Clearing Payments in Financial Networks with Credit Default Swaps*

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Abstract

We consider the problem of clearing a system of interconnected banks that have been exposed to a shock on their assets. Eisenberg and Noe (2001) and Rogers and Veraart (2013) showed that when banks can only enter into debt contracts with each other, then there always exists a unique Pareto efficient clearing payment vector and it can be computed in polynomial time.

In the present paper, we show that the situation changes radically when banks can also enter into *credit default swaps* (*CDSs*). We first prove the surprising result that with CDSs, there may not even be a clearing vector at all. This implies that the value of a contract may not be well-defined. Furthermore, we prove that even determining whether a clearing vector exists is computationally infeasible in the worst case (NP-hard). We then develop a new analysis framework to derive constraints on the contract space under which these problems are alleviated. Our results can be used to inform the discussion on different policy proposals. We show that routing all contracts via a central counterparty would not even guarantee existence. In contrast, we show that banning "naked" (speculative) CDSs would re-establish an existence guarantee for a unique Pareto efficient clearing payment vector.

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JEL Classification: C62 (Existence and Stability Conditions of Equilibrium); G01 (Financial Crises); G12 (Asset Pricing)

1 Introduction

The 2008 financial crisis has revealed the tight interconnections among banks and the risk this can pose for financial markets. Specifically, we have seen that when a bank *defaults* (i.e., goes into bankruptcy), this also influences all other banks that had contracts with this bank. Having a reliable *clearing mechanism*, i.e., a way of determining who has to pay how much money to whom, is particularly important for such cases where a market participant defaults. Currently, the dominant clearing systems used in the US are CHIPS and Fedwire, while in Europe (under SEPA regime), these services are provided by EBA Clearing.

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A clearing mechanism computes a *clearing payment vector*, i.e., a vector of payments from each bank to each other bank that is in accordance with the standard bankruptcy regulations. The design of good clearing mechanisms is difficult because of the complex interdependencies in today's financial networks. The problem is particularly challenging because the dependencies between banks can be cyclical and because defaulting banks can typically only recover part of the value of their assets (i.e., they incur *default costs*).

In their seminal paper, Eisenberg and Noe (2001) devised an efficient clearing mechanism for financial systems without default costs, and Rogers and Veraart (2013) later provided a mechanism which also applies in situations *with* default costs. Both mechanisms rely on the assumption that banks can only enter into simple *debt contracts* (i.e., loans from one bank to another). We argue, however, that the growing importance of financial derivative contracts makes it necessary to reconsider the question if today's financial networks can still always be efficiently cleared. Specifically, *credit default swaps (CDSs)*, which are contracts that are only triggered when a reference entity goes into default, have received only little attention in a network context so far. Market participants use CDSs to insure themselves against a default of the reference entity or to place a speculative bet on this event. Because the reference entity can itself be a financial institution, CDSs create new dependencies that do not exist in pure debt networks.

Of course, when no banks default, computing clearing payments is trivial. But whenever a bank defaults in a networked system, computing clearing payments may be nontrivial and proper mechanisms are needed. While this problem does not show up regularly, it can have immense economic consequences when it *does* surface. One example is the 1997 East Asia crisis, when about 50% of a network of interconnected firms in Indonesia, South Korea, and Thailand defaulted. As then World Bank chief economist Joseph Stiglitz describes it, "you couldn't tell whether they were bankrupt or not, because that depended on whether they got paid money that was owed to them by other firms who might or might not be in default depending on whether the firms that owed them money went bankrupt" (Stiglitz, 2016). At the time, regulators were lacking a solid theory of clearing, and Stiglitz points out that this led to a paralysis (it took years to resolve it), resulting in large welfare losses because important measures could not be implemented quickly.

Fortunately, financial crises with one or multiple financial entities defaulting are still the exception rather than the norm. Thus, computing clearing payments is "usually" a simple problem. However, regulators such as the European Central Bank conduct *stress tests* to evaluate how likely certain banks are to default given a set of adverse economic scenarios. Future stress tests should take into account network effects (Battiston et al., 2016), which would require them to essentially compute clearing payments, thus facing the same challenges as in a financial crisis.

It is important to note that in any clearing mechanism, all contracts must be evaluated simultaneously. Otherwise, if the contracts between banks were evaluated sequentially, then there could be an incentive for some banks to take influence on the clearing system and change the evaluation order to their own advantage. In Section 4, we provide an example demonstrating this.

1.1 Desiderata for Financial Systems

In the design of financial systems that have good properties from a clearing perspective, we strive for the following three basic desiderata:

1. *Existence:* A clearing payment vector should always exist.
2. *Maximality:* There should be always a clearing vector that minimizes the "degree of being in default" of each individual bank. This is again to counter incentive problems: if the clearing mechanism traded off the well-being of one bank against

that of another, then the affected banks would have an incentive to influence the clearing mechanism in their favor.

3. *Computability:* There should be an algorithm that computes this clearing payment vector efficiently (i.e., in time polynomial in the number of banks) as even in a financial crisis (where multiple banks may default), authorities must be prepared to compute clearing payments within a few days.

Our overarching goal is to identify constraints on the space of possible contracts such that the financial system is guaranteed to satisfy these basic desiderata. Here, we would like to stress the importance of *ex-ante guarantees* regarding the ability to clear the market at all times. The mere possibility that there could be a situation in the future where the market cannot be cleared, or clearing payments cannot be computed quickly, could undermine trust of market participants and bring about a liquidity crisis in the present. Furthermore, a possible future incentive to manipulate could give the banks reason to take influence on regulatory institutions today.

1.2 Contributions

Eisenberg and Noe (2001) showed that when banks can only enter into debt contracts and defaulting banks do not incur any default costs, then the three desiderata are fulfilled: the authors proved that in this case, there is always a vector of clearing payments that uniformly maximizes the value of all claims and they also provided a polynomial-time algorithm to compute it. Rogers and Veraart (2013) extended Eisenberg and Noe's result to financial networks of debt contracts with default costs.

In this paper, we show that *none* of the three desiderata are fulfilled in general financial systems consisting of debt contracts and CDSs that have as reference entities other banks in the network. First, we prove that a clearing payment vector may not even exist, such that the first desideratum (and consequently all three desiderata) is violated (Section 4.1). The intuition for this is that with CDSs, a bank A can hold a "short" position on another bank B, i.e., A is better off if B is worse off. In a network, a closed chain of dependencies can give rise to a situation where a bank may find itself in a situation where it is holding a "short" position on itself, i.e., a bank A is better off if bank A is worse off, which is obviously a contradiction. In contrast, in a pure debt network, banks would only hold "long" positions on each other (if one bank is worse off, then the other is also worse off). That is why no such situation was observed in prior work.

Second, we prove that in the presence of CDS contracts, not only can there be multiple solutions as in the (Rogers and Veraart, 2013) model, but there can also be multiple solutions, none of which minimizes the "degree of being in default" of all banks at once, thus violating the second desideratum (Section 4.2). The intuition for this result is that CDS contracts can give rise to situations in which two banks happen to hold a "short" position on each other, i.e., one is better off if the other is worse off. In this case, a clearing payment vector in which the first bank is doing well and the second is doing poorly, and the opposite, are both possible. Again, since networks of debt obligations contain only "long" positions, this effect can only be observed in networks with CDS contracts. The origin of the multiplicity is therefore of a different nature from the one arising in the (Rogers and Veraart, 2013) model.

Third, for a given financial network with debt contracts and CDSs, we show that it is computationally infeasible (NP-hard) to determine if there exists a clearing payment vector (Section 5). Furthermore, even if one exists, it is still NP-hard to find it, which violates the third desideratum. The intuition for this result is that, while in pure debt networks, clearing payments can be found by solving a sequence of linear equation systems (see (Eisenberg and Noe, 2001) and (Rogers and Veraart, 2013)), networks of debt and CDSs essentially require solving arbitrary *polynomial* equation systems, which is a much harder problem.

Fourth, we study what kind of constraints on the contract space are sufficient to guarantee our desiderata. Towards this end, we introduce a formalism to analyze the dependencies between banks as a graph where edge colors reflect "long" and "short" positions. We give different conditions on the cycles in this graph under which some or all of our desiderata are fulfilled (Section 6).

Our results contribute to the debate on a possible regulation of the CDS market. We show that imposing the requirement that all contracts be routed through a central counterparty does not even guarantee existence of a clearing vector (Section 7). In contrast, when there are no "naked" CDSs (i.e., CDSs that are held without also holding a corresponding debt contract), then there always exists a maximal clearing vector, and we conjecture that it can also be found efficiently.

2 Related Work

Virtually all previous literature on financial networks has modeled them as some kind of edge-weighted graph, where edges represent debt obligations (Eisenberg and Noe, 2001; Rogers and Veraart, 2013; Cifuentes et al., 2005) or cross-ownership (Elliott et al., 2014; Vitali et al., 2011).

These models are typically used to analyze contagion, i.e., how small shocks may amplify to system-wide losses and which network topologies are particularly susceptible to such effects. Researchers have investigated trade-offs between (stabilizing) diversification and (destabilizing) contagion effects based on network completeness (Allen and Gale, 2000), portfolio diversification and integration (Elliott et al., 2014), segmentation (Cabrales et al., 2014), the magnitude of shocks (Acemoglu et al., 2015), as well as other factors such as liquidity (Roukny et al., 2013). Several measures for an individual bank's contribution to systemic risk have been proposed: a distance measure by Acemoglu et al. (2015) and two network algorithms: DebtRank by Battiston et al. (2012) and LASER by Hu et al. (2012).

There have also been negative results on the relevance of direct contagion: Glasserman and Young (2015) found bounds on the magnitude of network effects in the (Eisenberg and Noe, 2001) model. This contrasts with sensitivity results by Staum and Liu (2012) and Hemenway and Khanna (2015). Earlier, Elsinger et al. (2006) had seen that correlations between banks' asset portfolios can dominate direct contagion effects.

The present paper differs fundamentally from the above pieces of work in the nature of the financial relationships that are analyzed: while weighted-graph models can express the binary properties *debt* and *ownership*, they fail to fully capture the relationship between the *three* banks involved in a credit default swap: the holder, the writer, and the reference entity. In this work, we present a new model that can reflect this relationship accurately. Then a concern more fundamental than contagion arises: that we might not even *know* what the banks' financial situations are because there is no answer to that question, or there are multiple conflicting ones, or we cannot compute it.

The little theoretical work that has been done so far on networks of OTC derivatives contracts like credit default swaps (Duffie and Zhu, 2011; Atkeson et al., 2013; Puliga et al., 2014) has modeled CDSs as written on reference entities external to the financial system. However, in real markets, a considerable amount of CDSs *is* written on financial institutions.[1] A recent analysis of newly available transactions data by D'Errico et al. (2016) has further shown that reference entities are in fact tightly connected, implying the presence of cyclic writer–holder–reference entity relations that will be discussed below.

To the best of our knowledge, we are the first to provide an analytically tractable model for clearing in financial networks with credit default swaps where reference entities

[1] A report by the BIS (2015) shows that the total notional of single-name CDS instruments on financial firms was about USD 1.8 trillion in the first half of 2015.

are themselves banks in the network. While Heise and Kühn (2012) considered such networks, their model does not seem to lend itself to analytical examination; the authors obtain results on contagion by simulating a fixed number of time steps, but it is not explored whether or not their process would converge to clearing payments.

A field that has only developed recently is the application of computational complexity theory to financial markets. Arora et al. (2010) and Zuckerman (2011) investigated the cost of asymmetric information in financial derivatives markets with computationally bounded agents. Braverman and Pasricha (2014) provided computational hardness results on fair pricing of compound options. Hemenway and Khanna (2015) showed that in Elliott et al.'s model, it is computationally infeasible to determine the worst-case impact of a given total negative shock. In contrast, we prove that in financial networks with CDSs, a much simpler problem is already computationally infeasible, namely the problem to determine if there is a clearing payment vector when shocks are given.

3 Formal Model

In this section, we introduce our formal model. It is based on the model by Eisenberg and Noe (2001) and its extension to default costs by Rogers and Veraart (2013). Both of these prior models were restricted to debt contracts. We define an extension to credit default swaps. We adjust the notation where necessary. A detailed comparison to the two previous models can be found in Appendix A.

Consider a two-period model:

- *Period 0:* Each bank receives an initial endowment called its *external assets*. Banks enter into bilateral contracts with each other. No bank is in default.
- *Period 1:* Banks' external assets change due to an exogenous shock. All banks make payments according to their contractual commitments from period 0 and the new external assets.

We define the elements of the financial system in period 1.

Banks and external assets. We denote by N a finite set of n *banks*. For any bank $i \in N$ let $e_i \geq 0$ denote the *external assets* of i as of period 1. Let $e = (e_i)_{i \in N}$ denote the vector of all external assets. We only model the post-shock values of the external assets here since the pre-shock values will not be relevant for our analysis.

Contracts. There are two types of contracts: *debt contracts* and *credit default swap contracts (CDSs)*. Every contract gives rise to a conditional obligation to pay a certain amount, called a *liability*, from its *writer* to its *holder*. Banks that are unable to fulfill this obligation are said to be *in default*. The *recovery rate* r_i of a bank i is the share of its liabilities it is able to pay. Thus, $r_i = 1$ if i is not in default and $r_i < 1$ if i is in default. Let $r = (r_i)_{i \in N}$ denote the vector of all recovery rates.

A *debt contract* obliges the writer i to unconditionally pay a certain amount to the holder j in period 1. This amount is called the *notional* of the contract and is denoted by $c^{\emptyset}_{i,j}$. A *credit default swap* obliges the writer i to make a conditional payment to the holder j in period 1. The amount of this payment depends on the default of a third bank k, called the *reference entity*. Specifically, the payment amount of the CDS contract from i to j with reference entity k and *notional* $c^k_{i,j}$ is $c^k_{i,j} \cdot (1 - r_k)$.

Note that when banks enter contracts, there would typically be an initial payment. For example, debt contracts arise because the holder lends an amount of money to the writer, and holders of CDSs pay a premium to obtain them. In our model, any initial payments have been made in period 0 and are implicitly reflected by the external assets.

The contractual relationships between all the banks are represented by a 3-dimensional matrix $c = (c^k_{i,j})_{i \in N,\, j \in N,\, k \in N \cup \{\emptyset\}}$. The entry $c^{\emptyset}_{i,j}$ is the total notional of the debt contracts

from i to j and the entry $c^k_{i,j}$ for $k \in N$ is the total notional of CDS contracts from i to j with reference entity k. Zero entries indicate the absence of the respective contract.

We require that no bank enters a contract with itself (i.e., $c^k_{i,i} = 0$ for all $k \in N \cup \{\emptyset\}$ and $i \in N$). We further require that any bank that is a reference entity in a CDS must be a writer of a debt contract (i.e., if $\sum_{k,l \in N} c^i_{k,l} > 0$, then $\sum_{j \in N} c^\emptyset_{i,j} > 0$ for all $i \in N$). Both requirements are needed to rule out pathological cases.They are always assumed to hold in the following.

For any bank i, the *creditors of* i are the banks that are holders of contracts for which i is the writer, i.e., the banks to which i owes money. Conversely, the *debtors of* i are the writers of contracts of which i is the holder, i.e., the banks by which i is owed money. Note that the two sets can overlap: for example, a bank could hold a CDS on one reference entity while writing a CDS on another reference entity, both with the same counterparty.

Default Costs. We model default costs following (Rogers and Veraart, 2013): there are two *default cost parameters* $\alpha, \beta \in [0, 1]$. Defaulting banks are only able to pay to their creditors a share of α of their external assets and a share of β of their incoming payments. Thus, $\alpha = \beta = 1$ means that there are no default costs and $\alpha = \beta = 0$ means that assets held by defaulting banks are worthless. The values $1 - \alpha$ and $1 - \beta$ are the *default costs*. Default costs could be legal or administrative costs or operational losses that a bank incurs upon default. Another important source of default costs are fire sales: defaulting banks might be forced to sell a large portfolio of assets in a market of limited liquidity, thus having to accept a lower price. Details can be found in (Rogers and Veraart, 2013).

We assume default costs to be the same across all banks for simplicity of the exposition; however, our model as well as our results easily generalize to per-bank default cost parameters α_i and β_i for $i \in N$ with minor adjustments.

Financial System. A *financial system* is a tuple (N, e, c, α, β) where N is a set of banks, e is a vector of external assets, c is a 3-dimensional matrix of contracts, and α and β are default cost parameters.

Liabilities, Payments, and Assets. For two banks i, j and a vector of recovery rates r, the *liabilities of* i *to* j *at* r are the amount of money that i has to pay to j if recovery rates in the financial system are given by r, denoted by $l_{i,j}(r)$. They arise from the aggregate of all debt- and CDS contracts from i to j.

$$l_{i,j}(r) := c^\emptyset_{i,j} + \sum_{k \in N} (1 - r_k) \cdot c^k_{i,j}$$

The *total liabilities of* i *at* r are the aggregate of all liabilities that i has toward other banks given the recovery rates r, denoted by $l_i(r)$.

$$l_i(r) := \sum_{j \in N} l_{i,j}(r)$$

The actual *payment* $p_{i,j}(r)$ from i to j at r can be lower than $l_{i,j}(r)$ if i is in default. By the *principle of proportionality* (discussed below), a bank that is in default makes payments for its contracts in proportion to the respective liabilities.

$$p_{i,j}(r) := r_i \cdot l_{i,j}(r)$$

The *total assets* $a_i(r)$ of a bank i at r consist of its external assets e_i and the incoming payments to i.

$$a_i(r) := e_i + \sum_{j \in N} p_{j,i}(r)$$

Clearing Recovery Rate Vector. Following Eisenberg and Noe (2001), we call a recovery rate vector r *clearing* if the payments $p_{i,j}(r)$ conform with the following three principles of standard bankruptcy law:

1. *Absolute Priority:* Banks with sufficient assets pay their liabilities in full.
2. *Limited Liability:* Banks with insufficient assets to pay their liabilities are in default and pay all of their assets to creditors after default costs have been subtracted.
3. *Proportionality:* In case of default, payments to creditors are made in proportion to the respective liability.

The principle of proportionality is automatically fulfilled by the definition of the payments $p_{i,j}(r)$. The other two principles lead to the following formal definition.

Definition 3.1 (Clearing Recovery Rate Vector)**.** A recovery rate vector r is called *clearing* for a financial system $X = (N, e, c, \alpha, \beta)$ if for all banks $i \in N$ we have

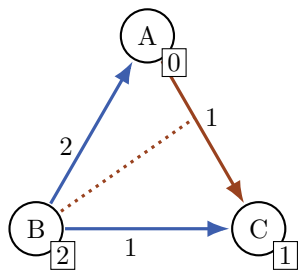
$$\sum_{j \in N} p_{i,j}(r) = \begin{cases} l_i(r) & \text{if } a_i(r) \geq l_i(r) \\ \alpha e_i + \beta \sum_{j \in N} p_{j,i}(r) & \text{if } a_i(r) < l_i(r). \end{cases} \tag{1}$$

We also call a clearing recovery rate vector a *solution*.

Remark 3.2 (Clearing Payments and Recovery Rates)*.* It is an equivalent point of view whether one considers clearing *payments* or clearing *recovery rates*. The previous definition is equivalent to requiring that the recovery rates of banks with positive liabilities are 1 in case they are not in default and $(\alpha e_i + \beta \sum_j p_{j,i}(r))/l_i(r)$ in case they are in default. The recovery rates of banks with zero liabilities are left unconstrained between 0 and 1. While this might seem unintuitive at first, it corresponds exactly to the definition of the recovery rate: r_i is the share of its total liabilities that bank i can pay. If there are no liabilities, then this value is not well-defined. Forcing the recovery rate to 1 in these cases would introduce an
artificial discontinuity.

Example and Visual Language. Figure 1 shows a visual representation of an example financial system. There are three banks $N = \{A, B, C\}$, drawn as circles, with external assets of $e_A = 0$, $e_B = 2$, and $e_C = 1$, drawn as rectangles on the banks. Debt contracts are drawn as blue arrows from the writer to the holder and they are annotated with the notionals $c^{\emptyset}_{B,A} = 2$ and $c^{\emptyset}_{B,C} = 1$. CDS contracts are drawn as orange arrows where a dashed line connects to the reference entity and are also annotated with notionals: $c^{B}_{A,C} = 1$. Default costs $\alpha = \beta = 0.5$ are given in addition to the picture.

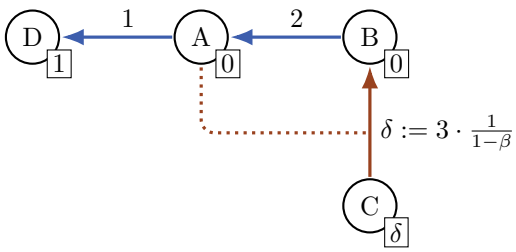
Figure 1 Example financial system. $\alpha = \beta = 0.5$



A clearing recovery rate vector for this example is given by $r_A = 1$, $r_B = \frac{1}{3}$, and $r_C = 1$. The liabilities arising from this recovery rate vector are $l_{B,A}(r) = 2$, $l_{B,C}(r) = 1$, and $l_{A,C} = \frac{2}{3}$. The clearing payments are $p_{B,A}(r) = \frac{2}{3}$, $p_{B,C} = \frac{1}{3}$, and $p_{A,C}(r) = \frac{2}{3}$. This is the only solution for this system.

We stress that we are not concerned with the question whether or not it is "rational" for the banks to form a certain financial system: contracts might have been entered for reasons exogenous to the system, or simply for cash transfers at time 0.

Figure 2 "Russell's Bank": Financial system with no solution for $\beta < 1$



4 Existence of Clearing Recovery Rates

In this section we explore the range of possible shapes of the set of solutions of a financial system with CDSs. An overview table is given at the end of the section.

4.1 No Solution

Whenever there are default costs, we can find a financial system where there are no clearing recovery rates, so that the first desideratum, existence, is not satisfied in general systems of debt and CDSs. We know from Rogers and Veraart (2013) that no such example can be constructed using only debt contracts. It also cannot be constructed using only CDSs because in a financial system consisting *only* of CDS contracts, the recovery rate vector $(1, \dots, 1)$ (nobody defaults) is always clearing: under this recovery rate vector, no liabilities arise and since no bank has any liabilities, really no bank defaults.

If there are any default costs, then we can construct a simple example for a financial system that does not have a solution, which we named after philosopher Bertrand Russell and the paradox in naive set theory also bearing his name.

Theorem 4.1 (No Solution with Default Costs: "Russell's Bank")**.** *For any pair of* (α, β) *with* $\alpha < 1$ *or* $\beta < 1$ *there exists a financial system* (N, e, c, α, β) *that has no clearing recovery rate vector.*

Proof outline (full proof in Appendix B). We distinguish two cases, $\beta < 1$ and $\alpha < \beta = 1$. For $\beta < 1$, consider Figure 2. We assume towards a contradiction that there is a clearing r. We first show that due to default costs, r_A can only fall into one of two cases: it can either have a low value (below the threshold β) or be equal to 1. r_A cannot be 1 because then B has assets 0, thus the payment from B to A is 0 and $r_A = 0$, a contradiction. But r_A can neither be low because then the payment from C to B is high, so the payment from B to A is high, so r_A is actually high. So in total, there is no possible value for r_A.

For $\alpha < \beta = 1$, we construct a variant of Figure 2 where A has positive external assets and the threshold between a low and a high value is given by the default cost parameter α. □

Remember that we require payments to be simultaneous so that contracts are evaluated at a single moment in time. In a situation where there is no solution, one might be tempted to compute *some* result by letting go of this principle and evaluating contracts in some order. However, payments can be different for different orders of evaluation, which introduces incentives for the banks to exert influence on the clearing system to change that order in their favor, as illustrated in the following example.

Example 4.2 (Incentives to Influence the Order of Evaluation without Simultaneity). Consider Figure 2 again. We show that when contracts are evaluated in some order, then the resulting payments to the banks depend on the chosen order. We need to assume that the creditors of defaulting banks become the new owners, thus receiving all incoming

payments of the defaulting bank, after default costs have been subtracted. This detail does not matter when we consider clearing payments, but it does matter here, as we will see. Assume that β is smaller than, but close to 1. We describe two possible orders of evaluation.

a) The debt contract from A to D is evaluated first. A, not yet having received a payment from B, defaults and D becomes the new owner of A. The CDS from C to B is triggered and B receives a payment of at least 3 so that it can pay its liability to $A = D$ in full. After default costs of A have been subtracted, D receives a payment of $\beta \cdot 2$. Only A is in default. Note that the profit D makes from the default of A is higher than the original liability from A.

b) The debt contract from B to A is evaluated first. B, not yet having received a payment from C, defaults and A becomes the new owner of B. When now the debt contract from $A = B$ to D is evaluated, $A = B$, still not having received any payments, defaults, and D becomes the new owner of $A = B$. The CDS is triggered and after default costs of A and B have been subtracted, D receives a payment of at least $\beta^2 \cdot 3 > \beta \cdot 2$. A and B are in default.

Thus, if evaluation is not simultaneous, but defaults are decided in some order by a committee or computed by a naive algorithm, then it is rational for bank B to lobby for an early evaluation of the debt from A, but it is rational for bank D to lobby against that and for an early evaluation of the debt from B.

The precondition of Theorem 4.1 is tight in the sense that any financial system with no default costs has a solution:

Theorem 4.3 (Existence of a Solution without Default Costs)**.** *Any financial system* $(N, e, c, 1, 1)$ *has a clearing recovery rate vector.*

Proof outline (full proof in Appendix B). Under the assumption of $\alpha = \beta = 1$, the right-hand side of equation (1) in Definition 3.1 is a continuous function of r. We use a formal version of Remark 3.2 to represent the clearing recovery rate vectors as fixed points of a certain set-valued function. We then apply the Kakutani fixed point theorem to obtain the result. □

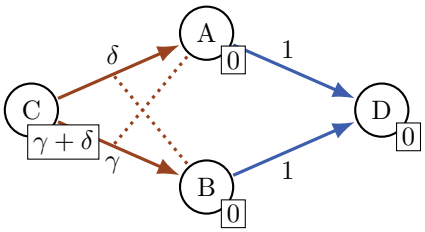
The theorem shows that financial systems that are completely "frictionless" in the sense that money is never lost, just redistributed, always have a solution. However, this does not imply that we also know what these solutions look like: the fixed point theorem we use is infamous for being non-constructive; it does not give rise to an algorithm to find the solution that we know exists. In fact, we show in a separate piece of work (Schuldenzucker et al., 2016a) that it is computationally infeasible in the worst case (PPAD-hard) to compute an (approximate) solution for a given financial system without default cost with high accuracy.

4.2 Multiple Solutions

In the previous section we have seen that as soon as default costs are not zero, a system with no solution may arise. But even when we have a solution (e.g., in the absence of default costs), there could be multiple ones and the set of solutions might have a structure that is economically desirable or not. To talk about the structure of solutions, we define a way to compare them:

Definition 4.4 (Dominance, Pareto Efficiency and Maximality)**.** Fix a financial system $X = (N, e, c, \alpha, \beta)$. A solution r *dominates* another solution r' (write $r \geq r'$) if $r_i \geq r'_i$ for all $i \in N$. r is called *Pareto efficient* if there is no other solution that dominates it. r is called *maximal* or a *maximum* if it dominates all solutions. In particular, then r is the

Figure 3 Financial system with multiple Pareto efficient solutions dependent on the choice of γ and δ



unique Pareto efficient solution. r is called *minimal* or a *minimum* if it is dominated by all solutions.

Our second desideratum, maximality, requires that a maximal solution exists. The following theorem gives a particularly severe example showing that maximality is not satisfied in general financial systems with debt and CDSs.

Theorem 4.5 (0-1 System)**.** *For any $\alpha \in [0,1]$ and $\beta \in [0,1]$ there exists a financial system (N, e, c, α, β) with four banks that has exactly two clearing recovery rate vectors $r^0 = (0,1,1,1)$ and $r^1 = (1,0,1,1)$.*

Proof outline (full proof in Appendix B). We use the financial system in Figure 3 with $\gamma = 1$ and a sufficiently large δ dependent on β. It is easily verified that r^0 and r^1 are indeed solutions of this system: if $r_A = 0$, then the CDS held by B pays 1, so $r_B = 1$, so the CDS to A pays 0 and then indeed $r_A = 0$. Likewise for $r_A = 1$. To see that no other solutions exist, we use a threshold argument similar to the proof of Theorem 4.1. □

To see why the solution structure in the previous theorem is economically undesirable, imagine a central authority faced with the problem of actually clearing the market: there are two solutions, one where A defaults (with $r_A = 0$) and B does not and one where B defaults (with $r_B = 0$) and A does not. It would not be clear which of the solutions to choose. Similarly to the case where there is no solution, this scenario could lead to severe incentive problems if timing were used. In today's financial practice, whether or not a CDS should be triggered is decided by private so-called *determinations committees* consisting of the most active participants in the CDS markets (ISDA, 2012). In Figure 3, it would be rational for A and B to compete for influence in the determination committee of the respective other bank, and whoever convinces their committee first that the other one is in a bad financial situation will indeed be right in hindsight.

Mathematically speaking, the problem we have observed here is that the two recovery rate vectors r^0 and r^1 are clearing, but their point-wise maximum $(1,1,1,1)$ is not. In financial systems where a maximal clearing recovery rate vector exists, this maximum can be chosen without worrying about putting one bank at a disadvantage and the resulting incentive problems.

Unfortunately, general systems with CDSs can be particularly ill-behaved beyond Theorem 4.5:

Theorem 4.6 (Non-Closed Solution Set)**.** *There exists a financial system such that the set of Pareto efficient clearing recovery rate vectors is a non-closed set of continuum size.*

Proof outline (full proof in Appendix B). We use a variant of Figure 3 with $\gamma = \delta = \frac{1}{\beta}$ for $\frac{1}{2} < \beta < 1$ and α arbitrary. We show that the set of solutions is

$$\{(r_A, 1 - r_A, 1, 1) \mid r_A \in \{0\} \cup (1-\beta, \beta) \cup \{1\}\}$$

by a case distinction similarly to the proof of Theorem 4.5. All solutions are Pareto efficient. □

Figure 4 Different possible cases for the set of solutions depending on the existence of CDSs (rows) and default costs (columns)

	No Default Costs	**Default Costs**
Debt only	Unique solution Trivial multiple solutions with maximum *(Eisenberg and Noe, 2001)*	Unique solution Multiple solutions with maximum *(Rogers and Veraart, 2013)*
Debt + CDSs		No solution
	Unique solution Multiple solutions with maximum Multiple solution without maximum *(this paper)*	

Using techniques from Section 5 below, it is also possible to extend the construction so that the possible recovery rates for r_A are exactly the open interval $(1-\beta, \beta)$.

Having a set of Pareto efficient solutions of continuum size means that it is impossible to enumerate all Pareto optima: they can only be represented as the solutions to certain polynomial equations. In the (stylized) case in the proof of Theorem 4.6, the respective equations are linear, but that cannot in general be assumed. Non-closedness implies that constrained optimization on the space of Pareto efficient solutions is not in general possible. In the above example, we might want to maximize the recovery rate of A while requiring that the recovery rate of all banks be above 0. However, that maximum does not exist. We leave as a question to future research if it is possible to have a financial system with a continuum of solutions, *none* of which are Pareto efficient. In such a case, any solution would leave room for Pareto improvement, which might put the clearing authority into legal difficulties in case they committed to always select a Pareto optimum.

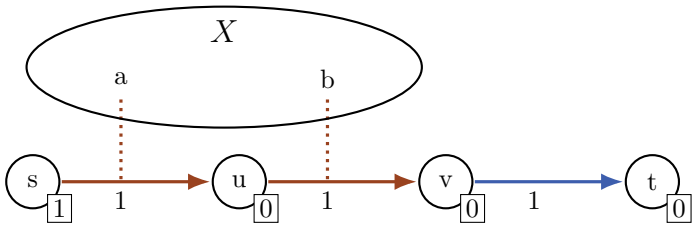
In contrast to the previous theorems, financial systems without CDSs always have a very well-behaved set of solutions: Rogers and Veraart (2013) have shown that in this case, there always exist a maximal and a minimal solution and they can be computed in polynomial time. Eisenberg and Noe (2001) have shown that if in addition $\alpha = \beta = 1$ and every bank is reachable by a path of debt from a bank with positive external assets, then there even is a unique solution. Without the path condition, the possible recovery rate vectors only differ on cycles of total external assets 0 if $\alpha = \beta = 1$.

Figure 4 gives an overview of the results obtained in this section. We remark again that while Rogers and Veraart (2013) and others observed multiple solutions due to default costs, the much more severe effects of having no solution or no maximal solution do not exist when all contracts are assumed to be simple debt. The figure lists all we can say about financial systems with an unconstrained contract space. We obtain stronger characterizations for systems where the space of possible contracts has been restricted in some way, but more sophisticated techniques are needed. We will come back to this topic in Section 6.

5 Computational Complexity

Given a financial system, we are interested in the algorithmic problem of distinguishing between the different cases in Figure 4. Specifically, to decide whether a given financial system has a solution or not. We will show that this problem is NP-hard.

Figure 5 Existing financial system X with input banks a and b and new banks s, t, u, v and contracts that translate a NOR gate. We have $r_v = r_a \,\mathrm{NOR}\, r_b$ whenever r is clearing.



We show NP-hardness via reduction from the Circuit Satisfiability problem.[2] We proceed in three steps:

1. We show how a single logic gate can be translated into a financial system.
2. We combine copies of these financial gates to translate a whole boolean circuit into a financial system.
3. We use this translation to show how, assuming that we have an efficient algorithm to decide if a financial system has a solution, we could obtain an efficient algorithm to solve Circuit Satisfiability.

We start by defining the problem we use for reduction:

Definition 5.1 (Circuit Satisfiability)**.** *Circuit Satisfiability* is the following decision problem: given a boolean circuit, decide if there exists an assignment of inputs such that the output is 1.

It is well known that Circuit Satisfiability is NP-complete (Papadimitriou, 1995). We can assume WLOG that the involved boolean circuits are constructed entirely of NOR gates, defined by $a \,\mathrm{NOR}\, b = \neg(a \vee b)$. That is because NOR forms a basis of propositional logic: $\neg a = a \,\mathrm{NOR}\, a$, $a \vee b = \neg(a \,\mathrm{NOR}\, b)$, and $a \wedge b = \neg(a \vee b)$. To translate a boolean circuit into a financial system, we need thus only be able to translate its basic building block, a NOR gate. We then combine these gates to larger circuits in a second step.

The translation of a NOR gate cannot be described as a single financial system because financial systems lack the notion of an "input". Instead, the following lemma shows how to *extend* an *existing* financial system with two *input banks* a and b by adding four new banks in such a way that the recovery rate of one of the new banks is always exactly $a \,\mathrm{NOR}\, b$. This extension is illustrated in Figure 5.

Lemma 5.2 (Financial NOR Gate)**.** *Let $X = (N, e, c, \alpha, \beta)$ be a financial system and let $a, b \in N$ be banks (not necessarily different). Assume that for all r clearing for X and all $i \in N$ we have $r_i \in \{0, 1\}$. Then there exists an extension X' obtained from X by adding four banks s, t, u, v such that*

1. *If r is clearing for X, then there exists r' clearing for X' such that $r'|_N = r$.*
2. *If r' is clearing for X', then $r'|_N$ is clearing for X and $r'_v = r'_a \,\mathrm{NOR}\, r'_b$.*

Here, $r'|_N$ is the vector r' with only the indices from N, i.e., without the indices s, t, u, v.

Proof. Let X' result from X by adding the banks and contracts in Figure 5. It is clear that the construction does not change the solution structure of the sub-system X of X'

[2] Recap that a boolean circuit is an abstract description of a digital logic circuit. An acyclic graph structure connects the inputs of the circuit and a set of logic gates, encoding boolean functions such as AND, OR, or NOT. A single node is marked as the *output*. The inputs receive an *assignment* (a tuple of 0s and 1s) and the value of the output is computed by evaluating all gates. The *size* of a boolean circuit is the number of gates plus the number of inputs.

because X is not affected by the recovery rates of the new banks s, t, u, v and the subsystem consisting of the new banks always has a solution. Hence, solutions of X can be extended to solutions of X', proving statement 1.

If r' is clearing for X', then r'_a and r'_b can only be 0 or 1 by assumption. If $r_a = 0$ and $r_b = 0$, then v has assets of 1 and thus a recovery rate of 1. Otherwise, the flow of money from s is "blocked" before either u or v and v has assets of 0 and thus a recovery rate of 0. This corresponds exactly to the definition of the NOR operation, proving statement 2. □

Remark 5.3 (Translating Logic Gates). We can express any logic gate as a financial system by first transforming it into a circuit of NOR gates and then repeatedly applying the previous lemma. Often however, more direct encodings can be found. For example, we can replace bank u in Figure 5 by a CDS on a directly from s to v to encode a $\neg r_a$ gate.

We now show how to translate a complete boolean circuit.

Lemma 5.4 (Financial Boolean Circuit)**.** *Let C be a boolean circuit with m inputs. For $\chi \in \{0,1\}^m$ write $C(\chi)$ for the value of the output of C given values χ of the inputs.*

For any $\alpha \in [0,1]$ and $\beta \in [0,1]$ there exists a financial system $X = (N, e, c, \alpha, \beta)$ of size linear in the size of C with distinguished input banks *$a_1, \dots, a_m \in N$ and a distinguished* output bank *$v \in N$ such that the following hold:*

1. *For any assignment $\chi \in \{0,1\}^m$ there exists a clearing recovery rate vector r for X such that $r_{a_i} = \chi_i$ for $i = 1, \dots, m$.*
2. *If r is clearing for X, then $r_i \in \{0,1\}$ for any bank $i \in N$.*
3. *If r is clearing for X, then $r_v = C(r_{a_1}, \dots, r_{a_n})$.*

Proof outline (full proof in Appendix C). WLOG we can assume that all gates are NOR gates. Boolean circuits are acyclic, so we can proceed in topological order, starting at the inputs and ending at the output. We translate inputs into a copy of the 0-1 system from Theorem 4.5 and we translate NOR gates by applying Lemma 5.2 to the part of the system that has been constructed so far. □

We can now prove our hardness result. We consider the problem of determining if a given financial system has a solution. Recap from Theorem 4.3 that financial systems with $\alpha = \beta = 1$ always have a solution, so this decision problem only makes sense for $\alpha < 1$ or $\beta < 1$.

Theorem 5.5 (NP-Hardness of Determining Existence)**.** *The following problem is NP-hard: Given a financial system (N, e, c, α, β) with $\alpha < 1$ or $\beta < 1$, decide if it has a clearing recovery rate vector.*

Proof outline (full proof in Appendix C). Reduction from Circuit Satisfiability. Given a boolean circuit C, we construct the translation into a financial system as of Lemma 5.4. On top of that, we add a copy of the "Russell's Bank" system from Theorem 4.1 (no solution), but with one of the debt contracts replaced by a CDS on the output bank of the circuit translation.

If now a falsifying assignment is supplied to the input banks, then the output bank has recovery rate 0, so the "no solution" system is active, and thus no such recovery rate vector can be clearing. In effect, the only possible clearing recovery rate vectors correspond to satisfying assignments of the circuit C, where the output bank has recovery rate 1 and one contract of the "no solution" system drops out, in which case it *has* a solution. Thus, determining whether there is a clearing recovery rate vector in this financial system is equivalent to determining whether there is a satisfying assignment for C, which is NP-hard. □

The previous theorem has an important implication for network stress tests: having seen the danger that a system with no solution brings, regulators may want to simulate an array of different scenarios and test each of them for whether it has a solution. They could then estimate the probability of having a solution and they would know which scenarios are particularly problematic and should therefore be prevented at all cost. However, Theorem 5.5 shows that this is computationally infeasible since checking whether a solution exists is NP-hard. We provide additional hardness results on related decision problems for individual banks in Appendix C.1.

Remark 5.6 (Complexity of the search problem, $\alpha < 1$ or $\beta < 1$). In practice, we are interested in the problem of finding a solution, given that one exists. Since solutions can be irrational (cf. Example C.4 in the appendix), this needs to be considered as an approximation problem. NP-hardness of this variant can be shown by generalizing the techniques from this section to approximate solutions: it is easy to see that our results still hold when recovery rates can deviate from 0 or 1 by a not too large ε.

Remark 5.7 (Complexity of the search problem, $\alpha = \beta = 1$). For $\alpha = \beta = 1$, the construction in Lemma 5.2 can be applied to systems X that do not necessarily fulfill the condition that all recovery rates are 0 or 1. One then receives a continuous version of the result where $r'_v = 1 - (1 - r'_a)(1 - r'_b)$. By connecting several of these "continuous NOR" systems, we receive financial systems that encode multiplication (a "continuous AND"), addition ("continuous OR"), and eventually one can build up financial systems that encode arbitrary polynomials with variables in $[0, 1]$. This observation illustrates another aspect of the additional complexity that CDSs introduce: while debt systems give rise to essentially *linear* equation systems, financial systems with CDSs can give rise to *polynomial* equation systems of arbitrary degree.

We remark however that this is not the source of the computational complexity of the search problem described in (Schuldenzucker et al., 2016a): here we show that the problem is PPAD-hard even when all CDS writers are highly capitalized, in which case the corresponding equations are of degree 1, but potentially non-monotonic.

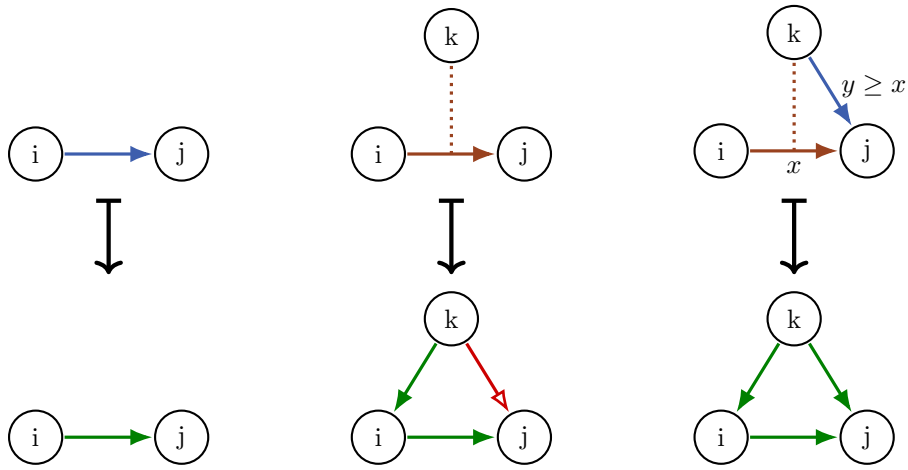
6 Restricted Contract Spaces

The previous two sections have shown that our three desiderata are not fulfilled in general financial systems with CDSs: there are systems that have no solution, or that have them, but without a maximal solution, so the clearing system would need to make trade-offs among different banks, and even the question if there *is* a solution is NP-hard to answer. In this section, we describe sub-classes of financial systems with CDSs that *do* fulfill some or all of our desiderata. We do this by analyzing the structure of dependencies arising from the contracts.

6.1 Dependency Analysis

There are two qualitatively different kinds of dependencies in financial systems with CDSs: "long" and "short" positions. The holder of a contract is always "long" on the writer of the contract, meaning that if the writer is doing worse, then also the holder might be doing worse in case the writer defaults. The writer of a CDS is also "long" on the reference entity: in case the reference entity is doing worse and goes into default, a liability arises for the writer. In contrast, the holder of a CDS is "short" on the reference entity: in case the reference entity is doing worse, the holder would receive the CDS payment and make a profit. Figures 6a and 6b illustrate our considerations graphically: we draw a green edge with a filled arrow head to indicate a "long" position and a red edge with an empty arrow head to indicate a "short" position. Edges are in direction of effect, i.e., if j is "long" or "short" on i, then there is an edge from i to j.

Figure 6 Different financial systems (first row) and their colored dependency graphs (second row). Green edges with filled arrow heads indicate "long" positions while red edges with empty arrow heads indicate "short" positions.



(a) Debt contract **(b)** Naked CDS **(c)** Covered CDS

One exception from the above rule needs to be made for the "short" relationship illustrated in Figure 6c: if the holder j of the CDS also holds a corresponding amount of debt written by the reference entity k, then it is not actually a short position: In case k fails to honor its liability, the gap between the debt notional and the actual payment is offset by a payment in the CDS; only if both k and i are in default, j receives less than the notional. In no case, j would make a profit from the ill-being of i. These uses of CDSs are called *covered* while the others are called *naked*. In general systems, we need to take into account the total notional of *all* CDSs a bank holds on a reference entity.

Definition 6.1 (Covered and Naked CDS Position)**.** Let $X = (N, e, c, \alpha, \beta)$ be a financial system. A bank j is said to have a *covered CDS position* towards another bank k if the total notional of CDSs j is holding on k does not exceed the notional of debt j is holding from k. Formally, j has a covered CDS position towards k if

$$\sum_{i \in N} c^k_{i,j} \leq c^\emptyset_{k,j}.$$

j is said to have a *naked CDS position* towards k otherwise.

The *colored dependency graph* of a financial system captures the above discussion on "long" and "short" positions formally.

Definition 6.2 (Colored Dependency Graph)**.** Let $X = (N, e, c, \alpha, \beta)$ be a financial system. The *colored dependency graph* $\mathrm{CD}(X)$ is the graph with nodes N and edges of colors red and green constructed as follows for each two banks $i, j \in N$:

1. if $c^k_{i,j} > 0$ for any $k \in N \cup \{\emptyset\}$ (j is a creditor of i), then add a green edge $i \to j$.
2. if $c^j_{i,k} > 0$ for any $k \in N$ (j is writer of a CDS on i), then add a green edge $i \to j$.
3. if j has a naked CDS position towards i, then add a red edge $i \to j$.

We call the first kind of edges *liability edges* (as they go from the debtor to the creditor of a contract, i.e., in direction of liabilities) and the second and third kind *reference edges* (as they go from a reference entity to one of the two counterparties of a CDS).

Note that the colored dependency graph can have parallel edges. This happens, for example, when a bank holds a naked CDS (red edge), but also holds a smaller amount of debt from the reference entity (green edge). These edges do not "net out"! Rather, a "long" or a "short" effect could be present, depending on the solvency of the CDS writer.

The colored dependency graph of a financial system containing only debt contracts has a (green) liability edge $i \to j$ whenever $c^{\emptyset}_{i,j} > 0$ and otherwise no edges. This structure was introduced by Eisenberg and Noe (2001) under the name "financial system graph". CDSs introduce new kinds of dependencies and we need the more sophisticated colored dependency graph to capture them.

6.2 Acyclic Financial Systems

The following theorem shows that if there are no cycles in the colored dependency graph, then all three desiderata are fulfilled.

Theorem 6.3 (Acyclic Financial Systems)**.** *Let X be a financial system such that* $\mathrm{CD}(X)$ *has no cycles. Then X has a clearing recovery rate vector, it is unique up to banks with no liabilities, and it can be found in polynomial time.*

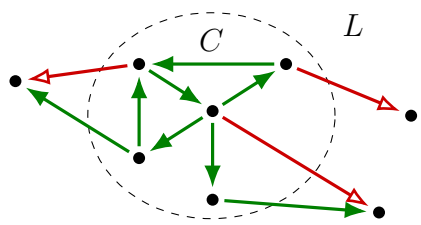
Proof outline (full proof in Appendix D). Our algorithm proceeds in *topological order* of dependencies. That is, we sort the banks into an order where dependencies only go from earlier to later banks, but never in the other direction. This is possible by assumption. We then iterate over the banks in this order, starting at a bank without incoming dependencies. For each bank i, we compute the total assets $a_i(r)$, liabilities $l_i(r)$, payments to other (later) banks $p_{i,j}(r)$, and the recovery rate. This is possible because the recovery rates of banks i depends on have already been determined by the time we reach i. Uniqueness follows because the recovery rate of a bank is determined by its assets and liabilities (unless liabilities are 0), which are determined by the recovery rates of preceding banks. Clearly, the process can be performed in polynomial time. □

The above theorem gives a formal argument that the complexity in financial systems is due to cyclic dependencies. We notice that with CDSs, we need to take *all* dependencies into account. In particular, it is not sufficient even for existence that the *liabilities* be acyclic: all counterexamples in Section 4 had an acyclic liability graph.

6.3 Green Core Systems

The previous theorem requires a very strong assumption; in reality, financial systems *do* contain cycles and not all of them need to pose a problem. In fact, we know from Rogers and Veraart (2013) that the solutions of debt systems, cyclic or not, have a very desirable structure: clearing recovery rates always exist and there is even always a maximum. At the same time, we know that these systems always have a completely green dependency graph, i.e., banks are only "long" on each other. In this section, we show that the solution structure of *all* completely green systems looks like this. In fact, we show the statement for a slightly larger class of financial systems, called *green core systems*.

Figure 7 Colored dependency graph of a green core system with core C and leaf set L



Definition 6.4 (Green Core System)**.** A financial system X is called a *green core system* if in $\mathrm{CD}(X)$, banks with an incoming red edge (i.e., the holders of naked CDS positions) have no outgoing edges. In this case we call the set of banks with an incoming red edge the *leaf set* L and the other banks the *core* C. An example is given in Figure 7.

Banks in the leaf set L have no liabilities (otherwise there would be an outgoing liability edge) and hence cannot default: their recovery rate is always 1. We are thus only interested in the recovery rates of banks in the core C. This does not render the leaf set obsolete: allowing a leaf set keeps the definition of green core systems general enough so that core banks can be writers of arbitrary outgoing CDSs.

Our theorem states that green core systems have the same shape of their solution sets as debt-only systems, described in Section 4.2, if dominance on the core C is considered.

Definition 6.5 (Dominance, Pareto Efficiency, and Maximality on Subsets of Banks)**.** Fix a financial system $X = (N, e, c, \alpha, \beta)$ and let $C \subseteq N$. A solution r *dominates* another solution r' *on* C if $r_i \geq r'_i$ for all $i \in C$. r is called *Pareto efficient on* C if there is no other solution that dominates it on C. r is called *maximal on* C or a *maximum on* C if it dominates all solutions on C. In particular, then r is the unique solution Pareto efficient on C. r is called *minimal on* C or a *minimum on* C if it is dominated by all solutions on C.

Theorem 6.6 (Existence and Maximality / Minimality in Green Core Systems)**.** *In a green core system with core C there always exist clearing recovery rate vectors that are maximal and minimal for C, respectively. In particular, there always exists a clearing recovery rate vector.*

Proof outline (full proof in Appendix D). As mentioned above, it suffices to consider recovery rates in the core C. We show that for any two banks $i, j \in C$ assuming a lower recovery rate for i can in effect only lead to *lower* or equal assets and *higher* or equal liabilities of j. This implies that the function

$$F : [0,1]^C \to [0,1]^C$$
$$F_i(r) := \begin{cases} 1 & \text{if } a_i(r) \geq l_i(r) \\ \frac{\alpha e_i + \beta \sum_{j \in N} p_{j,i}(r)}{l_i(r)} & \text{if } a_i(r) < l_i(r) \end{cases}$$

is monotonic. We then apply the Knaster-Tarski fixed-point theorem to receive the desired structure for the fixed points of F. These fixed points correspond to the solutions of the financial system. □

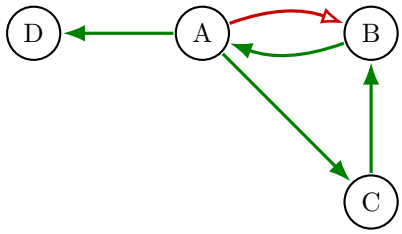
Corollary 6.7 (Existence and Maximality / Minimality without Naked CDSs)**.** *If no bank of a financial system has a naked CDS position towards another bank, then there exist a minimal and a maximal clearing recovery rate vector. In particular, there always exists a clearing recovery rate vector.*

Proof. In this case, the colored dependency graph contains no red edges at all. The financial system is hence trivially a green core system where the core contains all banks. Thus, Theorem 6.6 applies. □

The above theorem and corollary have an important implication for regulatory policy, which is discussed in Section 7.

Computing the Recovery Rates. The standard method for finding the maximal clearing recovery rate vector on C in Theorem 6.6 is to start at the highest possible recovery rate vector $r^0 := (1, \ldots, 1)$ and consider the iteration sequence defined by $r^{n+1} = F(r^n)$. This sequence is descending and bounded, so we can take the limit. The process may have to be repeated up to n times to skip over discontinuities introduced by default

Figure 8 Colored dependency graph for "Russell's Bank" (Figure 2)



costs when the recovery rate of a bank drops below 1. Finally, one arrives at a fixed point of F. This method was used by Rogers and Veraart (2013) to show existence and maximality.

Of course, this method, taking n times infinitely many iterations of the function F to converge, is not an algorithm. In systems of only debt contracts, one can avoid iterating F since the function is piecewise linear.

No such "shortcut" seems to be applicable to green core systems with CDSs, though: l_i can be linear in r when i is writer of a CDS and a_i can be nonlinear in r when i is holder of a covered CDS, and F_i contains a quotient of these functions. Recovery rates can be irrational even for very simple green core systems,[3] which implies that they can only be computed approximately. In particular, the recovery rates of a green core system are not in general the solutions of a series of linear equation systems, linear program, or mixed integer program.

We conjecture that one can still find the maximal and minimal clearing recovery rate vectors by iterating the function F:

Conjecture 6.8 (Polynomial-Time Algorithm for Green Core Systems)**.** Given a green core system X, an approximation of the maximal and the minimal clearing recovery rate vectors can be computed in polynomial time.

We leave the proof of this conjecture to future work. We imagine that the proof can be carried out by finding three components:

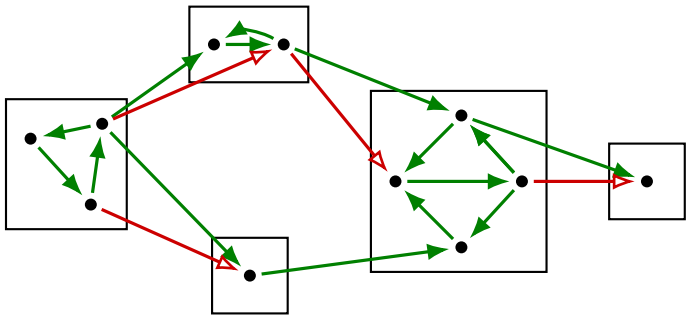
1. A threshold result to bound the precision needed to detect defaults (and thus discontinuity points).
2. A stopping criterion to detect when the iteration sequence $(r_n)_n$ is sufficiently close to a solution.
3. A precise estimate of the speed of convergence of the iteration sequence $(r_n)_n$ to show that only polynomially many steps are needed. Acceleration techniques could be used to speed up convergence between discontinuity points.

6.4 No-Red-Containing-Cycle Condition

We can combine the results from the previous two sections to receive an even weaker precondition under which existence is still fulfilled. Consider Figure 8 showing the colored dependency graph of the "Russell's Bank" system from Figure 2, which does not have a solution. The cycle A–B–A gives a concise argument why that is so: if A is doing well, then B is doing badly (red edge), so A is doing badly (green edge), which is a contradiction. At a higher level, the problem is that A is "long" on B, which is "short" on A, and thus A is "short on itself". If no such cycle exists, then we expect that a contradiction cannot arise and there is always a solution. We have already shown that if

[3] Example C.4 in the appendix in fact has a completely green dependency graph like in Corollary 6.7.

Figure 9 Dependency graph where no cycle contains a red edge like in Theorem 6.9. Strongly connected components are marked by black rectangles.



there is *no cycle whatsoever* in the colored dependency graph, then all three desiderata are fulfilled. We can extend part of this result by showing that systems where there are no cycles *that contain a red edge* always fulfill existence using Theorem 6.6 about green core systems.

Theorem 6.9 (No-Red-Containing-Cycle Condition)**.** *Let X be a financial system. Assume that in* $\mathrm{CD}(X)$*, no cycle contains a red edge. Then X has a clearing recovery rate vector.*

Proof outline (full proof in Appendix D). We proceed similarly to Theorem 6.3, but we operate on the strongly connected components (SCCs) of $\mathrm{CD}(X)$, defined as the largest sub-graphs where any two nodes are part of a cycle. An example dependency graph with SCCs is illustrated in Figure 9. By definition, the SCCs themselves form an acyclic graph and by assumption, the inner edges of each SCC are all green, which implies that the sub-system of an SCC plus its creditors is a green core system.

We now iterate over a topological ordering of SCCs like in the proof of Theorem 6.3, applying Theorem 6.6 to receive a clearing recovery rate vector of a green core network in each step. Since we do not specify an algorithm, but only a mathematical function for this step, the process does not give rise to an algorithm, but only to an inductive proof.

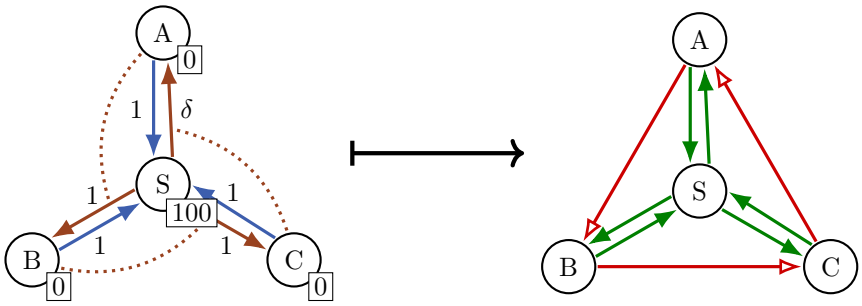
Updating the assets and liabilities of the banks is more complex than in Theorem 6.3 because several, possibly cyclically related banks need to be updated at once. The details are given in Appendix D.1. □

Intuition suggests that only cycles with an *odd* number of red edges could pose a problem to consistency since only they lead to the "short on itself" situations. However, the structure of the dependency graph is such that any red edge has two green edges connecting the same endpoints (cf. Figure 6). Thus, a cycle with an odd number of red edges exists iff any cycle with a red edge exists.

While a maximal solution could be chosen for each SCC in Theorem 6.9, the obtained solution is not in general maximal for the whole system. That is because superordinate components have a higher priority than dependent components: for the first component (with no incoming dependencies), we can choose a maximal solution, but the induced systems in dependent components might not be best possible any more. In general, a maximal solution need not exist.

From the proof of Theorem 6.9 it is clear that once Conjecture 6.8 is proven, we immediately receive an algorithm that approximates a clearing recovery rate vector in systems that do not contain a cycle with a red edge. It is then also easy to check whether the computed solution is unique: for green core systems, this can be done by comparing the maximal and the minimal solution; for systems without a red-containing cycle, this can then be done by performing this check in each recursive step.

Figure 10 Financial system with a central counterparty S where $\beta < 1$ is arbitrary and $\delta := \frac{1}{1-\beta+\beta^2-\beta^3}$. There is no clearing recovery rate vector. On the right the colored dependency graph.



We note that the converse of Theorem 6.9 is *not* true: for example, one can see that the system in Figure 3 has a cycle A–B–A with a red edge, but it also has a consistent default assignment. In general, a concise, exact characterization of existence that is also easy to check is not attainable: otherwise, we could transform it into an algorithm that checks existence in polynomial time, which would contradict NP-hardness.

7 Discussion: Policy Relevance

Our results from the previous section contribute to the discussion on a possible regulation of the CDS market: Corollary 6.7 shows that if naked CDS positions are not allowed, then maximal solutions always exist, and we conjecture that they are also efficiently computable. Banning naked CDSs has previously been proposed as a policy due to the problematic incentives arising from "short" positions: if a bank would profit from the default of another party, then it may have an incentive to promote this very default. At the peak of the European sovereign debt crisis, the European Commission (2011) in fact *did* put a ban on naked CDSs on sovereign states for this reason. We do not argue whether naked CDSs are "good" or "bad"; our analysis is not related to incentive arguments. Rather, we have shown that naked CDSs can increase the *complexity* of a financial system up to a point where it cannot be cleared any more under the assumptions of our model.

Our result on naked CDSs contrasts with another policy that has been proposed, and for a subset of contracts set into place (FSB, 2010), after the financial crisis: using a central counterparty. In its most extreme form, this means that all contracts are routed via a central node: a bank A would not any more write a CDS to a bank B directly, but rather bank A would write a CDS to a central counterparty S and S would write a CDS to bank B. The desired effect is that the central counterparty, being highly capitalized, would absorb a shock on counterparties, so that shocks would not spread through the network, thus preventing contagion. However, in Figure 10, we give an example showing that using a central counterparty is not an effective policy to guarantee even existence of a solution in our model: there are three banks that hold CDSs and write debt, and a central counterparty S. The other counterparties (CDS writers and debt holders) are assumed to be highly capitalized banks not shown in the picture. We express this by assuming high external assets of S.

This system does not have a solution for any $\beta < 1$ (the proof is given in Appendix E) and the reason becomes apparent when we look at the colored dependency graph on the right: there is still a red-containing (in this case completely red) cycle A–B–C. At a higher level, we see that while a central counterparty can help reduce *counterparty risk* (i.e., the risk to a bank that a debtor cannot pay its liability), *fundamental risk* (i.e., the risk that

the reference entity in a CDS has a higher or lower recovery rate than expected) still flows between the banks directly. This is enough to lead to non-existence of a solution.

The existence of scenarios where there is no solution poses a systemic risk beyond clearing because clearing payments have another interpretation: they are the payment the holder receives with all network effects taken into account, which equals the *value of a contract at maturity*. Thus, in situations without a clearing payment vector, the essential concept of the value of a contract is not well-defined. Relatedly, also the *ex-ante value* at some earlier point in time, being the discounted expected payment at maturity, is not necessarily well-defined any more. This implication is even more problematic in the context of a dynamically changing network: two banks might set up a contract and, given a good estimate of the network structure, be able to compute its value. As other banks set up more contracts, the network changes, and cycles might be closed. This could lead to situations where banks know the value of a contract when it is set up, but later, the value is not well-defined any more.

8 Conclusion

In this paper, we have studied the problem of clearing a network of banks, where some of them may be in default, such that computing clearing payments becomes a non-trivial problem. In contrast to prior work, we have considered financial networks with debt contracts as well as credit default swaps (CDSs), and we have shown that the addition of CDSs adds a surprising layer of complexity. In particular, while in pure debt networks it had previously been shown that a maximal clearing payment vector always exists and can be computed in polynomial time, we have shown that the situation is radically different in networks with CDSs. Perhaps most surprisingly, with CDSs there may not even be a clearing vector at all. Furthermore, we have shown that CDSs can also cause situations where multiple clearing vectors exist, with none of them being maximal, such that the authorities would have a hard time deciding in favor of any of them. From a computational point of view, we have proven that even determining whether a clearing vector exists is already NP-hard. However, we have also shown that our three desiderata (existence, maximality, and efficient computability) can be achieved if the contract space is sufficiently restricted. In particular, banning so-called naked CDSs re-establishes an existence guarantee for a unique maximal clearing vector.

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APPENDIX

A Comparison with Previous Models

Our introduction of CDS contracts changes the model substantially compared to the one by Rogers and Veraart (2013).

Most crucially, the liabilities $l_{i,j}(r)$ now depend on the recovery rate vector r, while they are just constants if no CDSs are present. Thus, the payments $p_{i,j}(r)$ depend on r in a non-linear way while without CDSs, the dependency is always linear. This in turn implies that the total assets a_i of a bank i depend on the recovery rate vector r in a way that is both *non-linear* and *non-monotonic*: an increase in the recovery rate of a bank j could lead to higher assets of a bank i (e.g., if j is a debtor of i) or lower assets of i (e.g., if i holds a CDS on j). What is common to both models is that the right-hand side of equation (1) is *not continuous* in r unless there are no default costs.

If there are no CDSs, then the above remarks do not apply and our model reduces to Rogers and Veraart's 2013 model. If there are no CDSs and in addition $\alpha = \beta = 1$ (no default costs), then our model reduces to the one by Eisenberg and Noe (2001).

B Proofs from Section 4

Lemma B.1. *Let $X = (N, e, c, \alpha, \beta)$ be a financial system, r clearing for X and let $i \in N$ be a bank. If $r_i < 1$ and $l_i(r) > 0$, then the following hold:*

1. *If i has only external assets ($\sum_j p_{j,i}(r) = 0$), then $r_i \leq \alpha$. If $\alpha > 0$, then even $r_i < \alpha$.*
2. *If i has only interbank assets ($e_i = 0$) then $r_i \leq \beta$. If $\beta > 0$, then even $r_i < \beta$.*
3. *In any case $r_i \leq \max(\alpha, \beta)$. If $\alpha > 0$ or $\beta > 0$, then even $r_i < \max(\alpha, \beta)$.*

Proof. Part 3: Since $r_i < 1$ we must have that $a_i(r) < l_i(r)$. And thus

$$\begin{aligned} r_i l_i(r) = \sum\nolimits_j p_{j,i}(r) &= \alpha e_i + \beta \sum\nolimits_j p_{j,i}(r) \\ &\leq \max(\alpha, \beta) \left(e_i + \sum\nolimits_j p_{j,i}(r) \right) \;=\; \max(\alpha, \beta) a_i(r) \\ &\leq \max(\alpha, \beta) l_i(r). \end{aligned}$$

If $\alpha > 0$ or $\beta > 0$, then the inequality in the last line is strict. The conclusion follows by division by $l_i(r)$. By assumption $l_i(r) \neq 0$.

Parts 1 and 2 are similar, except for that one of the two summands αe_i or $\beta \sum_j p_{j,i}(r)$ drops out in line 1. □

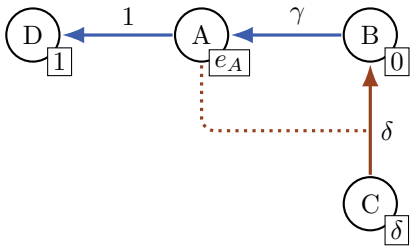
Proof of Theorem 4.1, $\beta < 1$. If $\beta < 1$, then let $\delta = 3 \cdot \frac{1}{1-\beta}$ and consider the system in Figure 2.

Assume towards a contradiction that there is a clearing recovery rate vector r.

- If $r_A = 1$, then $l_{C,B}(r) = \delta(1 - r_A) = 0$, hence by equation (1) from Definition 3.1 $p_{C,B}(r) = 0$, so $a_B(r) = 0 < 2 = l_B(r)$ and then again by equation (1) $p_{B,A}(r) = 0$. This implies $a_A(r) = 0 < 1 = l_A(r)$ and thus $r_A < 1$. Contradiction.
- If $r_A < 1$, then $r_A \leq \beta$ by Lemma B.1. Thus $p_{C,B}(r) = l_{C,B}(r) = \delta(1 - r_A) \geq \delta(1 - \beta) = 3$. Now $a_B(r) = 3 \geq l_B(r)$, so $p_{B,A}(r) = l_{B,A}(r) = 2$. Hence $a_A(r) \geq l_A(r)$ and so $r_A = 1$. Contradiction. □

□

Figure 11 Financial system with no solution for $\alpha < \beta = 1$ for appropriately chosen e_A, γ, and δ



Proof of Theorem 4.1, $\alpha < \beta = 1$. Consider Figure 11, a variant of Figure 2, with values for e_A, γ, and δ chosen as follows:

- Let $e_A \in (0, 1)$ arbitrary.
- Let $\gamma = 1 - \frac{1+\alpha}{2} e_A$.
- Let $\delta \geq \frac{\gamma}{1-\alpha e_A - \gamma}$.

It is easy to see that

$$e_A < 1 \tag{2}$$
$$e_A + \gamma > 1 \tag{3}$$
$$\alpha e_A + \gamma < 1. \tag{4}$$

We have $\gamma > 0$ because $\alpha < 1$ by assumption and $\delta > 0$ by (4), so this is a well-defined financial system.

We now perform a case-wise analysis like in the case for $\beta < 1$. Assume towards a contradiction that r is clearing.

- If $r_A = 1$, then $p_{C,B}(r) = 0$, so $a_B(r) = 0$ and $p_{B,A}(r) = 0$. Thus, $a_A(r) = e_A < 1 = l_A(r)$, implying that $r_A < 1$. Contradiction.
- If $r_A < 1$, then A is in default, so

$$\begin{aligned} r_A &= \frac{\alpha e_A + p_{B,A}(r)}{1} \\ &\leq \alpha e_A + \gamma. \end{aligned}$$

  Thus

$$\begin{aligned} p_{C,B}(r) &= \delta(1 - r_A) \\ &\geq \delta(1 - \alpha e_A - \gamma) = \gamma. \end{aligned}$$

  Thus, B is not in default and $p_{B,A}(r) = \gamma$. Now $a_A(r) = e_A + \gamma > 1 = l_A(r)$ by (3), so A is not in default and $r_A = 1$. Contradiction. □

□

Proof of Theorem 4.3. Assume WLOG that $N = \{1, \dots, n\}$. Consider the function ρ defined by

$$\begin{aligned} \rho : [0,1]^n &\to 2^{[0,1]^n} \\ \rho(r) &:= \bigtimes_{i=1}^{n} \rho_i(r) \end{aligned}$$

where for each $i \in \{1, \dots, n\}$

$$\rho_i : [0,1]^n \to 2^{[0,1]}$$
$$\rho_i(r) := \begin{cases} \{1\} & \text{if } l_i(r) > 0 \text{ and } a_i(r) \geq l_i(r) \\ \left\{ \frac{\alpha e_i + \beta \sum_j p_{j,i}(r)}{l_i(r)} \right\} & \text{if } l_i(r) > 0 \text{ and } a_i(r) < l_i(r) \\ [0,1] & \text{if } l_i(r) = 0. \end{cases}$$

The clearing recovery rate vectors are exactly the fixed points of the set-valued function ρ, i.e., the vectors $r \in [0,1]^n$ for which $r \in \rho(r)$. This is a formal version of Remark 3.2 and it is easy to see for general α and β from the definition of a clearing recovery rate vector.

By the Kakutani fixed point theorem (Kakutani, 1941), a fixed point of ρ exists if $\rho(r)$ is convex for each r and the graph of ρ

$$G_\rho := \{(r,s) \mid r, s \in [0,1]^n, s \in \rho(r)\}$$

is closed (and the domain $[0,1]^n$ of ρ is compact and convex, which is obviously true).

1. $\rho(r)$ is convex for any r: For any r and i the set $\rho_i(r)$ is either a point or an interval and thus trivially convex. Then $\rho(r)$, being the product of convex sets, is also convex. Convexity is fulfilled for any value of α and β.

2. ρ_i has a closed graph for any i: We first show that each of the sets

$$G_{\rho_i} := \{(r, s_i) \mid r \in [0,1]^n, s_i \in [0,1], s_i \in \rho_i(r)\}$$

is closed. Let $L_i = \{r \mid l_i(r) > 0\}$. L_i is open since l_i is a continuous function. Since $\alpha = \beta = 1$ we can write ρ_i as

$$\rho_i(r) = \begin{cases} \left\{ \min\left(1, \frac{a_i(r)}{l_i(r)}\right) \right\} & \text{if } r \in L_i \\ [0,1] & \text{if } r \notin L_i. \end{cases}$$

To see that G_{ρ_i} is closed let $((r^k, s_i^k))_{k \in \mathbb{N}}$ be a sequence in $[0,1]^n \times [0,1]$ converging to some point (r, s_i) such that $s_i^k \in \rho_i(r^k)$ for each k. We need to show that $s_i \in \rho_i(r)$. If $r \notin L_i$, then trivially $s_i \in \rho_i(r) = [0,1]$. If $r \in L_i$, then $s_i \in \rho_i(r)$ because the function $(r \mapsto \min(1, \frac{a_i(r)}{l_i(r)}))$ is continuous on the open set L_i.

3. ρ has a closed graph: Let

$$\iota : [0,1]^n \times [0,1]^n \to ([0,1]^n \times [0,1])^n$$
$$(r, s_1, s_2, \dots, s_n) \mapsto (r, s_1, r, s_2, \dots, r, s_n),$$

let $\tau : ([0,1]^n \times [0,1])^n \xrightarrow{\sim} ([0,1]^n)^n \times [0,1]^n$ be the obvious coordinate transformation and let

$$\Delta = \{(r, \dots, r) \mid r \in [0,1]^n\} \subseteq ([0,1]^n)^n$$

be the diagonal. The functions ι and τ are continuous, Δ is closed, and

$$\begin{aligned} G_\rho &= \{(r, s_1, \dots, s_n) \mid (r, s_i) \in G_{\rho_i} \, \forall i\} \\ &= \iota^{-1}\left(\left(\bigtimes_{i=1}^n G_{\rho_i} \right) \cap \tau^{-1}(\Delta \times [0,1]^n) \right). \end{aligned}$$

Thus, G_ρ is closed. □

Proof of Theorem 4.5. We use the system in Figure 3 with $\gamma = 1$ and a choice of δ dependent on β.

Case 1: If $\beta = 1$, then choose $\delta > 1$ arbitrary. If r is clearing, then

$$\begin{aligned} r_B &= \min\left(1, \frac{a_i(r)}{l_i(r)}\right) = \min(1, 1 - r_A) = 1 - r_A \\ r_A &= \min\left(1, \frac{a_i(r)}{l_i(r)}\right) = \min(1, \delta(1 - r_B)) = \min(1, \delta r_A). \end{aligned}$$

That is because A and B always have positive liabilities and only interbank assets and $\beta = 1$. Since $\delta > 1$, the second equation is true exactly if $r_A = 0$ or $r_A = 1$ and it is easy to check that these values indeed give rise to the two described clearing recovery rate vectors.

Case 2: If $\beta < 1$, then choose $\delta = \frac{1}{1-\beta}$. If r is clearing, then the following hold:

- If $r_A = 0$, then $a_B(r) = 1 - r_A = 1 = l_B(r)$, so $r_B = 1$ and $a_A(r) = \delta(1 - r_B) = 0$. So $(0, 1, 1, 1)$ is the unique clearing recovery rate vector with $r_A = 0$.
- If $r_A > 0$, then $a_B(r) < 1$ and so $r_B \leq \beta$ by Lemma B.1. Hence $a_A(r) = \delta(1-r_B) \geq \delta(1 - \beta) = 1 = l_A(r)$. So $r_A = 1$ and thus $a_B(r) = 0$, so $r_B = 0$. In total, $(1, 0, 1, 1)$ is the unique clearing recovery rate vector with $r_A > 0$. □

□

Proof of Theorem 4.6. Let $\frac{1}{2} < \beta < 1$ and α arbitrary. Like in Theorem 4.5, we use the system in Figure 3, but this time we set $\gamma = \delta = \frac{1}{\beta}$.

We claim that the set of solutions is

$$\{(r_A, 1 - r_A, 1, 1) \mid r_A \in \{0\} \cup (1 - \beta, \beta) \cup \{1\}\}.$$

To see this, notice that $a_B(r) = \gamma(1 - r_A)$ and $l_B(r) = 1$, so if r is clearing, then

$$\begin{aligned} r_B &= \begin{cases} 1 & \text{if } \gamma(1 - r_A) \geq 1 \\ \beta\gamma(1 - r_A) & \text{if } \gamma(1 - r_A) < 1 \end{cases} \\ \Leftrightarrow \quad r_B &= \begin{cases} 1 & \text{if } r_A \leq 1 - \beta \\ 1 - r_A & \text{if } r_A > 1 - \beta \end{cases} \end{aligned}$$

where the second line is by choice of γ. By symmetry, A and B can be exchanged to receive

$$r_A = \begin{cases} 1 & \text{if } r_B \leq 1 - \beta \\ 1 - r_B & \text{if } r_B > 1 - \beta. \end{cases}$$

To see that the equations for r_A and r_B hold exactly on the described set, we distinguish three cases:

- $r_A \leq 1 - \beta$. Then $r_B = 1$ and thus $r_A = 0$. So $(0, 1, 1, 1)$ is the unique solution with $r_A \leq 1 - \beta$.
- $r_A \geq \beta$. Then in particular $r_A > 1 - \beta$, so $r_B = 1 - r_A \leq 1 - \beta$, so $r_A = 1$ and $r_B = 0$. So $(1, 0, 1, 1)$ is the unique solution with $r_A \geq \beta$.
- $1 - \beta < r_A < \beta$. Then, since $r_A > 1 - \beta$, $r_B = 1 - r_A < 1 - \beta$ and $r_A = 1 - r_B$. So the solutions with $1 - \beta < r_A < \beta$ are exactly the recovery rate vectors

$$\{(r_A, 1 - r_A, 1, 1) \mid r_A \in (1 - \beta, \beta)\}. \quad \square$$

□

C Proofs from Section 5

Proof of Lemma 5.4. WLOG let C consist of NOR gates only. We model a boolean circuit as a directed acyclic graph with two types of nodes: INPUT nodes have no predecessors. They encode inputs. There are exactly m input nodes. NOR nodes have two predecessors. They encode NOR gates. We assume some ordering on the nodes to identify different inputs.

In the following, we construct a financial system that encodes the circuit C. All nodes of the circuit have a corresponding bank in the financial system (that uses the same label). We also add other banks that do not correspond to nodes in the circuit. For technical reasons, we do not consider a special output node. Instead, we replace property 3 by the following, stronger property:

(3a) Let k be any node in the circuit. For an assignment $\chi \in \{0,1\}^m$ let $C_k(\chi)$ be the value of node k given inputs χ. If r is clearing for X, then $r_k = C(r_{a_1}, \dots, r_{a_m})$.

We prove the theorem by induction on the number of nodes in the circuit. If there are no nodes, then the financial system with no banks has the required properties. So assume that there is at least one node.

Since the graph is acyclic, there is at least one node k with no successors. Let C' be the circuit without k. By induction hypothesis, there is a financial system X' that encodes C' in the sense of this theorem. We distinguish two cases based on the type of the node k:

Case 1: k is an INPUT node. Then k has no predecessors. Let X be the disjoint union of X' and a copy of the 0-1 system from Theorem 4.5. Identify k with bank A in Figure 3. Let the input banks of X' be the input banks of X together with k. Since the two financial systems X' and the copy of Figure 3 do not interact, the solutions of the compound system X are the unions of the solutions of the two components, i.e.,

$$\{(r, 0, \dots) \mid r \text{ clearing for } X'\} \cup \{(r, 1, \dots) \mid r \text{ clearing for } X'\}$$

where "..." marks fixed, but unimportant values for the three new banks other than k. By induction hypothesis, all clearing recovery rates are 0 or 1, so property 2 holds and, since both 0 and 1 for the input k occur, also property 1 holds. Towards property 3a, let j be a node of C and let r be clearing. If $j = k$, then j is an input, so $r_j = C_j(r_{a_1}, \dots, r_{a_m})$ trivially. If $j \neq k$, then $r_j = C_j(r_{a_1}, \dots, r_{a_m})$ by induction hypothesis.

Case 2: k is a NOR node. Then k has two predecessors a and b in C'. Let X result by applying Lemma 5.2 to X', a, and b. This is allowed since by induction hypothesis, any clearing recovery rates of X' are 0 or 1. Identify k with bank v in the lemma. Let the input banks of X be the input banks of X'. The solutions of X are

$$\{(r, r_a \operatorname{NOR} r_b, \dots) \mid r \text{ clearing for } X'\}.$$

This implies that property 2 holds and property 1 holds by induction hypothesis because the input banks of X are the same as those of X'. Property 3a holds by induction hypothesis and Lemma 5.2.

The construction adds exactly four banks in each step and there are as many steps as there are gates plus inputs, thus the size of X is linear in the size of C. □

C.1 Computational Hardness of Decision Problems for Individual Banks

We consider the problem of determining if there is a solution where a specific bank is in default. This problem can also be stated for $\alpha = \beta = 1$. Regulators might be interested in solving this problem because it would show if a bank is at risk of going into default if a for it unfortunate solution is selected. Unfortunately, as the following two theorems

show, it is NP-hard to get *any* information of this kind about a specific bank: neither can we determine efficiently if there is a solution where a given bank defaults, or not defaults, or if it defaults in all solutions, or if its recovery rate is dependent on the chosen solution.

Definition C.1 (Circuit Problems)**.** Define the following decision problems:

- *Circuit Satisfiability (Falsifiability):* Given a boolean circuit, decide if there exists an assignment of inputs such that the output is 1 (0).
- *Circuit Non-Constancy:* Given a boolean circuit, decide if it is true that there are two assignments of inputs: one that makes the output 1 and one that makes it 0.

We know that Circuit Satisfiability is NP-complete. Circuit Falsifiability is NP-complete because it is just Circuit Satisfiability applied to a circuit plus a NOT gate.

Lemma C.2. *Circuit Non-Constancy is NP-complete.*

Proof. Write CSAT for Circuit Satisfiability and CNC for Circuit Non-Constancy.

CNC is in NP because a witness can be given as a pair of a satisfying and a falsifying assignment.

CNC is NP-hard because CSAT can be reduced to it: given an instance of CSAT, first check CNC. If CNC is true, then CSAT is true and the satisfying assignment provides a witness. Otherwise, evaluate the circuit for a single assignment, say $(1, \dots, 1)$. If this assignment yields 1, then return it as a witness. Otherwise, the circuit is not satisfiable. □

Theorem C.3 (NP-hardness, individual bank)**.** *The following problems are NP-hard: Given a financial system X and a bank i in X, decide if*

a) Possible Default: *there exists r clearing such that $r_i < 1$.*

b) Possible Non-Default: *there exists r clearing such that $r_i = 1$.*

c) Certain Default: *for all r clearing we have $r_i < 1$.*

d) Certain Non-Default: *for all r clearing we have $r_i = 1$.*

e) Multiplicity: *there exist r and r' clearing such that $r_i \neq r'_i$*

Proof. a): Reduction from Circuit Falsifiability. Given an instance C of Circuit Falsifiability, let X be the financial system resulting from Lemma 5.4 applied to C and let $i := v$ be the output bank. By construction, if r is clearing for X and $r_i < 1$, then $r_i = 0$ and $(r_{a_1}, \dots, r_{a_n})$ is a falsifying assignment. Vice versa, any falsifying assignment $\chi \in \{0,1\}^n$ gives rise to a clearing recovery rate vector r with $r_{a_i} = \chi_i$ for $i \in \{1, \dots, n\}$ and $r_i = C(\chi) = 0 < 1$.

b): Reduction from Circuit Satisfiability like for a).

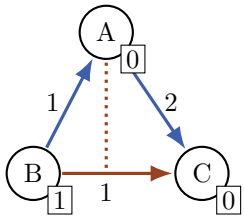
c) and d): these are the complements of b) and a), respectively, and are thus NP-hard as well.[4]

e): Reduction from Circuit Non-Constancy like for a). For our particular construction we know that $r_i \neq r'_i$ iff $r_i = 0$ and $r'_i = 1$ or vice versa. □

The previous theorems, like the theorem on NP-hardness of checking existence, have an important implication for network stress tests: regulators may want to simulate an array of different scenarios and test for each one if specific banks are at risk of defaulting. This should take all solutions (or at least all Pareto efficient solutions) into account since it is not clear a priori which of these would be chosen. However, theorem C.3 shows that such a stress test is computationally infeasible in a general network of debt and CDSs.

[4] It is easy to see that complements of NP-hard problems are NP-hard: the reduction is simply by interchanging the "Yes" and "No" answers. This is not to be confused with the (conjectured false) analogous proposition for NP-completeness. Cf. (Korte and Vygen, 2012) for a discussion.

Figure 12 Financial System where the unique solution is irrational. $\alpha = \beta = 1$.



C.2 The Proof of Theorem 5.5

Proof of Theorem 5.5. Reduction from Circuit Satisfiability. Given an instance C of Circuit Satisfiability, let X' result by application of Lemma 5.4 to C and let v be the output bank. Let X'' be the "Russell's Bank" system from Theorem 4.1. This corresponds either to Figure 2 (if $\beta < 1$) or to Figure 11 (if $\alpha < \beta = 1$). In any case there is a debt contract from B to A. Let X be the union of X' and X'' where this debt contract has been replaced by a CDS with reference entity v and the same notional. We claim that X has a solution iff C has a satisfying assignment.

Like in the proof of Theorem C.3, the solutions r of the X' component of X correspond to assignments $(r_{a_1}, \dots, r_{a_m})$ and output values $r_v = C(r_{a_1}, \dots, r_{a_m})$ of C. To see which of these extend to solutions for the whole of X, we distinguish two cases:

Case 1: $r_v = 0$. Then the CDS from B to A in the X'' component of X gives rise to a liability equal to its full notional and thus solutions of this component that extend r correspond to solutions of X'', which do not exist. So no extension of r can be a solution for X.

Case 2: $r_v = 1$. Then the CDS from B to A in the X'' component of X gives rise to a liability of 0. One now easily checks that $r_A = e_A$, $r_B = 1$, and $r_C = r_D = 1$ extend r to a solution on the whole of X.

Thus, the solutions of X correspond to the *satisfying* assignments $(r_{a_1}, \dots, r_{a_m})$ and output values $r_v = C(r_{a_1}, \dots, r_{a_m}) = 1$ of C. □

C.3 Irrational Solutions

Example C.4 (Irrational Solutions). Figure 12 shows a financial system the unique solution of which is irrational. To see this, note that by the contract structure r is clearing iff

$$r_A = \frac{1}{2} r_B, \qquad r_B = \frac{1}{2 - r_A}, \qquad r_C = 1.$$

One easily checks that the unique solution in $[0,1]^3$ to this system of equations is given by

$$r_A = 2 - \sqrt{2}, \qquad r_B = 1 - \frac{1}{\sqrt{2}}, \qquad r_C = 1.$$

D Proofs from Section 6

Proof of Theorem 6.6. Consider the function

$$F : [0,1]^C \to [0,1]^C$$
$$F_i(r) := \begin{cases} 1 & \text{if } a_i(r) \geq l_i(r) \\ \frac{a'_i(r)}{l_i(r)} & \text{if } a_i(r) < l_i(r). \end{cases}$$

where $a'_i(r) := \alpha e_i + \beta \sum_j p_{j,i}(r)$. By Remark 3.2, we know that

- any fixed point of F can be extended to a clearing recovery rate vector by setting the recovery rates of bank in the leaf set L to 1 and
- if r is a clearing recovery rate vector, then a fixed point of F is obtained from $r|_C$ by setting the recovery rates of all banks i with $l_i(r) = 0$ to 1.

It is thus sufficient to show that the fixed points of F have the desired structure.

By the Knaster-Tarski fixed-point theorem (Tarski, 1955), it suffices to show that F is monotonic in the sense that if $r_i \leq r'_i$ for all $i \in C$, then $F_i(r) \leq F_i(r')$ for all $i \in C$. Then by the theorem F has a maximal and a minimal fixed point. (in fact, the fixed points of F form a complete lattice)

For monotonicity of F it suffices that the functions a_i and a'_i be monotonically increasing and the function l_i be monotonically decreasing for all $i \in C$. This is easy to see.

$l_i(r)$ **is monotonically decreasing in** r**:** This follows directly from the definition and it is true in any financial system:

$$l_i(r) \;=\; \sum\nolimits_j c^{\emptyset}_{i,j} + \sum\nolimits_{j,k} (1 - r_j) c^k_{i,j}$$

$a_i(r)$ **and** $a'_i(r)$ **are monotonically increasing in** r**:** We show that the total incoming payments of a bank i, $\sum_j p_{j,i}(r)$, are monotonically increasing. The statement for a_i and a'_i then follows.

Towards the incoming payments, let

$$q_{k,i}(r) = r_k c^{\emptyset}_{k,i} + (1 - r_k) \sum\nolimits_j r_j c^k_{j,i}.$$

One checks that

$$\sum\nolimits_j p_{j,i}(r) \;=\; \sum\nolimits_j q_{k,i}(r)$$

by rearranging summands.

Each individual summand $q_{k,i}$ for $i \in C$ is monotonically increasing in r by the green core condition: let $r_i \leq r'_i \, \forall i \in C$. Then

$$\begin{aligned}
q_{k,i}(r') - q_{k,i}(r) &= r'_k c^{\emptyset}_{k,i} - r_k c^{\emptyset}_{k,i} + (1 - r'_k) \sum\nolimits_j r'_j c^k_{j,i} - (1 - r_k) \sum\nolimits_j r_j c^k_{j,i} \\
&\geq r'_k c^{\emptyset}_{k,i} - r_k c^{\emptyset}_{k,i} + (1 - r'_k) \sum\nolimits_j r_j c^k_{j,i} - (1 - r_k) \sum\nolimits_j r_j c^k_{j,i} \\
&= (r'_k - r_k) \cdot \Big(c^{\emptyset}_{k,i} - \sum\nolimits_j r_j c^k_{j,i} \Big) \\
&\geq 0
\end{aligned}$$

where the last line is because $r'_k - r_k \geq 0$ by assumption and

$$\begin{aligned}
& c^{\emptyset}_{k,i} - \sum\nolimits_j r_j c^k_{j,i} \\
\geq\; & c^{\emptyset}_{k,i} - \sum\nolimits_j c^k_{j,i} \\
\geq\; & 0
\end{aligned}$$

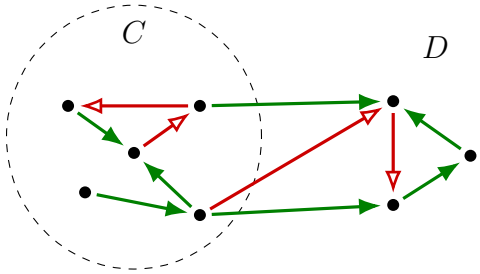
because X is a green core system and so i must have a covered CDS position towards k. □

D.1 Splitting Systems at Dependencies

To show Theorems 6.3 and 6.9 formally, we need some technical machinery for splitting networks at dependencies, introduced in this section.

Imagine a system consisting of two components C and D where we only have dependencies from C to D, but not vice versa. An example of such a structure is given

Figure 13 Superordinate component C with dependent component D



in Figure 13. Intuition suggests that we can find the solutions of such systems by first considering the "superordinate" component C, then the "dependent" component D.

While the approach seems simple enough, care needs to be taken to handle all the details, such as the different default cost parameters α and β, correctly. We prove that all solutions of a system like in Figure 13 can be found in two steps: first one finds a solution of the *out-restriction* of the system to C. Then one considers the financial system *induced* by this solution on D and finds a solution for that.

Definition D.1 (Out-Restriction)**.** Let $X = (N, e, c, \alpha, \beta)$ be a financial system and let $C \subseteq N$. Let $D = N \setminus C$.

The *out-restriction* X_C of X to C arises from X by considering only contracts where the writer is in C. Formally, we define $X_C = (N, e, d, \alpha, \beta)$ where

$$d^k_{i,j} := \begin{cases} c^k_{i,j} & \text{if } i \in C \\ 0 & \text{otherwise.} \end{cases}$$

for any $i, j \in N$ and $k \in N \cup \{\emptyset\}$. We label recovery rate vectors for X_C by s.

Definition D.2 (Induced System)**.** Let X, C, and D be like above and let s be a recovery rate vector for X_C. The system $X^{C,s}$ *induced* by s on C arises from X by considering only contracts where the holder is in D. We convert assets of the banks in C to external assets and we convert CDSs with reference entities in C to debt of appropriate notional as of s. Intuitively, this corresponds to a situation when it is known that C has recovery rates s and payments by banks in C to other banks in C have already been made. However, adjustments need to be made to correctly treat the two different default cost factors α and β. Formally, we define $X^{C,s} = (N, e^{C,s}, f, \alpha, \beta)$ where

$$e^{C,s}_i := \begin{cases} e_i + \sum_{j \in C} p_{j,i}(s) & \text{if } i \in C \text{ and } s_i = 1 \\ \alpha e_i + \beta \sum_{j \in C} p_{j,i}(s) & \text{if } i \in C \text{ and } s_i < 1 \\ e_i & \text{if } i \notin C \end{cases}$$

$$f^{\emptyset}_{i,j} := \begin{cases} 0 & \text{if } j \in C \\ s_i(c^{\emptyset}_{i,j} + \sum_{k \in C}(1 - s_k)c^k_{i,j}) & \text{if } i \in C \text{ and } j \notin C \\ c^{\emptyset}_{i,j} + \sum_{k \in C}(1 - s_k)c^k_{i,j} & \text{if } i, j \notin C \end{cases}$$

$$f^k_{i,j} := \begin{cases} 0 & \text{if } j \in C \text{ or } k \in C \\ s_i c^k_{i,j} & \text{if } i \in C \text{ and } j, k \notin C \\ c^k_{i,j} & \text{if } i, j, k \notin C \end{cases}$$

We label recovery rate vectors for $X^{C,s}$ by t.

Remark D.3.

- In X_C, the banks in D cannot default because they have no liabilities. Thus, only the recovery rates of banks in C are relevant. The banks in D further have no dependency on each other.
- In $X^{C,s}$, the banks in C cannot default because their liabilities have been reduced according to their recovery rate. Thus, only the recovery rates of banks in D are relevant. The banks in C further have no dependency on each other.

Lemma D.4 (Splitting Lemma)**.** *Let $X = (N, e, c, \alpha, \beta)$ be a financial system and let $C \subseteq N$ be a subset of banks with no incoming edges in* $\mathrm{CD}(X)$*. Let $D = N \setminus C$. Then the clearing recovery rate vectors r of X are exactly the recovery rate vectors of form $r = s|_C \cup t|_D$ where*

- *s is a clearing recovery rate vector for X_C and*
- *t is a clearing recovery rate vector for $X^{C,s}$.*

Proof. **First Direction:** Let r be clearing for X. We need to show that $s := r|_C \cup (1)_{i \in D}$ is clearing for X_C and that $t := r|_D \cup (1)_{i \in C}$ is clearing for $X^{C,s}$.

s is clearing for X_C because by assumption the assets and liabilities of banks in C only depend on the recovery rates of other banks in C. Thus, the other contracts can be safely ignored.

To see that t is clearing for $X^{C,s}$, we consider three different cases for a bank i: $i \in C$ with $r_i = 1$, $i \in C$ with $r_i < 1$, and $i \in D$. We use the notation $a_C(s)$ for assets calculated in X_C, $a^C(t)$ for assets calculated in $X^{C,s}$, and $a(r)$ for assets in X. Likewise for external assets e, liabilities l, and payments p.

We first consider banks $i \in C$ with $r_i = 1$. Then also $s_i = r_i = 1$. Hence, these banks have assets and liabilities in $X^{C,s}$ of

$$\begin{aligned}
a_i^C(t) = e_i^C &= e_i + \sum_{j \in C} p_{j,i,C}(s) \\
&= e_i + \sum_{j \in C} p_{j,i}(r) = a_i(r) \\
l_i^C(t) &= s_i \left(\sum_{j \in D} c_{i,j}^{\emptyset} + \sum_{\substack{j \in D \\ k \in C}} (1 - s_k) c_{i,j}^k \right) \\
&= \sum_{j \in D} c_{i,j}^{\emptyset} + \sum_{\substack{j \in D \\ k \in C}} (1 - r_k) c_{i,j}^k = l_i(r).
\end{aligned}$$

We use the assumption that there are no dependencies from D to C. For $l_i^C(t)$ we use the fact that $s_i = r_i = 1$ and $s_k = r_k$ for $k \in C$. Since r is clearing and $r_i = 1$ we have $a_i(r) \geq l_i(r)$. Thus, $t_i = 1$ is also clearing in $X^{C,s}$.

We second consider banks $i \in C$ with $r_i < 1$. Then also $s_i = r_i < 1$. Hence, these

banks have assets and liabilities in $X^{C,s}$ of

$$\begin{aligned}
a_i^C(t) = e_i^C &= \alpha e_i + \beta \sum_{j \in C} p_{j,i,C}(s) \\
&= \alpha e_i + \beta \sum_{j \in C} p_{j,i}(r) \\
l_i^C(t) &= s_i \left(\sum_{j \in D} c_{i,j}^{\emptyset} + \sum_{\substack{j \in D \\ k \in C}} (1 - s_k) c_{i,j}^k \right) \\
&= r_i \left(\sum_{j \in D} c_{i,j}^{\emptyset} + \sum_{\substack{j \in D \\ k \in C}} (1 - r_k) c_{i,j}^k \right) = r_i l_i(r).
\end{aligned}$$

We thus have

$$l_i^C(t) = r_i l_i(r) = \alpha e_i + \beta \sum_{j \in C} p_{j,i}(r) = a_i^C(t),$$

the middle equality because r is clearing. Thus, i is not in default in $X^{C,s}$ and $t_i = 1$ is clearing.

We third consider banks $i \in D$. These banks have external assets of $e_i^C = e_i$ and interbank assets (incoming payments) of

$$\begin{aligned}
\sum_{j \in N} p_{j,i}^C(t) = &\sum_{j \in C} t_j l_{j,i}^C(t) + \sum_{j \in D} t_j l_{j,i}^C(t) \\
= &\sum_{j \in C} t_j s_j \left(c_{j,i}^{\emptyset} + \sum_{k \in C} (1 - s_k) c_{j,i}^k \right) \\
&+ \sum_{j \in D} t_j \left(c_{j,i}^{\emptyset} + \sum_{k \in C} (1 - s_k) c_{j,i}^k + \sum_{k \in D} (1 - t_k) c_{j,i}^k \right) \\
= &\sum_{j \in C} r_j \left(c_{j,i}^{\emptyset} + \sum_{k \in C} (1 - r_k) c_{j,i}^k \right) \\
&+ \sum_{j \in D} r_j \left(c_{j,i}^{\emptyset} + \sum_{k \in C} (1 - r_k) c_{j,i}^k + \sum_{k \in D} (1 - r_k) c_{j,i}^k \right) \\
= &\, a_i(r)
\end{aligned}$$

where the first line is by definition of the payments, the second is by definition of the contracts in $X^{C,s}$, and the third is by definition of s and t. The last line follows simply by combining the sums. We use the fact that there are no payments or CDS references from D to C. For the liabilities we have

$$\begin{aligned}
l_i^C(t) &= \sum_{j \in D} \left(c_{i,j}^{\emptyset} + \sum_{k \in C} (1 - s_k) c_{i,j}^k + \sum_{k \in D} (1 - t_k) c_{i,j}^k \right) \\
&= \sum_{j \in D} \left(c_{i,j}^{\emptyset} + \sum_{k \in C} (1 - r_k) c_{i,j}^k + \sum_{k \in D} (1 - r_k) c_{i,j}^k \right) \\
&= l_i(t)
\end{aligned}$$

using the same technique. Altogether, external assets, interbank assets, and liabilities under t in $X^{C,s}$ equal the respective values under r in X. Hence, $t_i = r_i$ is clearing at i.

Second Direction: For the other direction, we need to show that if s and t are like in the theorem, then $r = s|_C \cup t|_D$ is clearing for X. This can be shown by just reversing

the arguments above: they are, at their core, all based on just equalities between assets, liabilities, and payments in X and $X^{C,s}$. □

D.2 Acyclic Financial Systems

Proof of Theorem 6.3. The statement follows directly from the following lemma for $C = N$. Notice that $X_N = X$. □

Lemma D.5. *Let $X = (N, e, c, \alpha, \beta)$ be a financial system such that $\mathrm{CD}(X)$ is acyclic. Let $C \subseteq N$ be a subset of banks without incoming edges in $\mathrm{CD}(X)$. Then there is a clearing recovery rate vector for X_C, it is unique up to banks with no liabilities, and it can be found in polynomial time.*

Proof. Proof by induction on $|C|$. For $|C| = 0$, i.e., $C = \emptyset$, X_C has no contracts, so $(1, \dots, 1)$ is clearing. So let $|C| > 0$.

Since $\mathrm{CD}(X)$ is acyclic, there is a bank $k \in C$ such that there is no edge in $\mathrm{CD}(X)$ from k to any other bank in C. Let $C' = C \setminus \{k\}$. By induction hypothesis, we can find in polynomial time the unique clearing recovery rate vector s' of $(X_C)_{C'} = X_{C'}$. The financial system $X_C^{C',s'}$ has only debt contracts and only bank k is both holder and writer of a contract. Hence, it is trivial to compute a solution t' for this system by comparing assets and liabilities of k and setting the other recovery rates to 1. The solution is unique unless k has no liabilities. By the Splitting Lemma D.4 $s'|_{C'} \cup t'|_{N \setminus C'}$ is a solution of X_C and unique up to banks with no liabilities. □

D.3 No-Red-Containing-Cycle Condition

Proof of Theorem 6.9. We follow the approach in the proof of Theorem 6.3. The statement follows directly from the following lemma for $C = N$. □

Lemma D.6. *Let $X = (N, e, c, \alpha, \beta)$ be a financial system such that in $\mathrm{CD}(X)$, no cycle contains a red edge. Let $C \subseteq N$ be a subset of banks without incoming edges in $\mathrm{CD}(X)$. Then there is a clearing recovery rate vector for X_C.*

Proof. Proof by induction on $|C|$. For $|C| = 0$, i.e., $C = \emptyset$, X_C has no contracts, so $(1, \dots, 1)$ is clearing. So let $|C| > 0$.

Let $D' \subseteq C$ be a subset of banks such that D' is strongly connected with respect to $\mathrm{CD}(X)$ and there is no edge in $\mathrm{CD}(X)$ from D' to $C' := C \setminus D'$. D' exists because the graph of strongly connected components is acyclic. By induction hypothesis, there is a clearing recovery rate vector s' of $X_{C'}$. The financial system $X_C^{C',s'}$ is a green core system: its dependency graph consists of

- the green liability edges from C' to D' in $X_C^{C',s'}$.
- the inner dependencies of D' in X. These must be all green because D' is strongly connected and no cycle contains a red edge.
- the dependencies from C to $D := N \setminus C$. These can be red. By construction, no bank in D has any outgoing dependencies in $X_C^{C',s'}$.

By Theorem 6.6, there is a clearing recovery rate vector t' of $X_C^{C',s'}$. By the Splitting Lemma D.4, $s'|_{C'} \cup t'|_{D'}$ is a solution of X_C. □

E Proofs from Section 7

Proof that Figure 10 does not have a solution for $\beta < 1$. Let $\beta < 1$. First note that the system is well-defined because

$$\begin{aligned}\delta &= 1 - \beta + \beta^2 - \beta^3 \\ &= 1 - \beta + \beta^2(1-\beta) > 0.\end{aligned}$$

Assume towards a contradiction that r is a clearing recovery rate vector. Then it is not possible that $r_A = 1$ because

$$r_A = 1 \Rightarrow r_B = 0 \Rightarrow r_C = 1 \Rightarrow r_A = 0$$

as is easily seen from the contracts. To see that it is also not possible that $r_A < 1$, we do a case distinction.

Case 1: If $\beta = 0$ and $r_A < 1$, then $r_A = 0$ by Lemma B.1. And by the analogous argument as above

$$r_A = 0 \Rightarrow r_B = 1 \Rightarrow r_C = 0 \Rightarrow r_A = 1.$$

Case 2: If $\beta > 0$ and $r_A < 1$, then $r_A < \beta$ by Lemma B.1. Then $a_B(r) > 1 - \beta$, so $r_B > \beta(1-\beta)$. Then $a_C(r) < 1 - \beta(1-\beta)$, so $r_C < \beta(1-\beta(1-\beta))$. Finally,

$$\begin{aligned}a_A(r) &> \delta\left(1 - \beta(1 - \beta(1-\beta))\right) \\ &= \delta\left(1 - \beta + \beta^2 - \beta^3\right) \\ &= 1 = l_A(r).\end{aligned}$$

Contradiction. □