}
\]
Thus, agent 1’s gain in expected utility from misreporting is
\[
-1/4 \cdot u_1(a) + 1/2 \cdot u_1(b) - 1/4 \cdot u_1(d). \quad (58)
\]
This is weakly negative if and only if

\[
\frac{u_1(b) - u_1(c)}{u_1(a) - u_1(c)} \leq 1/2. \quad (59)
\]

Next, consider the preference profile

\[
P_1, P_2 : a > b > c > d, \quad (60)
P_3, P_4 : a > c > d > b, \quad (61)
\]

and the misreport

\[
P'_1 : a > c > b > d. \quad (62)
\]

Then

\[
\begin{align*}
PS_1(P_1, P_1) &= (1/4, 1/2, 0, 1/4), \\
PS_1(P'_1, P_1) &= (1/4, 1/3, 1/3, 1/12),
\end{align*}
\]

so that the gain in expected utility for agent 1 from misreporting is

\[
-1/6 \cdot u_1(b) + 1/3 \cdot u_1(c) - 1/6 \cdot u_1(d). \quad (65)
\]

This is weakly negative if and only if

\[
\frac{u_1(c) - u_1(d)}{u_1(b) - u_1(d)} \leq 1/2. \quad (66)
\]

Observe that conditions (59) and (66) are precisely equivalent to the requirement that \( u_1 \) satisfies URBI\((1/2)\). Thus, since \( \rho_{(N,M,q)}(PS) = 1/2 \), every agent with a utility function that satisfies URBI\((1/2)\) has a dominant strategy to report truthfully. However, the two examples show that every agent with a utility function that violates URBI\((1/2)\) can beneficially manipulate the mechanism in some situations. This implies that, in the setting with 4 agents and 4 objects in unit capacity, the set of utility functions for which PS makes truthful reporting a dominant strategy is precisely URBI\((1/2)\).
5-by-5 Settings (adapted from (Abächerli, 2017)): In a setting with 5 agents \( N = \{1, 2, 3, 4, 5\} \) and 5 objects \( M = \{a, b, c, d, e\} \) with unit capacity, we again consider PS. Using the same algorithm as in Section 5.1, we can determine the degree of strategyproofness in this setting to be \( \rho(N,M,q)(\text{PS}) = 1/2 \). The utility function \( u_i \) with

\[
  u_i(a) = 7.99, \quad u_i(b) = 4, \quad u_i(c) = 2, \quad u_i(d) = 1, \quad u_i(e) = 0
\]

violates URBI(1/2) because

\[
  \frac{u_i(a) - \min_{j \in M} u_i(j)}{u_i(b) - \min_{j \in M} u_i(j)} = \frac{4}{7.99} > 1/2.
\]

However, exhaustive search over all possible preference profiles and misreports (using a computer) reveals that an agent with this utility function can not benefit from misreporting. Thus, in the setting with 5 agents and 5 objects, the set of utility functions for which PS makes truthful reporting a dominant strategy is strictly larger than URBI(1/2). This contrasts tightness of this set of the 3-by-3 and 4-by-4 settings.

References


