Hybrid Mechanisms: Trading off Efficiency and Strategyproofness in One-Sided Matching

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Abstract

We study one-sided matching mechanisms where agents have vNM utility functions and report ordinal preferences. Strong impossibility results restrict the design space of strategyproof mechanisms in this domain. Improving on efficiency beyond the ex-post efficiency of Random Serial Dictatorship (RSD) generally requires trade-offs. In this paper, we introduce hybrid mechanisms, which are convex combinations of existing mechanisms, and we show that they are a powerful yet simple method for trading off strategyproofness and efficiency. We present conditions under which hybrid mechanisms remain partially strategyproof with respect to a given bound for the degree of strategyproofness. At the same time, these hybrids have better efficiency properties than the less efficient component. Our approach can be successfully applied to create hybrids of RSD and the Probabilistic Serial mechanism (PS), as well as hybrids of RSD and the adaptive Boston mechanism (ABM). We provide numerical results demonstrating that the improvements can be significant.

1. Introduction

An economic setting where indivisible goods or resources (objects) are allocated to self-interested agents without the use of monetary transfers poses a one-sided matching problem. Examples include the allocation of students to schools, houses to tenants, and teachers to training programs (Niederle, Roth and Sönmez, 2008). Hylland and Zeckhauser (1979) first presented a mechanism for this problem, and it has since attracted much attention by mechanism designers (e.g., Abdulkadiroğlu and Sönmez (1998); Bogomolnaia and Moulin (2001); Abdulkadiroğlu and Sönmez (2003); Featherstone (2011)).

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It is often desired or even required that one-sided matching mechanisms perform well on multiple dimensions, such as efficiency, fairness, and strategyproofness. However, severe impossibility results prevent them from excelling on all these dimensions simultaneously (e.g., Zhou (1990)). This makes the one-sided matching problem an interesting mechanism design challenge. The folklore Random Serial Dictatorship mechanism is strategyproof and anonymous, but only ex-post efficient. If strategyproofness is dropped, the more demanding ordinal efficiency can be achieved by the Probabilistic Serial mechanism (PS). However, no ordinally efficient mechanism can simultaneously satisfy strategyproofness and symmetry (Bogomolnaia and Moulin, 2001). Rank efficiency, an even stronger efficiency concept, can be achieved via Rank-value mechanisms (Featherstone, 2011). However, while PS is at least weakly strategyproof, rank efficiency is even incompatible with weak strategyproofness. Obviously, trade-offs are necessary and have been the focus of recent research (e.g., see Budish (2012); Azevedo and Budish (2012); Aziz, Brandt and Brill (2013a)).

1.1. Hybrid Mechanisms

This paper introduces a new method for trading off strategyproofness and efficiency of one-sided matching mechanisms. The key idea is to “mix” a strategyproof mechanism \( f \) with a non-strategyproof mechanism \( g \) that has better efficiency properties. Our goal is to design hybrids that inherit a portion of the desirable properties of both mechanisms, i.e., higher efficiency than \( f \), but also better incentives for truth-telling than \( g \). Hybrid mechanisms are convex combinations of existing mechanisms: for mechanisms \( f \) and \( g \), the \( \beta \)-hybrid is given by the convex combination

\[
h_\beta(f, g) = (1 - \beta)f + \beta g,
\]

where \( \beta \in [0, 1] \) is called the mixing factor. This paper explores how these hybrids can be used to facilitate a parametric trade-off between efficiency and strategyproofness.

1.2. Partial Strategyproofness and Imperfect Dominance

To describe a parametric trade-off between strategyproofness and efficiency, we require two concepts: first, a parametric relaxation of strategyproofness, and second, a notion of how to compare mechanisms by their efficiency.

Towards the first question, we employ the URBI\((r)\)-partial strategyproofness concept, which we have recently introduced in Mennle and Seuken (2014b). It yields a parametric measure for the degree of strategyproofness: for mechanisms that are swap consistent and weakly invariant, the degree of strategyproofness can be measured by means of a single number \( r \in [0, 1] \). Intuitively, a mechanism is URBI\((r)\)-partially strategyproof if it makes truthful reporting a dominant strategy as long as an agent’s values for different objects differ by at least a factor \( r \) (i.e., bounded away from indifference by \( r \)).

In order to construct URBI\((r)\)-partially strategyproof hybrid mechanisms, we require two conditions. First, we need that \( g \) is weakly less varying than \( f \), i.e., if whenever \( g \) changes the allocation upon a change of report by some agent, then so does \( f \). Our second condition is weak invariance, a form of truncation robustness introduced by Hashimoto et al. (2013). We
say that the pair \((f, g)\) is \textit{hybrid-admissible} if \(f\) is strategyproof and \(g\) satisfies both conditions. We show that hybrid-admissibility is sufficient and that the construction may fail if any of the conditions for hybrid-admissibility are violated.

Towards the second question, i.e., comparing mechanisms by their efficiency, we employ the well-known concepts of \textit{ordinal} and \textit{rank dominance}. We present conditions under which the newly defined hybrids dominate the component \(f\). One concern is that while the allocation from \(g\) may dominate the allocation from \(f\) at some type profiles, the comparison may be ambiguous for other type profiles. To address this concern, we introduce \textit{imperfect dominance}, which requires that a mechanism is better (i.e., the allocations from \(g\) dominate the allocations from \(f\)) whenever the allocations are comparable by dominance.

1.3. Overview of Contributions

In this paper, we show that hybrid mechanisms are a powerful yet simple method for trading off strategyproofness and efficiency in the design of one-sided matching mechanisms. The following results are obtained for any fixed (but arbitrary) setting, i.e., number of agents, objects, and capacities.

1) \textbf{Construction of URBI(r)-partially Strategyproof Hybrid Mechanisms (Thm. 1):} We show that for any hybrid-admissible pair \((f, g)\) and any bound \(r \in (0, 1]\), there exists \(\beta \in (0, 1]\) such that \(h_\beta(f, g)\) is URBI(r)-partially strategyproof.

2) \textbf{Parametric Trade-off Between Efficiency and Strategyproofness (Prop. 6):} We show that for any hybrid-admissible pair \((f, g)\), where \(g\) is manipulable and imperfectly dominates \(f\), the hybrid \(h_\beta(f, g)\) becomes more efficient, but also less strategyproof for a higher mixing factor \(\beta\).

3) \textbf{Computability of the Maximum Mixing Factor (Prop. 8):} We show that for any hybrid-admissible pair \((f, g)\) and bound \(r \in [0, 1]\) the mechanism designer’s problem of finding the maximum mixing factor \(\beta_{\text{max}}\) is computable (although the algorithm we present has exponential complexity).

4) \textbf{Instantiations (Thms. 2 & 3):} We show that PS and the adaptive Boston mechanism (ABM) are weakly less varying than RSD. Consequently, these mechanisms can be combined with \(f = \text{RSD}\) to construct non-trivial hybrids that trade off strategyproofness for efficiency in a meaningful way. Numerical analysis shows that a significant share of the more efficient component can be included.

Trade-offs between strategyproofness and efficiency have recently attracted a lot of attention from researchers (e.g., Budish (2012); Aziz, Brandt and Brill (2013a)). This paper demonstrates how and under what conditions hybrid mechanisms can be used to facilitate this trade-off for one-sided matching mechanisms. We show that the conditions are not trivial, but satisfied by some well-known mechanisms, such as PS and ABM in combination with RSD. The hybrids also inherit the fairness properties that both components satisfy, e.g., anonymity and symmetry.
While we do not expect our hybrids to be applied by practitioners directly, they provide a new understanding of trade-offs between strategyproofness and efficiency in one-sided matching. In particular, they may serve as benchmark mechanisms for future research. Our parametric approach goes beyond comparing discrete points in the mechanism design space, as we show precisely in what sense strategyproofness is relaxed in order to achieve efficiency gains. In particular, this is the first paper to present a constructive and parametric method to perform such trade-offs in this domain.

2. Related Work

While the seminal paper on one-sided matching mechanisms by Hylland and Zeckhauser (1979) proposed a mechanism that elicits agents' cardinal utilities, this approach has proven problematic because it is difficult if not impossible to elicit cardinal utilities in settings without money. For this reason, recent work has focused on ordinal mechanisms, where agents submit ordinal preference reports, i.e., rankings over all objects (for an example see (Abdulkadiroğlu, Pathak and Roth, 2005), or (Carroll, 2011a) for a systematic argument). Throughout this paper, we only consider ordinal mechanisms.

For the deterministic case, strategyproofness of one-sided matching mechanisms has been studied extensively (Papai, 2000; Ehlers and Klaus, 2006, 2007; Hatfield, 2009; Pycia and Ünver, 2014). For non-deterministic mechanisms, Abdulkadiroğlu and Sönmez (1998) showed that RSD is equivalent to the Core from Random Endowments mechanism for house allocation, if agents' initial houses are drawn uniformly at random. Erdil (2013) showed that for settings where capacity exceeds demand, RSD is not the only strategyproof, ex-post efficient mechanism that satisfies symmetry. Bade (2013) extended their result by showing that taking any ex-post efficient, strategyproof, non-bossy, deterministic mechanism and assigning agents to roles in the mechanism uniformly at random is equivalent to RSD. However, it is still an open conjecture whether RSD is the unique mechanism that is anonymous, ex-post efficient, and strategyproof.

The research community has also introduced stronger efficiency concepts, such as ordinal efficiency. The Probabilistic Serial mechanism (PS) was originally introduced by Bogomolnaia and Moulin (2001) for strict preferences. Katta and Sethuraman (2006) introduced an extension that allows agents to be indifferent between objects. Recently, Hashimoto et al. (2013) showed that the unique mechanism that is ordinally fair and non-wasteful is PS with uniform eating speeds. Bogomolnaia and Moulin (2001) had already shown that PS is not strategyproof, but Kesten and Ekici (2012) found that its Nash equilibria can lead to ordinally dominated outcomes.

While ex-post efficiency and ordinal efficiency are the two most well-studied efficiency concepts for one-sided matching mechanisms, some mechanisms used in practice aim to maximize rank efficiency, which is a further refinement of ordinal efficiency (Featherstone, 2011). However, no rank efficient mechanism can even be weakly strategyproof. Other popular mechanisms, like the Boston Mechanism (see Ergin and Sönmez (2006)), are highly manipulable but are nevertheless in frequent use. Budish and Cantillon (2012) show practical evidence from combinatorial course allocation, suggesting that using a non-strategyproof mechanism may
lead to higher social welfare than using a strategyproof, ex-post efficient mechanism such as RSD. This challenges whether strategyproofness should be a hard constraint for mechanism designers.

Given that strategyproofness is such a strong restriction, many researchers have tried to relax it, using various notions of approximate strategyproofness. For example, Carroll (2011b) took this approach in the voting domain and quantified the incentives to manipulate (but only for certain normalized utilities). Budish (2011) proposed the interesting Competitive Equilibrium from Approximately Equal Incomes mechanism for the domain of combinatorial assignments. For the single-object assignment domain, this reduces to RSD. Finally, Dütting et al. (2012) used a machine learning approach to design mechanisms with “good” incentive properties. Instead of requiring strategyproofness, they minimize the agents’ ex-post regret, i.e., the utility increase an agent could gain from manipulating. For the randomized social choice domain, Aziz, Brandt and Brill (2013a) considered sure thing (ST) dominance and showed that while RSD is SD-strategyproof, it is merely ST-efficient. They contrasted this to strictly maximal lotteries, which are SD-efficient, but only ST-strategyproof. The concept of hybrid mechanisms presented in this paper differs from these approaches. Rather than comparing discrete points in the mechanism design space, we enable a continuous trade-off between strategyproofness and efficiency that can be described in terms of two parameters: the degree of strategyproofness for incentives and the mixing factor for efficiency.

Recently, Azevedo and Budish (2012) proposed a desideratum called Strategyproof in the Large (SP-L), which formalizes the intuition that as the number of agents in the market gets large, the incentives for individual agents to misreport should vanish in the limit. Kojima and Manea (2010) showed that for a fixed number of objects and a fixed agent, PS makes truthful reporting a dominant strategy for that agent if there are sufficiently many copies of each object. Liu and Pycia (2013) showed that when there are many agents relative to the number of object types, all symmetric, asymptotically efficient, and asymptotically strategyproof mechanism are asymptotically equivalent, which includes RSD. While these approaches only provide certain guarantees in the limit, our hybrid mechanisms enable a conscious trade-off with precise strategyproofness and efficiency properties for settings of any size.

3. Model

A setting \((N, M, q)\) consists of a set \(N\) of \(n\) agents, a set \(M\) of \(m\) objects, and a vector \(q = (q_1, \ldots, q_m)\) of capacities, i.e., there are \(q_j\) units of object \(j\) available. We assume that supply satisfies demand, i.e., \(n \leq \sum_{j \in M} q_j\), since we can always add dummy objects. Agents are endowed with von Neumann-Morgenstern (vNM) utilities \(u_i, i \in N\), over the objects. If \(u_i(a) > u_i(b)\), we say that agent \(i\) prefers object \(a\) over object \(b\), which we denote by \(a \succ_i b\). We work with the standard model, which excludes indifferences, i.e., \(u_i(a) = u_i(b)\) implies \(a = b\). A utility function \(u_i\) is consistent with preference ordering \(\succ_i\) if \(a \succ_i b\) whenever \(u_i(a) > u_i(b)\). Given a preference ordering \(\succ_i\), the corresponding type \(t_i\) is the set of all utilities that are consistent with \(\succ_i\), and \(T\) is the set of all types, called the type space. We use types and preference orderings synonymously.
An allocation is a (possibly probabilistic) assignment of objects to agents. It is represented by an \( n \times m \)-matrix \( x = (x_{i,j})_{i \in N, j \in M} \) satisfying the fulfillment constraint \( \sum_{j \in M} x_{i,j} = 1 \), the capacity constraint \( \sum_{i \in N} x_{i,j} \leq q_j \), and \( x_{i,j} \in [0,1] \) for all \( i \in N, j \in M \). The entries of the matrix are interpreted as probabilities, where \( i \) gets \( j \) with probability \( x_{i,j} \). An allocation is deterministic if \( x_{i,j} \in \{0,1\} \) for all \( i \in N, j \in M \). The Birkhoff-von Neumann Theorem and its extensions (Budish et al., 2013) ensure that given any allocation, we can always find a lottery over deterministic assignments that implements these marginal probabilities. Finally, let \( X \) denote the space of all allocations.

A mechanism is a mapping \( f : T^n \to X \) that chooses an allocation based on a profile of reported types. We let \( f_i(t_i,t_{-i}) \) denote the allocation that agent \( i \) receives if it reports type \( t_i \) and the other agents report \( t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) \in T^{n-1} \). Note that mechanisms only receive type profiles as input. Thus, we consider ordinal mechanisms, where the allocation is independent of the actual vNM utilities. If \( i \) and \( t_{-i} \) are clear from the context, we may abbreviate \( f_i(t_i,t_{-i}) \) by \( f(t_i) \). The expected utility for \( i \) is given by the dot product \( \langle u_i, f(t_i) \rangle \), i.e.,

\[
E_{f_i(t_i,t_{-i})}(u_i) = \sum_{j \in M} u_i(j) \cdot f_i(t_i)(j) = \langle u_i, f(t_i) \rangle.
\]

For strategyproof mechanisms, reporting truthfully maximizes agents’ expected utility:

**Definition 1 (Strategyproofness).** A mechanism is strategyproof if for any agent \( i \in N \), any type profile \( t = (t_i,t_{-i}) \in T^n \), any misreport \( t'_i \in T \), and any utility \( u_i \in t_i \) we have

\[
\langle u_i, f(t_i) - f(t'_i) \rangle \geq 0.
\]

Our model encompasses classical one-sided matching problems, such as house allocation and school choice markets, where only one side of the market has preferences. It is also straightforward to accommodate outside options. Priorities over the agents can be included implicitly in the mechanism.

### 4. URBI\((r)\)-partial Strategyproofness

The goal of designing hybrid mechanisms is to sacrifice “some” strategyproofness to obtain “some” efficiency gains. In order to make a conscious trade-off decision, a mechanism designer needs an understanding of “how much” strategyproofness she is giving up. Towards this end, we need a parametric relaxation of strategyproofness. Until recently, no such concept existed for the one-sided matching domain. However, in (Mennle and Seuken, 2014) we have presented an axiomatic approach to relaxing strategyproofness. Dropping one of three axioms that characterize strategyproof mechanisms leads to URBI\((r)\)-partially strategyproof mechanisms. This new concept enables a parametric analysis of strategyproofness for manipulable mechanisms. We now briefly describe this concept and its axiomatic derivation (see Appendix A for details).

To understand the axioms, suppose an agent is considering to report truthfully or swap two adjacent objects in its report, e.g., \( a \succ b \) to \( b \succ a \). A mechanism is swap consistent if upon such a swap, the reporting agent’s allocation either does not change at all, or the allocation
for \( b \) strictly increases and the allocation for \( a \) strictly decreases. The mechanism is \textit{weakly invariant} if the allocation does not change for any object that the agent strictly prefers to \( a \), and it is \textit{lower invariant} if it does not change for any object that the agent likes strictly less than \( b \). These axioms characterize strategyproof mechanisms: a mechanism is strategyproof \textit{if and only if} it is swap consistent, weakly invariant, and lower invariant (Theorem 1 in (Mennle and Seuken, 2014b)).

Dropping lower invariance (arguably the least intuitive and least important of the axioms), the larger class of URBI\((r)\)-partially strategyproof mechanisms emerges, and the incentive properties of these mechanisms can be described in a parametric fashion: for any setting \((N, M, \mathbf{q})\), a mechanism is swap consistent and weakly invariant \textit{if and only if} it is URBI\((r)\)-partially strategyproof for some bound \( r \in (0, 1) \) (Theorem 2 in (Mennle and Seuken, 2014b)). We now define URBI\((r)\)-partial strategyproofness and provide some intuition. Loosely speaking, URBI\((r)\) is a domain restriction that requires agents to have sufficiently different values for different objects.

**Definition 2** (Uniformly Relatively Bounded Indifference). A utility function \( u \) satisfies uniformly relatively bounded indifference \textit{with respect to} bound \( r \in [0, 1] \) \((\text{URBI}(r))\) if for any pair of objects \( a \) and \( b \) with \( u(a) > u(b) \) we have

\[
    r \cdot (u(a) - \min(u)) \geq u(b) - \min(u). \tag{4}
\]

Intuitively, an agent’s utility function satisfies URBI\((r)\), if the agent values any object \( b \) by a factor \( r \) less than the next better choice \( a \), i.e., \( r \cdot u(a) \geq u(b) \) (after normalization, i.e., \( \min u = 0 \)). Figure 1 provides a geometric intuition of the condition: the agent’s utility function cannot be arbitrarily close to the indifference hyperplane \( H(t, t') \) between the types \( t \) and \( t' \), i.e., it must lie within the shaded area of type \( t \). While utility \( u \) satisfies URBI\((r)\), \( v \) violates it. A mechanism is URBI\((r)\)-\textit{partially strategyproof} if it is strategyproof for the domain restricted by URBI\((r)\). Formally:
Definition 3 (URBI(r)-partial Strategyproofness). Given a setting \((N, M, q)\) and a bound \(r \in [0, 1]\), the mechanism \(f\) is URBI(r)-partially strategyproof in the setting \((N, M, q)\) if for any agent \(i \in N\), any type profile \(t = (t_i, t_{-i}) \in T^n\), any misreport \(t'_i \in T\), and any utility \(u_i \in \text{URBI}(r) \cap t_i\) we have
\[
\langle u_i, f(t_i) - f(t'_i) \rangle \geq 0. \tag{5}
\]

For strategyproof mechanisms, the bound \(r\) is 1, and otherwise it is in the interval \([0, 1]\). Obviously, strategyproofness implies URBI(r)-partial strategyproofness for any bound \(r\) and any setting: for strategyproofness the incentive constraint (5) must hold for all possible utility functions, and the relaxation requires the incentive constraint to hold only for a subset of the utility functions, the set that satisfies URBI(r). For convenience and if the setting is clear from the context, we will denote by URBI(r) the set of all utility functions in a particular setting that satisfy uniformly relatively bounded indifference with respect to the bound \(r\).

In (Mennle and Seuken, 2014b) we have shown that strategyproof mechanisms (e.g., RSD) and a number of mechanisms with reasonable incentive properties (e.g., PS and the adaptive Boston mechanism (ABM)) are URBI(r)-partially strategyproof. More manipulable mechanisms (e.g., the naïve Boston (NBM) and Rank-value mechanisms (RV)), however, are generally not URBI(r)-partially strategyproof for any \(r \in (0, 1]\).

In (Mennle and Seuken, 2014b) we have also shown that the URBI(r) domain restriction is maximal: consider a mechanism \(f\) that is URBI(r)-partially strategyproof for some bound \(r \in (0, 1]\), and thus swap consistent and weakly invariant. Maximality means that, unless we are given additional structural information about \(f\), URBI(r) is in fact the largest set of utilities for which partial strategyproofness can be guaranteed. This maximality makes the degree of strategyproofness a useful measure for “how strategyproof” a mechanism is.

5. Construction of URBI(r)-partially Strategyproof Hybrids

We now formally define hybrid mechanisms and show under what conditions they are URBI(r)-partially strategyproof.

5.1. Definition of Hybrid Mechanisms

Definition 4. For two mechanisms \(f\) and \(g\), and \(\beta \in [0, 1]\), the mechanism \(h_\beta(f, g)\) is called the \(\beta\)-hybrid of \(f\) and \(g\) and is defined for any type profile \(t \in T^n\) by
\[
h_\beta(f, g)(t) = (1 - \beta)f(t) + \beta g(t). \tag{6}
\]

\(\beta\) is called the mixing factor, and \(f\) and \(g\) are called the components of \(h_\beta(f, g)\). We say that the hybrid is non-trivial if \(\beta \in (0, 1]\).

A hybrid mechanism is well-defined, since the convex combination of two allocations (represented by matrices of equal dimensions) is again an allocation of the same dimensions (capacity and fulfillment constraint are easy to verify). To enable interesting trade-offs, the main idea is to use a strategyproof mechanism \(f\) for good incentives and a more efficient mechanism
for good efficiency. Since the hybrid mechanism may violate strategyproofness, we use URBI(r)-partial strategyproofness to measure the resulting incentive properties.

5.2. Auxiliary Concepts

Not all pairs of mechanisms are suitable for the construction of URBI(r)-partially strategyproof hybrids. In this section, we give a formal definition of the weak invariance axiom and introduce the technical constraint of weakly less varying mechanisms, two conditions for the design of non-trivial, URBI(r)-partially strategyproof hybrids.

5.2.1. Weak Invariance

Weak invariance is the second axiom we have used to characterize strategyproof and URBI(r)-partial strategyproof mechanisms in Mennle and Seuken (2014b). It is also a form of truncation robustness: an agent cannot improve its allocation by untruthfully claiming preference for an outside option over some objects (see Hashimoto et al. (2013), who used weak invariance as one of the central axioms to characterize PS). Weak invariance requires that if an agent swaps two adjacent objects in its reported preference order, the allocation for any object it strictly prefers to both cannot change. For a formal definition, we need the following: the neighborhood $N_t$ of a type $t$ is the set of all types whose corresponding preference orders differ by just a swap of two adjacent objects. The upper contour set of an object $a$ with respect to some type $t$ is the set of objects that an agent of type $t$ strictly prefers to $a$, denoted by $U(a,t)$.

Axiom 1 (Weak Invariance). A mechanism $f$ is weakly invariant if for any agent $i \in N$, any type profile $t = (t_1, \ldots, t_n) \in T^n$, and any type $t_i' \in N_{t_i}$ with $a_k > a_{k+1}$ under $t_i$ and $a_{k+1} > a_k$ under $t_i'$, $i$'s allocation for the objects in the upper contour set is unaffected by the swap, i.e., for all $j \in U(a_k, t_i)$

$$f(t_i)(j) = f(t_i')(j).$$

(7)

Many mechanisms from the literature satisfy weak invariance, such as RSD, PS, the naïve and the adaptive Boston mechanism, and Student-proposing Deferred Acceptance. However, as we will see in Section 8.2.1, RV does not.

5.2.2. Weakly Less Varying Mechanisms

In the construction of hybrid mechanisms, the component $g$ will typically be manipulable, and $f$ will be strategyproof. Even if we require $g$ to satisfy weak invariance, it may introduce incentives to manipulate into the hybrid against which any share of $f$ is powerless. To see how this problem can arise, suppose that agent $i$ has true type $t_i$ and finds it beneficial to report $t_i'$ instead under $g$. If the allocation from $f$ does not change between the two reports $t_i$ and $t_i'$, then for any $\beta > 0$, $i$ will want to manipulate the hybrid $h_{\beta}(f,g)$. The following additional constraint is needed to prevent this situation.
Definition 5 (Weakly Less Varying). A mechanism $g$ is weakly less varying than another mechanism $f$ if for any agent $i \in N$, any type profile $t = (t_i, t_{-i}) \in T^n$, and any misreport $t'_i \in N_{t_i}$ the following holds:

$$g_i(t_i, t_{-i}) \neq g_i(t'_i, t_{-i}) \Rightarrow f_i(t_i, t_{-i}) \neq f_i(t'_i, t_{-i}).$$

(8)

5.3. Existence of URBI($r$)-partially Strategyproof Hybrid Mechanisms

For the construction of URBI($r$)-partially strategyproof hybrid mechanisms, the pair of component mechanisms $(f, g)$ must satisfy the following requirement:

Definition 6 (Hybrid-admissible). The pair $(f, g)$ is hybrid-admissible if $f$ is strategyproof, and $g$ is weakly invariant and weakly less varying than $f$.

We are now ready to present our first main result.

Theorem 1. For any setting $(N, M, q)$, any hybrid-admissible pair $(f, g)$, and any bound $r \in [0, 1)$, there exists a positive mixing factor $\beta \in (0, 1]$ such that the hybrid $h_{\beta}(f, g)$ is URBI($r$)-partially strategyproof in the setting $(N, M, q)$.

Proof outline (formal proof in Appendix B.1). Consider an agent of type $t : a_1 > \ldots > a_m$ and a misreport $t' : a_1 > \ldots > a_K > a_{K+1} > \ldots > a_m$, where the positions of the first $K$ objects remain unchanged. The key insight is that we only need to consider cases where the allocation of $a_{K+1}$ strictly decreases under $f$. If less of $a_{K+1}$ is allocated, this has a negative effect on the overall expected utility from misreport $t'$. We show that for any agent with utility in URBI($r$) and for sufficiently small $\beta > 0$, this negative effect is sufficient to make the expected utility from reporting $t'$ lower than the expected utility from reporting $t$. Finally, we show that $\beta > 0$ can be chosen uniformly for all type profiles and misreports, while it may depend on the mechanism and the setting.

For a mechanism designer, Theorem 1 has the following important interpretation: given any setting, any desired bound $r \in [0, 1)$, and any hybrid-admissible pair of mechanisms, a non-trivial hybrid can be constructed that is URBI($r$)-partially strategyproof for the desired bound in that setting. Intuitively, we would like to choose a mixing factor $\beta$ as large as possible as long as URBI($r$)-partial strategyproofness is not violated, because the higher $\beta$, the more of the more efficient component $g$ is included. Theorem 1 yields that a non-trivial $\beta$ can be found. In Section 9, we present numerical results for instantiations with RSD and PS, as well as RSD and ABM, which show that the maximal mixing factor can be significant.

Regarding tightness of Theorem 1, none of the requirements can be dismissed: $g$ must be weakly less varying than $f$, otherwise a beneficial manipulation may persist for any mixing factor (as discussed in Section 5.2.2). Propositions 1 and 2 show that strategyproofness of $f$ and weak invariance of $g$, respectively, are also essential.

Proposition 1. If $f$ is not strategyproof, there exists a mechanism $g$ that is weakly invariant and weakly less varying than $f$, and a bound $r \in (0, 1)$ such that no non-trivial hybrid of the pair $(f, g)$ will be URBI($r$)-partially strategyproof.
Proof outline (formal proof in Appendix B.2). The key idea is to choose \( g \) as a constant mechanism that yields the same allocation, independent of the agents’ reports. If \( f \) is manipulable by some agent with utility \( \tilde{u} \), we choose \( \tilde{r} \) such that \( \tilde{u} \in \text{URBI}(\tilde{r}) \). Then any non-trivial hybrid will be manipulable for \( \tilde{u} \). □

**Proposition 2.** For any strategyproof \( f \) and any \( g \) that is weakly less varying than \( f \), but not weakly invariant in a setting \((N, M, q)\), no non-trivial hybrid of the pair \((f, g)\) is \( \text{URBI}(r) \)-partially strategyproof for any bound \( r \in (0, 1] \) in the setting \((N, M, q)\).

Proof outline (formal proof in Appendix B.2). To see this, we show that if \( g \) is not weakly invariant, then neither is any non-trivial hybrid. Consequently, the hybrid is not \( \text{URBI}(r) \)-partially strategyproof either. □

We have now obtained a good understanding of the incentive properties of hybrid mechanisms. However, we are interested in the trade-off between efficiency and strategyproofness. To this end, we analyze the efficiency properties of hybrid mechanisms in Sections 6.2 and 6.3 and describe the trade-offs that can be realized via hybrids in more detail in Section 6.4.

**Remark 1.** Fairness is also a desirable property for one-sided matching mechanisms. It is straightforward to see that a hybrid of two anonymous or symmetric mechanisms will also be anonymous or symmetric, respectively. However, this paper is concerned with trade-offs between strategyproofness and efficiency, and we leave trade-offs between fairness and strategyproofness or efficiency to future research.

## 6. Efficiency of Hybrid Mechanisms

In this section, we analyze the efficiency properties of hybrid mechanisms, and we obtain the following four results: first, ex-post efficiency is inherited by the hybrid if both components are ex-post efficient. Second, if the component \( g \) dominates \( f \), then any non-trivial hybrid \( h_\beta(f, g) \) also dominates \( f \). Third, in situations where \( f \) and \( g \) are comparable by dominance only for some type profiles, we define imperfect dominance for mechanisms, and we show that if \( g \) imperfectly dominates \( f \), then any non-trivial hybrid \( h_\beta(f, g) \) also imperfectly dominates \( f \). Fourth, if \( g \) is more efficient than \( f \), but manipulable, then as the mixing factor \( \beta \) increases, the hybrid becomes more efficient, but the degree of strategyproofness decreases.

### 6.1. Ex-post Efficiency of Hybrid Mechanisms

Ex-post efficiency is ubiquitous in matching. Most one-sided matching mechanisms considered in theory and applications are ex-post efficient, such as RSD, PS, RV mechanisms, and Boston mechanisms. Ex-post efficiency means that after implementation (i.e., when every agent holds exactly one object), no group of agents can agree on a trade cycle that gives all agents in the group an object they weakly prefer to their previous allocation, and at least one agent gets an object it strictly prefers. Formally:
Definition 7 (Ex-post Efficiency). Given a type profile \( t = (t_i, t_{-i}) \in T^n \), a deterministic allocation \( x \) ex-post dominates another deterministic allocation \( y \) at \( t \) if all agents weakly prefer their allocation under \( x \) to their allocation under \( y \). The dominance is strict if at least one agent strictly prefers its allocation under \( x \). A deterministic allocation \( x \) is ex-post efficient at \( t \) if it is not strictly ex-post dominated by any other deterministic allocation at \( t \). Finally, a probabilistic allocation is ex-post efficient if it has a lottery decomposition consisting only of ex-post efficient deterministic allocations.

Proposition 3 ensures that hybrids inherit ex-post efficiency from their components.

Proposition 3. If both components \( f \) and \( g \) are ex-post efficient, so is \( h_\beta(f, g) \).

Proof. If the two allocations \( f(t) \) and \( g(t) \) are ex-post efficient, they can be written as lotteries over ex-post efficient, deterministic allocations. The convex combination of these lotteries is again a lottery over ex-post efficient, deterministic allocations. \( \square \)

6.2. Comparison by Dominance

To compare allocations by their efficiency, we draw on first order-stochastic dominance.

Definition 8 (Ordinal Efficiency). For a type \( t : a_1 > \ldots > a_m \) and two allocation vectors \( v = v_{j \in M} \) and \( w = w_{j \in M} \), the allocation vector \( v \) first order-stochastically dominates \( w \) if for all ranks \( k \in \{1, \ldots, m\} \):

\[
\sum_{j \in M: j > a_k} v_j \geq \sum_{j \in M: j > a_k} w_j.
\]  

For a type profile \( t \), an allocation \( x \) ordinarily dominates another allocation \( y \) at \( t \) if for all agents \( i \in N \) their respective allocation vector \( x_i \) first order-stochastically dominates \( y_i \) (for type \( t_i \)). \( x \) strictly ordinarily dominates \( y \) if in addition inequality (9) is strict for some agent \( i \in N \) and some rank \( k \in \{1, \ldots, m\} \).

\( x \) is ordinarily efficient at \( t \) if it is not strictly ordinarily dominated by any other allocation at \( t \). We let \( \succeq_O \) and \( >_O \) denote weak and strict ordinal dominance, respectively.

If \( x \) ordinarily dominates \( y \) at \( t \) and \( t \) is the true type profile of the agents, then all agents will prefer \( x \) to \( y \), independent of their actual utility functions. Bogomolnaia and Moulin (2001) show that PS produces ordinarily efficient allocations, which may strictly ordinarily dominate the allocations of RSD.

Featherstone (2011) introduced a strict refinement of ordinal efficiency, called rank efficiency. He developed Rank-value mechanisms that produce rank efficient allocations, but he also showed that rank efficiency and strategyproofness are incompatible.

Definition 9 (Rank Efficiency). For a type profile \( t \) let \( ch(k, t_i) \) denote the \( k \)th choice object of an agent of type \( t_i \). The rank distribution of an allocation \( x \) at \( t \) is the vector \( d^x \) with:

\[
d_k^x = \sum_{i \in M} x_{i, ch(k, t_i)} \text{ for } k \in \{1, \ldots, m\}.
\]
Thus, $d_k$ is the expected number of $k$th choices allocated under $x$ with respect to type profile $t$. An allocation $x$ rank dominates another allocation $y$ at $t$ if the rank distribution $d^x$ first order-stochastically dominates $d^y$. $x$ strictly rank dominates $y$ if in addition inequality (10) is strict for some rank $r \in \{1, \ldots, m\}$. $x$ is rank efficient at $t$ if it is not strictly rank dominated by any other allocation at $t$. Let $\succeq_R$ and $>_R$ denote weak and strict rank dominance, respectively.

Proposition 4 shows how hybrid mechanisms capture efficiency gains in the sense of ordinal or rank dominance.

**Proposition 4.** For allocations $x$ and $y$ let $z = (1 - \beta)x + \beta y$. For any type profile $t \in T^n$ and any $\beta \in (0, 1]$ the following hold:

1) $y \succeq_O x$ at $t$ $\Rightarrow$ $z \succeq_O x$ at $t$, and $y >_O x$ at $t$ $\Rightarrow$ $z >_O x$ at $t$,

2) $y \succeq_R x$ at $t$ $\Rightarrow$ $z \succeq_R x$ at $t$, and $y >_R x$ at $t$ $\Rightarrow$ $z >_R x$ at $t$.

The proof is technical, but straightforward and given in Appendix B.3.

Suppose that $f$ is strategyproof and ex-post efficient and that the allocations from $g$ ordinally (or rank) dominate the allocations from $f$. Proposition 3 ensures that any hybrid produces ex post-efficient allocations. Furthermore, by Proposition 4, the hybrid $h_\beta(f, g)$ ordinally (or rank) dominates $f$ for all type profiles for which $g$ ordinally (or rank) dominates $f$, and this dominance is strict for the type profiles $t$ where the dominance of $g(t)$ over $f(t)$ is strict.

### 6.3. Comparison by Imperfect Dominance

It is possible that $g$ dominates $f$ at some type profiles, while at other type profiles, the allocations from both mechanisms are incomparable by dominance. This is the case, for example, for PS and RSD (see Examples 1 and 2 in Appendix B.4). To be able to compare mechanisms that are not always comparable via the dominance relation, we introduce a new relaxation of the dominance concepts: instead of requiring dominance for all type profiles, we only require dominance for those type profiles where a comparison is possible.

**Definition 10 (Imperfect Dominance).** Consider two mechanisms $f$ and $g$. $g$ weakly imperfectly ordinally dominates $f$ if $g(t) \succeq_O f(t)$ for all type profiles $t \in T^n$ where $g(t)$ and $f(t)$ are comparable by the dominance relation. $g$ strictly imperfectly ordinally dominates $f$ if in addition $g(t) >_O f(t)$ for some type profiles $t \in T^n$. Weak and strict imperfect rank dominance are defined analogously. We denote weak and strict imperfect ordinal dominance by $\succeq_{IO}$ and $>_IO$ and weak and strict imperfect rank dominance by $\succeq_{IR}$ and $>_IR$, respectively.

Proposition 5 sheds light on the efficiency gains of hybrids when the components are not fully comparable by dominance.

**Proposition 5.** For mechanisms $f$ and $g$ and any $\beta \in (0, 1]$ the following hold:

1) $g \succeq_{IO} f$ $\Rightarrow$ $h_\beta(f, g) \succeq_{IO} f$ and $g >_{IO} f$ $\Rightarrow$ $h_\beta(f, g) >_{IO} f$,

2) $g \succeq_{IR} f$ $\Rightarrow$ $h_\beta(f, g) \succeq_{IR} f$ and $g >_{IR} f$ $\Rightarrow$ $h_\beta(f, g) >_{IR} f$.  

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Proof outline (formal proof in Appendix B.5). From Proposition 4 it follows that if \( g(t) \) weakly or strictly ordinally or rank dominates \( f(t) \) at \( t \), then so does \( h_\beta(f,g)(t) \) for any \( \beta > 0 \). We show that if \( f(t) \) does not dominate \( g(t) \), then it does not dominate \( h_\beta(f,g)(t) \) either.

Note that \( h_\beta(f,g) \) strictly dominates the component \( f \) for exactly the same type profiles where \( g \) strictly dominates \( f \), and the hybrid is incomparable to \( f \) for exactly the same type profiles where \( g \) is incomparable to \( f \). Intuitively, this means that \( h_\beta \) has higher efficiency than \( f \) whenever \( g \) has higher efficiency than \( f \). Thus, if a mechanism designer prefers \( g \) over \( f \) for efficiency, she also prefers the hybrid to \( f \) for the same reasons.

6.4. Hierarchies of Efficiency and Manipulability

In Sections 6.2 and 6.3, we have evaluated how hybrid mechanisms can be used to harness efficiency gains in the sense of perfect or imperfect dominance. We will now combine these observations with URBI(r)-partial strategyproofness from Theorem 1.

6.4.1. Trade-offs of Dominance and Degree of Strategyproofness

In Mennle and Seuken (2014b) we have shown that the strategyproofness of an URBI(r)-partially strategyproof mechanism can be measured by the maximum value of the bound \( r \). For any setting and mechanism \( h \), the degree of strategyproofness is given by

\[
\rho_{(N,M,q)}(h) = \max \{ r \in [0,1] \mid h \text{ is URBI}(r)-\text{PSP in the setting } (N,M,q) \}.
\]  

(11)

For URBI(r)-partially strategyproof mechanisms, this is finite and well-defined.

From the mechanism designer’s perspective, the trade-off between strategyproofness and efficiency can be viewed as follows: she decides on an acceptable degree of strategyproofness and chooses the mixing factor \( \beta \) as high as possible, such that efficiency is “maximized,” given strategyproofness on the required set URBI(r). The following proposition, our second main result, provides a formalization of this trade-off between efficiency and strategyproofness. The intuition is that as the mixing factor \( \beta \) increases and more of \( g \) is incorporated, the hybrid mechanism becomes more efficient, but also less strategyproof (if \( g \) is manipulable).

**Proposition 6.** Consider any setting \((N,M,q)\) and any hybrid-admissible pair \((f,g)\), where \( g \succ_{IO} f \). If \( g \) is not strategyproof in the setting, then for any two mixing factors \( 0 \leq \beta < \beta' \leq 1 \),

1) \( h_\beta(f,g) \succ_{IO} h_\beta'(f,g) \),

2) \( \rho_{(N,M,q)}(h_\beta(f,g)) > \rho_{(N,M,q)}(h_{\beta'}(f,g)) \).

The same holds if the \( \succ_{IO} \)-relation is replaced by any of \( \succ_{IO} \succ_{IR} \succ_{OR} \succ_{O} \succ_{R} \succ_{R} \).

**Proof outline (formal proof in Appendix B.6).** To see 1), apply Proposition 5 to \( f \) and \( g \), and then to \( h_\beta(f,g) \) and \( g \), noting that \( h_\beta(f,g) \) is a hybrid of \( h_\beta(f,g) \) and \( g \). To show 2), we use the fact that \( \rho_{(N,M,q)} \) is determined by finitely many constraints and at least one of these constraints must bind for \( h_\beta(f,g) \). Then we show that this constraint is violated by \( h_{\beta'}(f,g) \).
Remark 2. We now have two values that capture the properties of hybrid mechanisms. The degree of strategyproofness $\rho$ yields a parametric measure for incentives. Similarly, the mixing factor $\beta$ provides a quantitative notion for the efficiency improvements that a hybrid yields: if $f$ is a strategyproof mechanism and $g$ is a manipulable alternative that is more attractive in terms of efficiency, then $\beta$ is the extent to which the hybrid captures these efficiency improvements with respect to the total possible improvements of $g$ over $f$.

6.4.2. Vulnerability to Manipulation

Pathak and Sönmez (2013) proposed an interesting method for comparing mechanisms by their vulnerability to manipulation. For the expected utility case their comparison states that $g$ is as intensely and strongly manipulable as $f$ if whenever an agent with utility $u$ finds a beneficial manipulation to $f$, the same agent in the same situation would find a manipulation for $g$ that yields a weakly greater increase in expected utility. The trade-off realized by our hybrid mechanisms can also be formulated in terms of this vulnerability to manipulation concept:

**Corollary 1.** In Proposition 6, $h_{\beta}(f, g)$ is as intensely and strongly manipulable as $h_{\beta}(f, g)$.

This follows from the proof of Proposition 6. Note that the strict comparison of intensely and strongly more manipulable from Pathak and Sönmez (2013) does not hold, because hybrid mechanisms do not necessarily become manipulable for more type profiles when $\beta$ increases. A significant advantage of the degree of strategyproofness is that it is parametric, rather than purely relative. Furthermore, the degree of strategyproofness is computable, as we will show in the next section.

7. Computability of the Maximum Mixing Factor $\beta_{\text{max}}$

So far, we have shown that hybrid mechanisms trade off efficiency and strategyproofness in a meaningful way. But what about the mechanism designer’s problem of determining a maximal mixing factor, i.e., finding

$$\beta_{\text{max}} = \beta_{\text{max}}^{(N, M, q)}(f, g) = \max \{\beta \in [0, 1] \mid h_{\beta}(f, g) \text{ is URBI}(r)\text{-PSP in } (N, M, q)\} \quad (12)$$

for a setting $(N, M, q)$, a hybrid-admissible pair $(f, g)$, and an acceptable degree of strategyproofness $r \in [0, 1]$? It is not obvious that $\beta_{\text{max}}$ is computable, because URBI$(r)$-partial strategyproofness imposes infinitely many constraints:

$$\langle u_i, f(t_i) - f(t'_i) \rangle \geq 0. \quad (13)$$

must hold for any utility function $u_i \in t_i \cap \text{URBI}(r)$, i.e., uncountably many. It turns out that the computation is indeed possible. In Section 7.1, we present an algorithm that is complete and correct for this problem. The algorithm makes no structural assumptions about the input parameters (other than that $f$ and $g$ are computable), i.e., it shows computability of the mechanism designer’s maximization problem (12) for any pair of components $(f, g)$. Note that our main goal is to show computability, not computational efficiency. In Section 7.2, we briefly discuss the complexity and potential improvements.
ALGORITHM 1: Compute $\beta_{\text{max}}$

Input: setting $(N, M, q)$, mechanisms $f, g$, inverse bound $s = \frac{1}{r}$

Variables: agent $i$, type profile $(t_i, t_{-i})$, type $t'_i$, vectors $\delta^f, \delta^g$, polynomials $x^f, x^g$, counter $k$, choice function $\text{ch}$, real $\beta$

begin
  $\beta \leftarrow 1$
  for $i \in N, (t_i, t_{-i}) \in T^n, t'_i \in T$ do
    $\forall j \in M : \delta^f_j \leftarrow f(t_i)(j) - f(t'_i)(j), \delta^g_j \leftarrow g(t_i)(j) - g(t'_i)(j)$
    $x^f_1(s) \leftarrow \delta^f_{\text{ch}(t_i, 1)}$, $x^g_1(s) \leftarrow \delta^g_{\text{ch}(t_i, 1)}$
    for $k \in \{2, \ldots, m - 1\}$ do
      $x^f_k(s) \leftarrow x^f_{k-1}(s) \cdot s + \delta^f_{\text{ch}(t_i, k)}, x^g_k(s) \leftarrow x^g_{k-1}(s) \cdot s + \delta^g_{\text{ch}(t_i, k)}$
      if $x^f_k(s) - x^g_k(s) > 0$ then
        $\beta \leftarrow \min\left\{\beta, \frac{x^f_k(s)}{x^g_k(s)}\right\}$
      end
    end
  end
return $\beta$
end

7.1. Computability of $\beta_{\text{max}}$ Using Finite Constraint Sets

As mentioned above, the set of constraints (13) that determine $\beta_{\text{max}}$ is uncountable. To handle this problem, we provide a condition that is equivalent to URBI($r$)-partially strategyproofness, but involves only a finite set of constraints (Proposition 2 from Mennle and Seuken (2014b), repeated as Proposition 7 below). We then present our third main result, an algorithm that computes the maximal mixing factor $\beta_{\text{max}}$.

Proposition 7 (Proposition 2 from Mennle and Seuken (2014b)). Given a setting $(N, M, q)$ and a mechanism $f$, for any agent $i \in N$, any type profile $t = (t_i, t_{-i}) \in T^n$, any misreport $t'_i \in T$, and

1) for any object $j \in M$ let $\delta_j = f(t_i)(j) - f(t'_i)(j)$ be the change in the allocation of $j$ to $i$ as $i$ changes its report between $t_i$ and $t'_i$ while the other agents report $t_{-i}$, and
2) for $k \in \{1, \ldots, m - 1\}$, define polynomials (in $s$) recursively by $x_1(s) = \delta_{\text{ch}(t_i, 1)}$ and $x_k(s) = s \cdot x_{k-1}(s) + \delta_{\text{ch}(t_i, k)}$, where $\text{ch}(t_i, k)$ is the $k$th choice of an agent of type $t_i$.

Then $f$ is URBI($r$)-partially strategyproof if and only if for all agents $i \in N$, type profiles $t = (t_i, t_{-i}) \in T^n$, misreports $t'_i \in T$, ranks $k \in \{1, \ldots, m - 1\}$, and $s = \frac{1}{r}$ we have

$$x_k(s) \geq 0. \quad (14)$$

Algorithm 1 uses condition (14) to compute the maximum mixing factor. It takes as input a setting $(N, M, q)$, a hybrid-admissible pair of mechanisms $(f, g)$, and a bound $r \in [0, 1]$.  

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It works similarly to Algorithm 1 from Mennle and Seuken (2014b), which verifies that a mechanism is URBI(p)-partially strategyproof: it iterates through all type profiles and possible manipulations. However, instead of just verifying the positivity constraints for the polynomials \( x(s) \), it appropriately adjusts the parameter \( \beta \), so that the positivity constraint is guaranteed to be satisfied by the hybrid mechanism.

**Proposition 8.** For computable \( f \) and \( g \), Algorithm 1 is complete and correct.

*Proof outline (formal proof in Appendix B.7).* Completeness is straightforward. For correctness, we employ Proposition 7. We show that for URBI(p)-partial strategyproofness of \( h_\beta(f,g) \) with \( \beta \) computed by Algorithm 1, all constraints are satisfied and at least one constraint binds.

### 7.2. Complexity of Computing \( \beta_{\text{max}} \)

Algorithm 1 iterates through all possible type profiles, all agents, and all manipulative reports that agents may submit. For each of these scenarios, the algorithm must check for any rank \( k \in \{2, \ldots, m-1\} \) whether the mixing factor \( \beta \) needs adjustment. Computing the outcomes of the mechanisms \( f \) and \( g \) may also be complex. Let \( O(f) \) and \( O(g) \) denote the complexity of determining \( f \) and \( g \) for a single type profile, respectively. The overall complexity of Algorithm 1 is \( O \left( n \cdot m \cdot (m!)^{n+1} (O(f) + O(g)) \right) \).

#### 7.2.1. Lower Bound for Complexity

In the most general case (without any additional structure), a mechanism is specified in terms of a set of allocation matrices \( \{ f(t), t \in T^n \} \). This set will contain \( (m!)^n \) matrices of dimension \( n \times m \). Consequently, the size of the problem is \( S = (m!)^n \cdot n \cdot m \). In terms of \( S \), Algorithm 1 has complexity \( O \left( S \sqrt{S} \right) \). Thus, for the general case, there is not much room for improvement, since any correct and complete algorithm will have complexity \( O(S) \) (we must consider all type profiles).

#### 7.2.2. Improvements

Some reductions of this complexity are possible if more information is available about the mechanisms \( f \) and \( g \). For anonymous \( f \) and \( g \) the identity of the agents is irrelevant, and thus, the outcomes for the type profiles \( (t_1, t_2, t_{-1,-2}) \) and \( (t_2, t_1, t_{-1,-2}) \) are symmetric, i.e., only one of them must be considered. In this case, complexity reduces to \( O \left( n \cdot m! \left( \frac{m! + n - 1}{n} \right) (O(f) + O(g)) \right) \), as only \( \left( \frac{m! + n - 1}{n} \right) \) type profiles must be checked. Suppose, the mechanisms are also neutral, i.e., the allocation does not depend on the objects’ names. Then it suffices to consider only agent 1 with a fixed type, and the complexity reduces further to \( O \left( m! \left( \frac{m! + n - 2}{n - 1} \right) (O(f) + O(g)) \right) \). Even with these reductions, the

\footnote{Determining the probabilistic allocation of a mechanism may be computationally hard, even if implementing the mechanism is easy (e.g., see Aziz, Brandt and Brill (2013b)).}
computational effort to run Algorithm 1 is prohibitively high for larger settings. However, it is likely that more efficient algorithms exist for mechanisms with additional restrictions, and bounds may be derived analytically for certain interesting mechanisms, such as PS. Having shown computability, we leave the design of computationally efficient algorithms to future research.

8. Instantiations of Hybrid Mechanisms

So far, we have considered abstract hybrid mechanisms and derived general results. In this section, we consider instantiations of hybrids, using existing mechanisms as components. Indeed, our new concepts are applicable to some (but not all) well-known mechanisms. \( f = \text{RSD} \) is a canonical choice, because it is the only mechanism known to be strategyproof, ex-post efficient, and anonymous. In order to apply Theorem 1 (construction of URBI(r)-partially strategyproof hybrids), we must establish two requirements for the second component: (1) \( g \) must be weakly invariant, and (2) \( g \) must be weakly less varying than \( \text{RSD} \). Furthermore, to realize a trade-off between efficiency and strategyproofness, \( g \) must dominate \( \text{RSD} \) in some sense. Table 1 provides an overview of our results: trade-offs for imperfect ordinal dominance can be achieved via hybrids of \( \text{RSD} \) and \( \text{PS} \). Trade-offs for rank dominance are more difficult to obtain, but are possible via hybrids of \( \text{RSD} \) and \( \text{ABM} \), at least in the limit as \( n = m \to \infty \).

8.1. Hybrids of RSD and PS

By Theorem 2 of Hashimoto et al. (2013), \( \text{PS} \) is weakly invariant. Since \( \text{PS} \) is ordinally efficient, it is never ordinally dominated by \( \text{RSD} \) at any type profile. Furthermore, \( \text{PS} \) may (but does not always) ordinally dominate \( \text{RSD} \) (Examples 1 and 2). Thus, \( \text{PS} \) imperfectly ordinally dominates \( \text{RSD} \). To obtain hybrid-admissibility of the pair (RSD,PS), it remains to be shown that \( \text{PS} \) is weakly less varying than \( \text{RSD} \).

**Theorem 2.** \( \text{PS} \) is weakly less varying than \( \text{RSD} \).

*Proof outline (formal proof in Appendix B.8).* Consider an agent \( i \) that swaps two objects in its report, e.g., from \( a > b \) to \( b > a \). First, we show that \( \text{PS} \) changes the allocation if and only

<table>
<thead>
<tr>
<th>( f )</th>
<th>( g )</th>
<th>Dominance</th>
<th>WI</th>
<th>WLV</th>
<th>Construction of URBI(r)-partially strategyproof hybrids possible</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{RSD} )</td>
<td>( \text{PS} )</td>
<td>( g &gt;_{IO} f )</td>
<td>✔</td>
<td>✔</td>
<td>✔ (Corollary 2 in Sec. 8.1)</td>
</tr>
<tr>
<td>( \text{RSD} )</td>
<td>( \text{RV} )</td>
<td>( g &gt;_{IR} f )</td>
<td>✗</td>
<td>✗</td>
<td>✗ (Sec. 10.5 in (Mennle and Seuken, 2014b))</td>
</tr>
<tr>
<td>( \text{RSD} )</td>
<td>( \text{NBM} )</td>
<td>( g &gt;_{IR} f )</td>
<td>✔</td>
<td>✗</td>
<td>✗ (Example 4 in Sec. 8.2.2)</td>
</tr>
<tr>
<td>( \text{RSD} )</td>
<td>( \text{ABM} )</td>
<td>( g &gt;_{IR} f ) in the limit</td>
<td>✔</td>
<td>✔</td>
<td>✔ (Corollary 3 in Sec. 8.3)</td>
</tr>
</tbody>
</table>
if neither $a$ nor $b$ are exhausted when $i$ finishes consuming objects that it strictly prefers to both. Next, we show that RSD changes the allocation if and only if there exists an ordering of the agents such that all objects that $i$ prefers strictly to $a$ and $b$ are allocated before $i$ gets to pick, but neither $a$ nor $b$ are allocated by then. Finally, we show that the former condition (for PS) implies the latter condition (for RSD), using an inductive argument: the key idea is to show that if no such ordering of the agents exists for $m$ objects, then we can construct a case with $m - 1$ objects that exhibits the same issue.

\[\text{Corollary 2. The pair (RSD,PS) admits the construction of URBI(r)-partially strategyproof hybrids that imperfectly ordinally dominate RSD.}\]

8.2. Limitations for Hybrids of RSD with RV or NBM

A mechanism designer may also want to trade off strategyproofness for improvements of the rank distribution. Mechanisms that aim at some form of a good rank distribution are RV (Featherstone, 2011) and Boston mechanisms (Mennle and Seuken, 2014a). It turns out, however, that in general neither RV nor the naïve variant of the Boston mechanism (NBM) are suitable second components in combination with RSD.

8.2.1. Impossibility for RV

RV strictly imperfectly rank dominates RSD, since RV is rank efficient, i.e., never rank dominated, but RSD is not. However, in Section 10.5 of Mennle and Seuken (2014b), we show that no rank efficient mechanism can be weakly invariant. In addition, for some choices of rank valuation, RV is also not weakly less varying than RSD, as Example 3 in Appendix B.9 illustrates. Thus, hybrids of RSD and RV cannot be constructed by means of Theorem 1.\(^2\)

8.2.2. Impossibility for the Naïve Boston Mechanism

We consider the Boston mechanism with single uniform tie-breaking (Miralles, 2009). The “naïve” variant of the Boston mechanism (NBM) lets agents apply to their respective next best choices in consecutive rounds, even if the objects to which they apply have no more remaining capacity. NBM imperfectly rank dominates RSD (Theorem 2 in Mennle and Seuken (2014a)). NBM is also weakly invariant by Proposition 3 in Mennle and Seuken (2014a). However, it is not weakly less varying than RSD, as Example 4 in Appendix B.9 shows. Thus, the pair (RSD,NBM) is not hybrid-admissible and cannot be used to achieve meaningful trade-offs between strategyproofness and rank efficiency.

8.3. Hybrids of RSD and ABM

In (Mennle and Seuken, 2014a), we have introduced an adaptive variant of the Boston mechanism (ABM), which is sensitive to the fact that agents cannot benefit from applying at

\(^2\)The requirement that $g$ is weakly invariant may be dropped from Theorem 1 if instead a relative upper bound on the maximum utility of any agent is imposed. Then RV may be weakly less varying than RSD for some rank valuations. Finding such rank valuations is subject to future research.
objects that are already exhausted. Instead, in each round, agents who have not been allocated so far, apply to their most preferred object with remaining capacity.

We now show that hybrids of RSD and ABM can be used to trade off strategyproofness and rank efficiency, at least in the limit. ABM is weakly invariant (by Proposition 3 in (Mennle and Seuken, 2014)). By Theorem 4 in (Mennle and Seuken, 2014), ABM imperfectly rank dominates RSD in the limit (as \( n = m \to \infty \)). By Proposition 5, non-trivial hybrids of ABM and RSD will also imperfectly rank dominate RSD in the limit. To show that the pair (RSD, ABM) is hybrid-admissible, it remains to be shown that ABM is weakly less varying than RSD.

Theorem 3. ABM is weakly less varying than RSD.

Proof outline (formal proof in Appendix B.10). Both RSD and ABM are implemented by randomizing over orderings of agents \( \pi \). Suppose \( i \) manipulates by swapping \( a \) and \( b \). If ABM changes the allocation, then there exists an ordering \( \pi \) such that all objects that \( i \) prefers strictly to \( a \) and \( b \) are allocated in previous rounds to other agents. Then \( i \) gets to “pick” between \( a \) and \( b \) in some round. Starting with \( \pi \), we construct an ordering \( \pi' \) such that if the ordering \( \pi' \) is drawn under RSD, \( i \) gets to pick between \( a \) and \( b \), but no object it strictly prefers to \( a \) or \( b \). This is sufficient for RSD to also change the allocation under the swap of \( a \) and \( b \) by \( i \). □

Corollary 3. The pair (RSD, ABM) admits the construction of URBI\((r)\)-partially strategyproof hybrids that imperfectly rank dominate RSD in the limit.

9. Numerical Results

In Section 8, we have shown that we can construct interesting hybrids by combining RSD with PS or ABM, which lets a mechanism designer trade off strategyproofness for better efficiency. To illustrate the magnitude of these trade-offs, we now compute \( \beta_{\text{max}} \) for a variety of settings \((N, M, q)\) and acceptable degrees of strategyproofness \( r \in [0, 1] \).

Figure 2 shows plots of the maximal mixing factor \( \beta_{\text{max}} \) for settings with unit capacity and different numbers of objects and agents. Observe that as the acceptable degree of strategyproofness for the hybrid increases, the allowable share of \( g \) decreases and becomes 0 if full strategyproofness is required. We also see that the relationship between \( r \) and \( \beta_{\text{max}} \) is not linear. In particular, the first efficiency improvements (from \( \beta_{\text{max}} = 0 \) to \( \beta_{\text{max}} > 0 \)) are the most “costly” in terms of a reduction of the degree of strategyproofness \( r \). On the other hand, for mild strategyproofness requirements, the share of PS or ABM in the hybrid can be significant, e.g., more than 30% of PS or 17% of ABM for \( r = 0.75 \) and \( n = m = 4 \).

Figure 3 shows plots of \( \beta_{\text{max}} \) for hybrids of RSD and PS, where we hold the number of object types constant at \( m = 3 \), but vary the capacity of the objects \( q \in \{2, 3, 4\} \). We observe that

\footnote{Note that potential rank dominance of RSD over ABM is not a real concern, as the share of profiles for which this occurs is negligible, even in small settings and converges to zero at an exponential rate.}
for larger capacities, the hybrids can contain a larger share of PS. It is conceivable that the degree of strategyproofness of PS keeps increasing and converges to 1 in the limit as capacity increases, an interesting question for future research.

10. Conclusion

In this paper, we have presented a novel approach to trading off strategyproofness and efficiency for one-sided matching mechanisms. We have introduced hybrid mechanisms, which are convex

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4This is consistent with findings by Kojima and Manea (2010), who have shown that for some fixed agent and a fixed number of objects, PS makes truthful reporting a dominant strategy for that agent if the capacities of the objects are sufficiently high. We omitted the curve for $n = 3$ in Figure 3 to better illustrate the effect. It would lie close to the curve for $n = 9$, i.e., incentives get worse (for $n = 6$) before the convergence takes effect.
combinations of two component mechanisms, as a method to perform this trade-off. Typically, the first component $f$ introduces better incentives while the second component $g$ introduces better efficiency.

For our first result, we have employed a novel concept for relaxing strategyproofness in a parametric way, i.e., URBI($r$)-partial strategyproofness. Under the axiom of weak invariance and the technical assumption that $g$ be weakly less varying than $f$, we have shown that non-trivial hybrid mechanisms can be constructed that are URBI($r$)-partially strategyproof for any desired degree of strategyproofness.

For our second result, we have shown that hybrid mechanisms in fact trade off strategyproofness for efficiency: as the mixing factor $\beta$ (i.e., the share of $g$) increases, efficiency of the hybrid increases, but the degree of strategyproofness decreases, while fairness properties that both mechanisms satisfy (anonymity or symmetry) are preserved. From the mechanism designer’s perspective this means the following: if $f$ is a strategyproof mechanism, $g$ is a non-strategyproof alternative that is more appealing due to its efficiency properties, and a certain degree of strategyproofness (smaller than 1.0) is acceptable, then a hybrid can be used to maximize efficiency improvements subject to the URBI($r$)-partial strategyproofness constraint. As we have shown in Section 7, the mechanism designer’s problem of determining a maximal mixing factor can be solved algorithmically.

Finally, we have presented instantiations of hybrid mechanisms with $f = \text{RSD}$ as the strategyproof component. Using $g = \text{PS}$ yields better efficiency in an (imperfect) ordinal dominance sense, and using $g = \text{ABM}$, an adaptive variant of the Boston mechanism, yields better efficiency in an (imperfect) rank dominance sense in the limit. Numerically, we have illustrated the connection between the degree of strategyproofness $r$ and the maximal mixing factor $\beta_{\text{max}}$, and we have shown that the latter can be significant for mild restrictions on the degree of strategyproofness.

This paper contributes to an important area of research concerned with trade-offs between strategyproofness and efficiency in one-sided matching. Hybrid mechanisms break new ground, because the method is constructive, it enables a parametric trade-off, and the mechanism designer’s problem of determining a suitable mechanism is computable. Our hybrids shed light on the frontiers of such trade-offs and can serve as benchmark mechanisms for future research.

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Appendix

A. Axiomatic Characterization of Strategyproof and URBI(r)-partially Strategyproof Mechanisms

First, we review the axioms introduced in Mennle and Seuken (2014b). Then we restate the three main theorems, a characterization of strategyproofness, a characterization of URBI(r)-partial strategyproofness, and the maximality result for the URBI(r)-domain restriction.

The neighborhood $N_t$ of a type $t$ is the set of types $t'$ where the corresponding preference orders differ by just a swap of two adjacent objects. The upper contour set of an object $a$ with respect to some type $t$ is the set of objects that an agent of type $t$ strictly prefers to $a$, and the lower contour set contains all the objects that the agent likes strictly less than $a$. These are denoted by $U_{p_{a,t}}$ and $L_{p_{a,t}}$, respectively. The axioms swap consistency, weak invariance, and lower invariance limit the way in which a mechanism can change the allocation if an agent swaps two adjacent objects in its reported preference ordering.

**Axiom 1** (Swap Consistency). A mechanism $f$ is swap consistent if for any agent $i \in N$, any type profile $t = (t_i, t_{-i}) \in T^n$, and any type $t_i' \in N_{t_i}$ (i.e., in the neighborhood of $t_i$) with $a_k > a_{k+1}$ under $t_i$ and $a_{k+1} > a_k$ under $t_i'$, one of the following holds:

1) $i$’s allocation is unaffected by the swap, i.e., $f(t_i) = f(t_i')$, or

2) $i$’s allocation for $a_k$ strictly decreases and its allocation for $a_{k+1}$ strictly increases, i.e.,

$$f(t_i)(a_k) > f(t_i')(a_k) \text{ and } f(t_i)(a_{k+1}) < f(t_i')(a_{k+1}).$$

(15)

The intuition of swap consistency is that the mechanism should be responsive to the agent’s preferences: an agent’s chances of getting an object should increase if it is brought up in the preference order. The mechanism must also be direct by changing the allocation of the objects that are swapped, if any.

**Axiom 2** (Weak Invariance). A mechanism $f$ is weakly invariant if for any agent $i \in N$, any type profile $t = (t_i, t_{-i}) \in T^n$, and any type $t_i' \in N_{t_i}$ with $a_k > a_{k+1}$ under $t_i$ and $a_{k+1} > a_k$ under $t_i'$, $i$’s allocation for the upper contour set $U(a_k, t_i)$ is unaffected by the swap, i.e., for all $j \in U(a_k, t_i)$

$$f(t_i)(j) = f(t_i')(j).$$

(16)

Intuitively, under weak invariance, an agent cannot influence the allocation of one of its better choices by swapping two less preferred objects. Weak invariance is essentially equivalent to truncation robustness (see (Hashimoto et al., 2013)). Many mechanisms from the literature satisfy weak invariance, such as the Random Serial Dictatorship, Probabilistic Serial, the Boston mechanism, and Student-proposing Deferred Acceptance.

**Axiom 3** (Lower Invariance). A mechanism $f$ is lower invariant if for any agent $i \in N$, any type profile $t = (t_i, t_{-i}) \in T^n$, and any type $t_i' \in N_{t_i}$ with $a_k > a_{k+1}$ under $t_i$ and $a_{k+1} > a_k$
under $t'_i$, i’s allocation for the lower contour set $L(a_{k+1}, t_i)$ is unaffected by the swap, i.e., for all $j \in L(a_{k+1}, t_i)$

$$f(t_i)(j) = f(t'_i)(j).$$

(17)

Lower invariance complements weak invariance: it requires that an agent cannot influence the allocation for less preferred objects by swapping two more preferred objects. In combination, the three axioms characterize strategyproofness.

**Theorem 4** (Theorem 1 from Mennle and Seuken (2014b)). A mechanism $f$ is strategyproof if and only if it is swap consistent, weakly invariant, and lower invariant.

The requirement of swap consistency prevents manipulability in a first order-stochastic dominance sense and weak invariance is a popular form of truncation robustness, used by Hashimoto et al. (2013) to characterize the Probabilistic Serial mechanism. While lower invariance is the missing piece to guarantee strategyproofness, it does not have a strong, intuitive justification as an axiom. Dropping the lower invariance requirement leads to the class of URBI($r$)-partially strategyproof mechanisms.

**Theorem 5** (Theorem 2 from Mennle and Seuken (2014b)). Given a setting $(N, M, q)$, a mechanism $f$ is URBI($r$)-partially strategyproof for some $r \in (0, 1]$ if and only if $f$ is swap consistent and weakly invariant.

Finally, the URBI($r$)-domain restriction can be shown to be maximal for the set of swap consistent, weakly invariant mechanisms.

**Theorem 6** (Theorem 3 from Mennle and Seuken (2014b)). Given a setting $(N, M, q)$, a bound $r \in (0, 1]$, and a utility function $\tilde{u} \in t$ that violates URBI($r$), there exists a mechanism $\tilde{f}$ such that

- $\tilde{f}$ is URBI($r$)-partially strategyproof,
- $\tilde{f}$ is not $\{\tilde{u}\}$-partially strategyproof, i.e., there exists a type $t' \neq t$ and reports $t_- \in T^{n-1}$ such that

$$\langle \tilde{u}, f(t, t_-) - f(t', t_-) \rangle < 0.$$  

(18)

**B. Omitted Examples, Lemmas, and Proofs**

**B.1. Theorem 1 from Section 5.3**

*Proof of Theorem 1.* For any setting $(N, M, q)$, any hybrid-admissible pair $(f, g)$, and any bound $r \in [0, 1)$, there exists a positive mixing factor $\beta \in (0, 1]$ such that the hybrid $h_\beta(f, g)$ is URBI($r$)-partially strategyproof in the setting $(N, M, q)$.

Consider a strategyproof mechanism $f$ and a weakly less varying, weakly invariant mechanism $g$, a fixed setting $(N, M, q)$, and a fixed bound $r \in [0, 1)$. We must find a mixing factor $\beta \in (0, 1]$ such that no agent with a utility satisfying URBI($r$) will find a beneficial manipulation to the hybrid $h_\beta(f, g)$. 

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Let \( t = (t_i, t_{-i}) \in T^n \) be a type profile, \( u_i \in t_i \) a utility function for agent \( i \), and let \( t_i' \in T \) be a potential misreport. Suppose that \( g \) changes the allocation for \( i \) (otherwise the incentive constraint for the hybrid mechanism is trivially satisfied for this type profile and misreport). By Lemma 1, there exists a rank \( L \in \{1, \ldots, m-1\} \) such that the gain in expected utility from reporting \( t_i' \) instead of \( t_i \) under \( g \) is upper-bounded by

\[
\langle u_i, g(t_i') - g(t_i) \rangle \leq u_i(a_L) - u_i(a_m),
\]

and the utility gain from reporting \( t_i \) truthfully instead of the misreport \( t_i' \) under \( f \) is lower-bounded by

\[
\langle u_i, f(t_i) - f(t_i') \rangle \geq \epsilon u_i(a_L) - \epsilon u_i(a_{L+1}),
\]

where \( \epsilon > 0 \) depends only on the setting and the mechanism \( f \). Thus, the utility gain from reporting \( t_i \) truthfully instead of the misreport \( t_i' \) under the hybrid \( h_\beta(f, g) \) is lower bounded by

\[
\langle u_i, h_\beta(f, g)(t_i) - h_\beta(f, g)(t_i') \rangle = (1 - \beta) \langle u_i, f(t_i) - f(t_i') \rangle + \beta \langle u_i, g(t_i) - g(t_i') \rangle \geq (1 - \beta) \epsilon (u_i(a_L) - u_i(a_{L+1})) - \beta (u_i(a_L) - u_i(a_m)) \]

\[
= (u_i(a_L) - u_i(a_m)) (\epsilon(1 - \beta) - \beta) - (u_i(a_{L+1}) - u_i(a_m)) (\epsilon(1 - \beta)).
\]

If \( u_i \) satisfies URBI(\( r \)), we can lower bound \( (u_i(a_L) - u_i(a_m)) \) by \( r(u_i(a_{L+1}) - u_i(a_m)) \) and get

\[
\langle u_i, h_\beta(f, g)(t_i) - h_\beta(f, g)(t_i') \rangle \geq \frac{u_i(a_{L+1}) - u_i(a_m)}{r} (\epsilon(1 - \beta) - \beta - r \epsilon(1 - \beta)).
\]

Since \( \frac{u_i(a_{L+1}) - u_i(a_m)}{r} \geq 0 \), this is positive if and only if

\[
\epsilon(1 - \beta) - \beta - r \epsilon(1 - \beta) \geq 0 \iff \beta \leq \frac{\epsilon(1 - r)}{\epsilon(1 - r) + 1}.
\]

This upper bound for \( \beta \) is strictly positive and independent of the specific utility function \( u_i \), the type profile \( (t_i, t_{-i}) \), and the misreport \( t_i' \). Therefore, \( h_\beta(f, g) \) is URBI(\( r \))-partially strategyproof if \( \beta \) is chosen to satisfy (27).

**Lemma 1.** Consider a setting \((N, M, q)\), a strategyproof mechanism \( f \), a weakly less varying, weakly invariant mechanism \( g \), an agent \( i \in N \), a type profile \( t = (t_i, t_{-i}) \in T^n \), a misreport \( t_i' \in T \), and a utility function \( u_i \in t_i \). If \( f_i(t_i, t_{-i}) \neq f_i(t_i', t_{-i}) \), then there exists \( L \in \{1, \ldots, m-1\} \) such that the gain in expected utility from reporting \( t_i' \) instead of \( t_i \) under \( g \) is upper-bounded by

\[
\epsilon (u_i(a_L) - u_i(a_m)),
\]

and the gain in expected utility from reporting \( t_i \) truthfully instead of \( t_i' \) under \( f \) is lower-bounded by

\[
\epsilon (u_i(a_L) - u_i(a_{L+1})),
\]

where \( \epsilon > 0 \) depends only on the setting and the mechanism \( f \).
Proof. We first introduce the auxiliary concept of the canonical transition. Consider two types \( t \) and \( t' \). A transition from \( t \) to \( t' \) is a sequence of types \( \tau(t, t') = (t^0, \ldots, t^S) \) such that

- \( t^0 = t \) and \( t' = t^S \),
- \( t^{k+1} \in N_{t_k} \) for all \( k \in \{0, \ldots, S - 1\} \).

A transition can be interpreted as a sequence of swaps of adjacent objects that transform one type into another if applied in order. Suppose,

\[
t' : a_1 > a_2 > \ldots > a_m.
\]

Then the canonical transition is the transition that results from starting at \( t \) and swapping \( a_1 \) (which may not be in first position for \( t \)) up until it is in first position. Then do the same for \( a_2 \), until it is in second position, and so on, until \( t' \) is obtained.

Suppose \( t_i \) corresponds to the preference ordering

\[
t_i : a_1 > \ldots > a_{L-1} > a_L > \ldots > a_m,
\]

and let \( a_L \) be the best choice object (under \( t_i \)) for which the allocation under \( f \) changes, i.e.,

\[
f(t_i)(a_k) = f(t'_i)(a_k) \text{ for } k < L, \quad f(t_i)(a_L) \neq f(t'_i)(a_L). \tag{30}
\]

Consider the canonical transition from \( t'_i \) to \( t_i \). This will bring the objects \( a_k, k < L \) into position (as they are under \( t_i \)) first. By Theorem 1 in Mennle and Seuken (2014) and because \( f \) is strategyproof, the allocation for each of these objects can only weakly increase or weakly decrease. However, by (30) their allocations remain unchanged. Therefore, the allocation does not change for any of the swaps that bring the objects \( a_k, k < L \) into position. Using that \( g \) is weakly less varying than \( f \), we can assume that

\[
t'_i : a_1 > \ldots > a_{L-1} > a'_L > \ldots > a'_m
\]

without loss of generality.

By weak invariance of \( g \), the highest gain the agent could gain from reporting \( t'_i \) instead of \( t_i \) arises if all probability for its last choice is converted to probability for the best choice for which the allocation can change at all, i.e., \( a_L \). Thus, the utility gain is bounded by

\[
u_i(a_L) - u_i(a_m). \tag{31}
\]

Let

\[
e = \min \{ |f(t_i)(j) - f(t'_i)(j)| \mid j \in M, i \in N, (t_i, t_{-i}) \in T^m, t'_i \in T : f(t_i)(j) \neq f(t'_i)(j) \} \tag{32}
\]

be the smallest positive amount by which the allocation of some object to some agent can change upon a change of report by that agent under \( f \). In the canonical transition from \( t_i \) to \( t'_i \), the object \( a_L \) will only be swapped downwards, i.e., its allocation can not increase in any step. But since we assumed that it changes, it must strictly decrease. This decrease has at least
magnitude $\epsilon$ by definition. Thus, when misreporting, the agent looses at least $\epsilon$ probability for $a_L$ in some swap. From Theorem 1 in Mennle and Seuken (2014b) we know that the allocation for the other object involved in that swap must strictly increase by the same amount $\epsilon$. Since all other swaps reverse the order of objects from “right” (as under $t_i$) to “wrong” (as under $t'_i$), the allocation can only get weakly worse for the agent. Therefore, the gain from reporting $t_i$ truthfully instead of $t'_i$ under $f$ is at least $\epsilon(u_i(a_L) - u_i(a_{L+1}))$. 

### B.2. Propositions 1 and 2 from Section 5.3

**Proof of Proposition 1.** If $f$ is not strategyproof, there exists a mechanism $g$ that is weakly invariant and weakly less varying than $f$, and a bound $r \in (0,1)$ such that no non-trivial hybrid of the pair $(f, g)$ will be URBI($r$)-partially strategyproof.

Assume that $f$ is not strategyproof. Then there exists $t = (t_i, t_{-i}) \in T^n$, $t'_i \in T$ and a utility $u_i \in t_i$ such that

$$\langle u_i, f(t_i) - f(t'_i) \rangle < 0. \quad (33)$$

Choose as $g$ the constant mechanism that allocates equal shares of all objects to all agents, independent of their reported preferences. This $g$ is weakly invariant and weakly less varying than any mechanism. Then for any mixing factor $\beta \in [0,1)$, the resulting hybrid mechanism will be also be manipulable for $u_i$. In particular, let

$$\bar{r} = \max \left\{ \frac{u_i(b) - \min u_i}{u_i(a) - \min u_i} \mid a > b \text{ for } i \right\} < 1. \quad (34)$$

Then $u_i$ satisfies URBI($r$) for any $r > \bar{r}$. However, $h_\beta(f, g)$ is manipulable by $u_i$ independent of $\beta$. Thus, we find cannot a non-trivial mixing factor such that the hybrid is URBI($\bar{r}$)-partially strategyproof.

**Proof of Proposition 2.** For any strategyproof $f$ and any $g$ that is weakly less varying than $f$, but not weakly invariant in a setting $(N, M, q)$, no non-trivial hybrid of the pair $(f, g)$ is URBI($r$)-partially strategyproof for any bound $r \in (0, 1]$ in the setting $(N, M, q)$.

Consider a situation in which $g$ violates weak invariance, i.e., the setting $(N, M, q)$, a type profile $t = (t_i, t_{-i}) \in T^n$, a misreport $t'_i \in N_{t_i}$, where

$$t_i : \ldots > a > \ldots > x > y > \ldots, \quad (35)$$

$$t'_i : \ldots > a > \ldots > y > x > \ldots, \quad (36)$$

and $g(t_i)(a) \neq g(t'_i)(a)$. Since $f$ is strategyproof, $f(t_i)(a) = f(t'_i)(a)$ (by Theorem 1 from Mennle and Seuken (2014b)), and thus for any $\beta > 0$,

$$h_\beta(f, g)(t_i)(a) \neq h_\beta(f, g)(t'_i)(a). \quad (37)$$

But then the hybrid is not weakly invariant and thus (by Theorem 2 from Mennle and Seuken (2014b)) not URBI($r$)-partially strategyproof for any bound $r \in (0, 1]$ in the setting $(N, M, q)$. 

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B.3. Proposition 4 from Section 6.2

Proof of Proposition 4. For allocations \( x \) and \( y \) let \( z = (1 - \beta)x + \beta y \). For any type profile \( t \in T^n \) and any \( \beta \in (0, 1] \) the following hold:

1) \( y \succeq_O x \) at \( t \) \( \Rightarrow \) \( z \succeq_O x \) at \( t \), and \( y >_O x \) at \( t \) \( \Rightarrow \) \( z >_O x \) at \( t \),

2) \( y \succeq_R x \) at \( t \) \( \Rightarrow \) \( z \succeq_R x \) at \( t \), and \( y >_R x \) at \( t \) \( \Rightarrow \) \( z >_R x \) at \( t \).

If \( y \succeq_O x \) at \( t \), we have for all agents \( i \in N \) and objects \( j \in M \)
\[
\sum_{j' > j} y_{i,j'} \geq \sum_{j' > j} x_{i,j'}.
\]
Thus, for any agent \( i \) and any object \( j \) we also have
\[
\sum_{j' > j} z_{i,j'} = (1 - \beta) \sum_{j' > j} x_{i,j'} + \beta \sum_{j' > j} y_{i,j'} \geq \sum_{j' > j} x_{i,j'}.
\]
If \( \beta > 0 \) and (38) is strict for some \( i \) and \( j \), then so is (39). This yields 1. For 2. observe that the rank distribution of \( z \) is given by
\[
d^z = (1 - \beta)d^x + \beta d^y,
\]
and weak and strict rank dominance follow by a similar argument as above.

B.4. Examples from Section 6.3

Example 1 (Adapted from Bogomolnaia and Moulin (2001)). Consider a setting \( N = \{1, \ldots, 4\} \), \( M = \{a, b, c\} \), \( q_a = q_b = 1, q_c = 2 \). For the type profile
\[
t_1, t_2 : a > b > c, \quad t_3, t_4 : a > b > c,
\]
the resulting allocations are
\[
PS(t) = \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right) >_O \left( \begin{array}{ccc} 5/12 & 1/12 & 1/12 \\ 1/12 & 5/12 & 1/12 \\ 1/12 & 1/12 & 5/12 \end{array} \right) = RSD(t),
\]
i.e., \( PS(t) \) strictly ordinally dominates \( RSD(t) \) at \( t \).

Example 2. Consider a setting \( N = \{1, \ldots, 3\} \), \( M = \{a, b, c\} \), \( q_a = q_b = q_c = 1 \). For the type profile
\[
t_1 : a > b > c, \quad t_2 : b > a > c, \quad t_3 : b > c > a,
\]
the resulting allocations are
\[
\text{PS}(t) = \left( \begin{array}{cccc}
\frac{5}{3} & 0 & 0 & 0 \\
\frac{1}{2} & 1 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}
\end{array} \right), \quad \text{RSD}(t) = \left( \begin{array}{cccc}
\frac{3}{4} & 0 & 0 & 0 \\
\frac{1}{2} & 1 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}
\end{array} \right).
\]
(47)

Since RSD\(_1(t) >_O\) PS\(_1(t)\) and PS\(_2(t) >_O\) RSD\(_2(t)\), we have PS\(_t\) _⪰_ O RSD\(_t\) at \(t\) (and RSD\(_t\) _⪯_ O PS\(_t\) at \(t\)).

B.5. Proposition 5 from Section 6.3

Proof of Proposition 5. For mechanisms \(f\) and \(g\) and any \(\beta \in (0, 1]\) the following hold:

1) \(g \succeq IO f \implies h_\beta(f, g) \succeq IO f\) and \(g > IO f \implies h_\beta(f, g) > IO f\),
2) \(g \succeq IR f \implies h_\beta(f, g) \succeq IR f\) and \(g > IR f \implies h_\beta(f, g) > IR f\).

From Proposition 4 it follows that if \(g(t)\) weakly or strictly ordinally or rank dominates \(f(t)\) at \(t\), then so does \(h_\beta(f, g)(t)\) for any \(\beta > 0\). We only need to show that if \(f(t) >_O g(t)\), then \(f(t) >_O h_\beta(f, g)(t)\), and likewise for rank dominance.

Denote by “\(x \succeq y\)” the fact that \(x\) does not strictly dominate \(y\). Let \(z = (1 - \beta)x + \beta y\). For some allocations \(x, y\), if \(x >_O y\), then either \(x = y\), or there exists an agent \(i\) and an object \(j\) such that
\[
\sum_{j' > i} x_{i, j'} > \sum_{j' > i} y_{i, j'}.
\]
(48)

If \(x = y\), then \(x = z\), implying \(x >_O z\). Otherwise, \(x >_O z\) holds, because
\[
\sum_{j' > i} x_{i, j'} = (1 - \beta) \sum_{j' > i} x_{i, j'} + \beta \sum_{j' > i} y_{i, j'} = \sum_{j' > i} z_{i, j'}.
\]
(49)

Similarly, if \(x > IR y\), then either \(d^x = d^y\) or there exists a rank \(k\) such that
\[
\sum_{k' \leq k} d^x_{k'} < \sum_{k' \leq k} d^y_{k'}.
\]
(50)

\(x > IR z\) follows, because either \(d^z = d^x\), or \(d^z = (1 - \beta)d^x + \beta d^y\), and therefore
\[
\sum_{k' \leq k} d^x_{k'} < (1 - \beta) \sum_{k' \leq k} d^x_{k'} + \beta \sum_{k' \leq k} d^y_{k'} = \sum_{k' \leq k} d^y_{k'}.
\]
(51)

\[\square\]
B.6. Proposition 6 from Section 6.4

Proof of Proposition 6. Consider any setting \((N,M,q)\) and any hybrid-admissible pair \((f,g)\), where \(g >_\text{IO} f\). If \(g\) is not strategyproof in the setting, then for any two mixing factors \(0 \leq \beta < \beta' \leq 1\),

1) \(h_{\beta'}(f,g) >_\text{IO} h_{\beta}(f,g)\),

2) \(\rho_{(N,M,q)}(h_{\beta}(f,g)) > \rho_{(N,M,q)}(h_{\beta'}(f,g))\).

The same holds if the >_{IO}-relation is replaced by any of >_{IR}, >_{O}, \leq_{IR}, \leq_{O}, \geq_{IR}, \geq_{O}.

Under these assumptions, 1) is a direct consequence of Propositions 4 and 5. To see 2), let \(r = \rho_{(N,M,q)}(h_{\beta}(f,g))\). Then at least one of the positivity constraints in Proposition 1 in Mennle and Seuken (2014b) must bind, i.e., there exists an agent \(i \in N\), a type profile \(\tilde{t} = (\tilde{t}_i, \tilde{t}_{-i}) \in T^n\), a misreport \(p_i \in T\), and some \(k \in \{1, \ldots, m-1\}\) such that

\[
\frac{1}{r} \leq x_{k}^{h_{\beta}(f,g)}(1) = 0,
\]

and for some \(\epsilon > 0\) and any \(s \in (\frac{1}{r} - \epsilon, \frac{1}{r})\) we have

\[
x_{k}^{h_{\beta}(f,g)}(s) < 0.
\]

Since \(f\) is strategyproof, \(x_k^f(s) \geq 0\) for any \(s \geq 1\), and therefore \(x_k^g(s) < 0\). If we now consider the hybrid \(h_{\beta'}(f,g)\) for a larger mixing factor, we get that (52) is violated, i.e.,

\[
\frac{1}{r} < 0,
\]

since \(x_k^{h_{\beta'}(f,g)} = (1 - \beta')x_k^f + \beta' x_k^g\). (54) implies that \(h_{\beta'}(f,g)\) is not URBI\((r)\)-partially strategyproof, and 2) follows.

B.7. Proposition 8 from Section 7.1

Proof of Proposition 8. For computable \(f\) and \(g\), Algorithm 1 is complete and correct.

Since \(f\) and \(g\) are computable and all loops in Algorithm 1 are finite, it terminates on any admissible input, i.e., it is complete. To see correctness, we show that for the resulting value of \(\beta\), the hybrid \(h_{\beta}(f,g)\) satisfies the positivity constraints (14) from Proposition 2 in Mennle and Seuken (2014b), and at least on of the constraints is tight. We have that

\[
x_{k}^{h_{\beta'}(f,g)}(s) \geq 0 \iff (1 - \beta)x_k^f(s) + \beta x_k^g(s) \geq 0
\]

\[
\iff x_k^f(s) \geq \beta(x_k^f(s) - x_k^g(s)).
\]

Since \(f\) is strategyproof, we have \(x_k^f(s) \geq 0\). Then, if \(x_k^f(s) - x_k^g(s) \leq 0\), inequality (56) imposes no constraint on \(\beta\). Otherwise, the algorithm decreases \(\beta\) to an appropriate value such that the
constraint is satisfied with equality. Since \( f \) is strategyproof, none of the previous constraints get violated when \( \beta \) is decreased. Thus, the final value of \( \beta \) is no larger than \( \beta_{\text{max}} \). However, at least one constraint must be tight, i.e., non-trivially satisfied with equality. Therefore, the value of \( \beta \) will be exactly \( \beta_{\text{max}} \), which proves correctness.

\[ \square \]

### B.8. Theorem 2 from Section 8.1

**Proof of Theorem 2.** \( \text{PS} \) is weakly less varying than \( \text{RSD} \).

Suppose, \( n \) agents compete for \( m = m_a + 2 + m_b \) objects with capacities given by \( q \), and let \( M = \{a_1, \ldots, a_{m_a}, x, y, b_1, \ldots, b_{m_b}\} \). Agent 1 is considering the two type reports

\[
\begin{align*}
    t_1 & : a_1 > \ldots > a_{m_a} > x > y > b_1 > \ldots > b_{m_b}, \\
    t'_1 & : a_1 > \ldots > a_{m_a} > y > x > b_1 > \ldots > b_{m_b},
\end{align*}
\]

where the positions of \( x \) and \( y \) are reversed in the second report. The reports of the other agents are fixed and given by \( t_{-1} \).

Further suppose that with reports \( (t_1, t_{-1}) \), the objects where exhausted at times \( 0 < \tau_1 \leq \tau_2 \leq \ldots \leq \tau_m = 1 \) under \( \text{PS} \). Re-label the objects as \( j_1, \ldots, j_m \) in increasing order of the times at which they were exhausted. If two objects were exhausted at the same time, re-label them in arbitrary order. Denote by \( \tau_x \) and \( \tau_y \) the times at which \( x \) and \( y \) were exhausted, respectively.

Given these considerations, Lemma 2 yields equivalent conditions under which \( \text{PS} \) changes the allocation, Lemma 3 yields similar conditions under which \( \text{RSD} \) changes the allocation, and Lemma 4 shows that the former condition implies the latter.

**Lemma 2.** In Theorem 2, \( \text{PS}(t_1, t_{-1}) \neq \text{PS}(t'_1, t_{-1}) \) if and only if

1) there exists \( k \geq m_a \) such that \( \tau_1 \leq \ldots \leq \tau_k < \min(\tau_x, \tau_y) \leq 1 \) and

2) for all \( l \in \{1, \ldots, m_a\} \) there exists \( l' \in \{1, \ldots, k\} \) with \( a_l = j_{l'} \).

**Proof.** “\( \Rightarrow \)” Choose \( k \) such that \( j_k \) is the last of the \( a_1, \ldots, a_{m_a} \) to run out. Suppose, \( \tau_y \leq \tau_k \). Agent 1 is busy consuming shares of other objects until time \( \tau_k \), regardless of the reported order of \( x \) and \( y \). After \( \tau_k \) agent 1 consumes shares of \( x \) until it is exhausted. Because \( y \) was already exhausted before \( \tau_k \), agent 1 gets no shares of \( y \). Under report \( t'_1 \), it would finish consuming other objects at \( \tau_k \) and find objects \( y \) exhausted. Hence, it would begin consuming shares of \( x \) immediately, just as it did under report \( t_1 \). Thus, the order in which \( x \) and \( y \) are reported does not matter for the times at which it consumes objects \( x \) and \( y \). Because \( t_1 \) and \( t'_1 \) only differ in the order of \( x \) and \( y \), the remaining objects are also consumed in the same order and at the same times. Hence, agent 1’s allocation does not change.

The case for \( \tau_x \leq \tau_k \) is analogous.

Because \( \text{PS} \) is non-bossy (Kesten and Ekici, 2012), we know that if the switch from \( t_1 \) to \( t'_1 \) did not change the allocation for agent 1, it did not change the allocation at all.
Suppose the last of the objects \(a_1, \ldots, a_m\) to be exhausted is \(j_k\), and \(\tau_k < \tau_y \leq \tau_x\). Then agent 1 gets no shares of \(y\). If it switches its report to \(>_1'\), it will receive a non-trivial share of \(y\), hence the allocation changes.

Now suppose the opposite, namely \(\tau_y > \tau_x\). Agent 1 begins consumption of \(x\) at time \(\tau_k\) and then turns to \(y\) at time \(\tau_x\). Thus agent 1 receives \(\tau_x - \tau_k\) shares of \(x\) and \(\tau_y - \tau_x\) shares of \(y\). When it switches its report to \(>_1'\), it will consume shares of \(y\) between \(\tau_k\) and \(\tau_y'\). We need to show that \(\tau_y' - \tau_k > \tau_y - \tau_x\). If \(\tau_y' \geq \tau_y\), this is clear, because \(\tau_k < \tau_x\) by assumption. In the following we assume \(\tau_y' < \tau_y\).

Let \(n_y(\tau)\) be the number of agents other than agent 1 consuming shares of \(y\) at time \(\tau\). \(n_y\) is integer-valued and increasing in \(\tau\), and there must exist a \(\delta > 0\) such that \(n_y(\tau_y - \delta) \geq 1\). This means that agent 1 is not the only agent consuming shares of \(y\) before it is exhausted. Otherwise, agent 1 would exhaust \(y\) alone, which implies that agent 1 received no shares of \(x\), a contradiction.

If agent 1 reports \(>_1'\) instead, let \(n_y'(\tau)\) be the corresponding number of agents consuming \(y\) at times \(\tau\). We observe that \(x\) will be exhausted later, because agent 1 is no longer consuming shares of it. This means that agents who prefer \(x\) over \(y\) will arrive later at \(y\). Agents arriving at \(y\) from other objects than \(x\) may also arrive later, because they face less competition from the agents stuck at \(x\), etc. Therefore \(n_y' < n_y\).

Under report \(>_1\) from agent 1, \(y\) is exhausted by \(\tau_y\), i.e.,

\[
q_y = \int_{\tau_y}^{\tau_y} n_y(\tau) + \mathcal{K}_{(\tau \geq \tau_k)} d\tau ,
\]

and under report \(>_1'\), \(y\) is exhausted by \(\tau_y'\), i.e.,

\[
q_y = \int_{\tau_y}^{\tau_y'} n_y'(\tau) + \mathcal{K}_{(\tau \geq \tau_k)} d\tau \leq \int_{\tau_y}^{\tau_y} n_y(\tau) + \mathcal{K}_{(\tau \geq \tau_k)} d\tau.
\]

Equating (57) and (58) gives

\[
\int_{\tau_y}^{\tau_y} n_y(\tau) + \mathcal{K}_{(\tau \geq \tau_k)} d\tau \leq \int_{\tau_y}^{\tau_y'} n_y(\tau) + \mathcal{K}_{(\tau \geq \tau_k)} d\tau
\]

\[
\Rightarrow \int_{\tau_y}^{\tau_y'} n_y(\tau) + \mathcal{K}_{(\tau \geq \tau_k)} d\tau \leq \int_{\tau_y}^{\tau_y'} \mathcal{K}_{(\tau \geq \tau_k)} d\tau - \int_{\tau_y}^{\tau_y} \mathcal{K}_{(\tau \geq \tau_k)} d\tau + \int_{\tau_y'}^{\tau_y} \mathcal{K}_{(\tau \geq \tau_k)} d\tau
\]

\[
= \int_{\tau_y}^{\tau_y'} \mathcal{K}_{(\tau \geq \tau_k)} - \mathcal{K}_{(\tau \geq \tau_k)} d\tau
\]

\[
= \tau_x - \tau_k
\]

We know that \(j_k\) is exhausted before \(\tau_y'\) and hence \(n_y(\tau) + \mathcal{K}_{(\tau \geq \tau_k)} \geq 1\) for \(\tau \in [\tau_y', \tau_y]\), and \(\geq 2\) for \(\tau \in [\tau_y - \delta, \tau_y]\). This yields

\[
\tau_y - \tau_y' < \tau_x - \tau_k,
\]

or equivalently \(\tau_y - \tau_x < \tau_y' - \tau_k\).

\(\square\)
Lemma 3. In Theorem 2, RSD(\(t'_1, t_{-1}\)) \(\neq\) RSD(\(t_1, t_{-1}\)) if and only if there exists a sequence \((c_1, \ldots, c_{k_c})\) of \(k_c\) agents such that if RSD chose these agents first and in this order, they remove all objects \(a_1, \ldots, a_{m_a}\) (and possibly more), but neither \(x\), nor \(y\).

Proof. In the RSD mechanism, a permutations of agents is chose amongst all possible permutations with uniform probability. The probability for agent 1 to get some object \(j\) is

\[
P[1 \text{ gets } j] = \frac{|\{\pi \text{ permutation of } N : 1 \text{ gets } j \text{ under } \pi\}|}{|\{\pi \text{ permutation of } N\}|},
\]

(64)

where the denominator is \(n!\), and each permutation under which agent 1 gets \(j\) contributes \(\frac{1}{n!}\) to the total probability.

For some permutation \(\pi\) consider the turn of agent 1. There are 5 possible cases:

1) Agent 1 faces a choice set including some \(a_l\)’s. This makes no contribution to its chances of getting \(x\) or \(y\).

2) Agent 1 faces a choice set consisting only of \(b_l\)’s. Again, this makes no contribution to its chances of getting \(x\) or \(y\).

3) Agent 1 faces only \(b_l\)’s and \(x\), but not \(y\). This case contributes \(\frac{1}{n!}\) to its chances of getting \(x\). This contribution is independent of the order in which it ranked \(x\) and \(y\) in its report.

4) Agent 1 faces only \(b_l\)’s and \(y\), but not \(x\). This case contributes \(\frac{1}{n!}\) to its chances of getting \(y\) and the contribution is again independent of the ranking of \(x\) and \(y\).

5) Agent 1 faces \(x\), \(y\) and some \(b_l\)’s, but no \(a_l\)’s. This case contributes \(\frac{1}{n!}\) to either the probabilities for \(x\) or \(y\), depending on the ranking.

\[\Rightarrow\] If changing from \(t_1\) to \(t'_1\) influences the allocation, the allocation for agent 1 must have changed. This is because RSD is non-bossy (by Lemma 5). RSD also is strategyproof, hence by Theorem 1 in Mennle and Seuken (2014b) the probabilities for objects \(x\) and \(y\) must have changed. In all but the last case, the chances do not depend on the order in which \(x\) and \(y\) are reported. Thus, at least one permutation leads to case (5). This means that the sequence of agents chosen prior to agent 1 removes all \(a_l\)’s, but neither \(x\) nor \(y\).

\[\Leftarrow\] Under report \(t_1\), agent 1 will receive \(x\) any time case (5) occurs, while under \(t'_1\) it will receive \(y\). If a sequence \((c_1, \ldots, c_{k_c})\) as defined in Lemma 3 exists, it is also the beginning of at least one permutation. When this permutation is selected, case (5) occurs. Switching from report \(t_1\) to \(t'_1\) thus strictly increases agent 1’s chances of getting \(y\).

\[\Box\]

Lemma 4. In Theorem 2, 1. and 2. from Lemma 2 imply the existence of a sequence as described in Lemma 3.
Proof. We prove the claim by constructing a sequence of agents

\[(c_1, \ldots, c_k) = (c_1^{q_1}, \ldots, c_k^{q_k})\]

inductively. Under RSD this sequence will remove objects \(j_1, \ldots, j_k\) in this order.

**Selection of \(c_1^{q_1}, \ldots, c_k^{q_k}\)** By assumption \(j_k\) was consumed strictly before \(x\), hence \(\tau_k < 1\). Then at least \(q_k + 1\) agents receive non-trivial shares of \(j_k\). Otherwise, if only \(q_k\) agents received shares of \(j_k\), they would get the entire capacity and take time 1 to consume it, a contradiction. Select \(q_k\) of these agents other than agent 1 as \(c_1^{q_k}, \ldots, c_k^{q_k}\).

Because all \(c_1^{q_k}, \ldots, c_k^{q_k}\) actually received shares of \(j_k\) under PS, they must all prefer \(j_k\) to all other objects except for possibly \(j_1, \ldots, j_{k-1}\). In other words, suppose that \(j_1, \ldots, j_{k-1}\) were removed under RSD in previous turns, the selected agents would remove \(j_k\) completely if chosen next (in arbitrary order).

**Selection of \(c_1^{l}, \ldots, c_k^{q_k}, l < k\)** Suppose, \(c_1^{q_1}, \ldots, c_k^{q_k}\) have been selected. Suppose further that \(m_l\) agents (plus possibly agent 1) receive non-trivial shares of \(j_l\) under PS. There are two cases:

**Case 1** At least \(q_l\) of the \(m_l\) agents have not been selected as any of the \(c_1^{q_1}, \ldots, c_k^{q_k}\) so far. Then these agents are chosen as \(c_1^{q_1}, \ldots, c_k^{q_k}\).

**Case 2** Only \(n_l < q_l\) of the \(m_l\) agents have not been selected so far. The rest of the \(m_l\) agents have been selected at \(k'\) other objects. Let these objects be \(j_{\rho(1)}, \ldots, j_{\rho(k')}\) with \(\rho(k') \in \{1, \ldots, k\}\) for all \(k' \in \{1, \ldots, k\}\). At each of the objects \(j_{\rho(k')}\), \(q_{\rho(k')}\) agents are selected. Now there must be at least \(q_l - n_l + 1\) additional agents (possibly including agent 1) consuming non-trivial shares of the objects \(j_{\rho(k')}\), otherwise at most \(n_l + q_{\rho(1)} + \ldots + q_{\rho(k')} + q_l - n_l\) agents fully consume objects \(j_1, j_{\rho(1)}, \ldots, j_{\rho(k')}\). This will take them until time 1, a contradiction.

There are two possible cases for these additional \(q_l - n_l\) agents (excluding agent 1).

**Case 2.1** All of them are available for selection. Then they are selected for the objects \(j_{\rho(k')}\) of which they consume non-trivial shares, and the now free agents can be selected for \(j_l\).

**Case 2.1** Some of these agents are selected at some other objects \(j_{\rho(k' + 1)}, \ldots, j_{\rho(k' + k')}\). Then we use the free agents as in case 2.1, say \(n_{k'}\). Then we still need \(q_l - n_l - n_{k'}\) agents for \(j_l\). There must be at least \(q_l + q_{\rho(1)} + \ldots + q_{\rho(k')} + 1\) agents consuming non-trivial shares of the objects \(j_1, j_{\rho(1)}, \ldots, j_{\rho(k')}\). \(q_l - n_l - n_{k'}\) are not selected for any of these objects. Again there are two cases.

We repeat this argument inductively until enough agents are found who are still available and can replace agents such that the need at object \(j_l\) can be satisfied. This must happen, otherwise all agents selected so far as \(c_1^{l+1}, \ldots, c_k^{q_k}\), some \(n_l^{t} < q_l\) agents and possibly agent 1 fully consume objects \(j_l, j_{l+1}, \ldots, j_k\) objects, again a contradiction.
The fact that all selected agents \(c_1, \ldots, c_l, l \in \{1, \ldots, k\}\) receive a non-trivial share in the objects \(j_l\) implies that they each prefer \(j_l\) to all other objects, except possibly \(j_1, \ldots, j_{l-1}\). Thus, the sequence \((c_1, \ldots, c_k)\) has the properties needed for 3.

**Lemma 5.** For any distribution over orderings, the respective RSD mechanism is non-bossy.

**Proof.** Fix a distribution over orderings of the agents and let \(p_\pi\) be the probability that ordering \(\pi\) is chosen. Suppose that RSD is bossy, then there exists an agents \(i, t',\) types \(t_i, t_i'\), and \(t_{-i} \in T_{n-1}\) such that \(f(t_i, t_{-i}) = f(t_i', t_{-i})\), but \(f(t_i, t_{-i}) \neq f(t_i', t_{-i})\). For the sake of brevity, we write \(t\) and \(t'\) for \(t_i\) and \(t_i'\), respectively.

Let \(\text{Can}(t, t') = (t_0 = t, t_1, \ldots, t_{k-1}, t_k = t')\) be the canonical transition from \(t = t_i\) to \(t' = t_i'\). As in the proof of Lemma 1, the fact that the allocation is the same at the start and at the end of the transition implies that the allocation never changes during the transition, i.e., \(f(t_l) = f(t_{l+1})\) for all \(l \in \{0, \ldots, k-1\}\). Recall that under strategyproof mechanisms, the effect of swaps in the canonical transition is never undone by subsequent swaps and that swaps only effect the probabilities for adjacent objects (see Theorem 1 in Mennle and Seuken (2014b)). Let \(\text{Can}(t, t') = (t_0 = t, t_1, \ldots, t_{k-1}, t_k = t')\) be the canonical transition from \(t = t_i\) to \(t' = t_i'\). As in the proof of Lemma 1, the fact that the allocation is the same at the start and at the end of the transition implies that the allocation never changes during the transition, i.e., \(f(t_l) = f(t_{l+1})\) for all \(l \in \{0, \ldots, k-1\}\). Recall that under strategyproof mechanisms, the effect of swaps in the canonical transition is never undone by subsequent swaps and that swaps only effect the probabilities for adjacent objects (see Theorem 1 in Mennle and Seuken (2014b)).

But the allocation changed for agent \(i'\), hence it must have changed for agent \(i\) at some swap in the transition, say from \(t_i\) to \(t_{i+1} \in N_{t_i}\). Let \(j', j''\) be the objects that were swapped in this transition. Consider an ordering of the agents \(\pi\) with \(p_\pi > 0\). There are two cases.

- Agent \(i\) gets the same object under \(t_i\) as under \(t_{i+1}\). Then the swap had no effect on the allocation of any other agent, i.e., under \(\pi\) the swap does not change the allocation of the other agents.

- Agent \(i\) receives \(j'\) under \(t_i\), but \(j''\) under \(t_{i+1}\). Then the swap changes the allocation of the agent that received \(j''\) under \(t_i\). The magnitude of the change is \(-p_\pi < 0\). This agent can be \(i'\) by assumption.

However, the latter case is impossible, because this would also strictly increase agent \(i\)'s chances of receiving \(j''\) (by \(p_\pi > 0\)), implying \(f(t_i); \neq f(t_{i+1})\), a contradiction.

**B.9. Examples 3 and 4 from Section 8.2**

**Example 3.** Consider a setting \(N = \{1, \ldots, 3\}, M = \{a, b, c\}, q_a = q_b = q_c = 1\). For the type profile

\[
\begin{align*}
    t_1 & : a > b > c, \\
    t_2, t_3 & : c > a > b.
\end{align*}
\]
Suppose the rank valuation is $v = (v_1, v_2, v_3) = (10, 6, 0)$. Then RV will allocate $b$ to agent 1 with certainty. To see this suppose that agent 1 gets $a$ instead. Then some other agent $i$ received $b$. If agent 1 and agent $i$ trade, the objective increases by $6 - 10 + 6 - 0 = 2$. Now suppose that agent 1 gets $c$. Again some agent $i$ gets object $a$. If agent 1 and agent $i$ trade, this improves the objective by $10 - 0 + 6 - 10 = 6$. We have argued that agent 1 will get $b$ in any deterministic allocation chosen by RV with rank valuation $v$. Then by definition, agent 1 must get $b$ with certainty.

Suppose now that agent 1 reports $t'_1 : a > c > b$ instead, i.e., it swaps objects $b$ and $c$ in its report. Then under any rank efficient allocation (with respect to $(t'_1, t_{-1})$), agent 1 will receive object $a$. This is because whenever agent 1 gets another object in some deterministic allocation, the objective improves if agent 1 trades with the agent who received $a$ (independent of $v$). Since no rank efficient allocation will give agent 1 any other object than $a$, swapping $b$ and $c$ in its report is a beneficial manipulation for agent 1. This is independent of its actual utility, as long as the utility is consistent with $t_1$.

Now consider the outcome of RSD: it is easy to see that for any ordering of the agents, if agent 1 does not receive $a$ when it gets to choose, object $c$ will not be available. Therefore, $\text{RSD}_1(t_1, t_{-1}) = \text{RSD}_1(t'_1, t_{-1})$, i.e., RSD does not change the allocation. This means that RV with the specific choice of rank valuation $v$ is not weakly less varying than RSD, and agent 1 in the given situation would want to manipulate any non-trivial hybrid of RSD and RV.

Example 4. Consider a setting where $N = \{1, \ldots, 6\}$, $M = \{a, \ldots, f\}$, $q_j = 1$, and the type profile is

- $t_1, t_2 : a > b > c > d > e > f$
- $t_3, t_4, t_5, t_6 : c > b > f > d > a > e$.

Under RSD, the allocation for agent 1 is $(\frac{1}{2}, \frac{1}{10}, 0, \frac{7}{30}, \frac{1}{6}, 0)$ of objects $a$ through $f$, respectively. Swapping $c$ and $d$ in its report will not change the outcome for agent 1 under RSD. Under NBM and truthful reporting, the allocation for agent 1 is the same as under RSD. But if agent 1 changes its report by swapping $c$ and $d$, its allocation under NBM is $(\frac{1}{2}, \frac{1}{10}, 0, \frac{2}{5}, 0, 0)$ of objects $a$ through $f$, respectively, an allocation which it prefers in a first order-stochastic dominance sense.

**B.10. Theorem 3 from Section 8.3**

**Proof of Theorem 3.** ABM is weakly less varying than RSD.

Suppose the following manipulation by agent $i$ by a swap:

- $t_i : a_1 \succ \ldots \succ a_m > x \succ y > b_1 \succ \ldots \succ b_m$
- $\mapsto t'_i : a_1 \succ \ldots \succ a_m > y \succ x > b_1 \succ \ldots \succ b_m$. 

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By Lemma 3, RSD changes the allocation \( \text{RSD}_i(t_i, t_{-i}) \neq \text{RSD}_i(t'_i, t_{-i}) \) if and only if there exists an ordering of the agents \( \pi \) such that \( i \) gets to pick between \( x \) and \( y \) in its turn, but all objects \( a_1, \ldots, a_m \) are exhausted by higher-ranking agents. We show that if ABM changes the allocation, then such an ordering \( \pi \) exists. Thus, a change of allocation under ABM implies a change of allocation under RSD.

Suppose, the change of report by agent \( i \) from \( t_i \) to \( t'_i \) changes the outcome of ABM for \( i \), i.e., \( \text{ABM}_i(t_i, t_{-i}) \neq \text{ABM}_i(t'_i, t_{-i}) \). Then from the proof of swap consistency (Proposition 4 in Mennle and Seuken (2014a)) we know that there exists an ordering of the agents \( \pi' \) such that in some round (say \( L \)), \( i \) has not been allocated an object yet, all \( a_1, \ldots, a_m \) are exhausted, but neither \( x \) nor \( y \) are exhausted. Let \( r(i') \) be the round in which \( i' \) is allocated its object, and let

\[
R(r) = \{ i' \in N | r(i') = r \}
\]

be the set of agents who receive their allocation in round \( r \) (given ordering \( \pi' \)). If \( i' \) is allocated object \( j \) in round \( r \), \( i' \) has applied to \( j \) in that round. Thus, out of all the objects with capacity available at the beginning of round \( r \), \( i' \) must prefer \( j \). Facing the same set of choices under RSD, \( i' \) would also pick \( j \).

Consider an ordering \( \pi \) that ranks an agent \( i' \) before another agent \( i'' \) if \( r(i') < r(i'') \) and ranks them in arbitrary order if \( r(i') = r(i'') \). Additionally, let \( \pi \) rank \( i \) after all the agents in the set \( R(1) \cup \ldots \cup R(L-1) \). If RSD chooses \( \pi \) as the ordering of the agents, then all agents in \( R(1) \) receive their first choice (as under ABM). Next, all agents in \( R(2) \) face the choice sets out of which they most prefer the object they were allocated under ABM. This continues until finally \( i \) faces a choice set that includes none of the \( a_1, \ldots, a_m \), but both \( x \) and \( y \). Hence, \( \pi \) is the ordering we are looking for, and its existence concludes the proof. \( \Box \)