On the Difficulty of Comparing School Choice Mechanisms by Their Vulnerability to Manipulation

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PRELIMINARY OVERVIEW OF RESULTS

Abstract

This note gives three formal results that show the difficulty of comparing two variants of the Boston school choice mechanism via their vulnerability to manipulation (Pathak and Sönmez, 2013).

1 Preliminaries

1.1 Formal Model

A *school choice problem* is a tuple $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$, where

- $I = \{i_1, \ldots, i_n\}$ is the set of n *students*,
- $S = \{s_1, \ldots, s_m\}$ is the set of m *schools*,
- $\boldsymbol{q} = (q_s)_{s \in S}$ is the vector of *school capacities*, where q_s is the number of seats available at school s. There is at least one seat at each school (i.e., $q_s \geqslant 1$ for all $s \in S$),
- $\boldsymbol{P} = (P_i)_{i \in I}$ is the *preference profile* where P_i is the strict *preference order* of student i over the schools S,
- $\boldsymbol{\Pi} = (\pi_s)_{s \in S}$ is the *priority profile* where π_s is the *priority order* of school s over students I.

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If students can be assigned to an *outside option*, we model this by including a school $\emptyset$ in S with $q_\emptyset = n$. A *(deterministic) assignment* is represented by an $n \times m$-matrix $x = (x_{i,s})_{i \in I, s \in S}$, where $x_{i,s} = 1$ if student i is assigned a seat at school s and $x_{i,s} = 0$ otherwise. An assignment x is *feasible* if no school's seats are assigned beyond its capacity so that $\sum_{i \in I} x_{i,s} \leqslant q_s$ for all $s \in S$. A *mechanism* is a mapping φ that takes as input a school choice problem $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$ and determines a feasible assignment $\varphi(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$.

With slight abuse of notation we denote by $\varphi_i(\boldsymbol{P}, \boldsymbol{\Pi})$ the school to which student i is assigned by φ in the problem $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$.

1.2 Boston School Choice Mechanisms

Given a problem $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$, the *naïve Boston mechanism (NBM)* assigns students to schools in rounds. In the first round, each student $i \in I$ "applies" to the school that she prefers most according to P_i. Each school $s \in S$ accepts the applications of students according to the priority order π_s. If s has enough capacity to accommodate all applicants, all applications are accepted. Otherwise, the applications of the students with highest priority under π_s are accepted until all seats are filled. The remaining applications are rejected. All students who did not receive their first choice, enter the second round. In the kth round ($k \geqslant 2$), all unassigned students apply to their kth choice school. Again, the schools accept applications according to their priority orders up to capacity and reject all remaining applications.

Note that under the naïve Boston mechanism it is possible that a student applies to a school that has no more remaining seats available. Thus, the student's application at this school will definitely be rejected, independent of her priority at this school. By ranking this school last instead, this student would apply to all remaining schools earlier, which yields a simple heuristic to manipulate NBM. A second variant of the Boston mechanism lets students skip exhausted schools in the application process. This makes manipulation according to this heuristic obsolete because the mechanism treats students' preferences *as if* the student had ranked the exhausted school last. Formally, the *adaptive Boston mechanism (ABM)* proceeds exactly like NBM, except that in the kth round all unassigned students who have not been assigned so far, apply to the school that they prefer most out of all the schools with strictly positive remaining capacity.

1.3 Comparing Mechanisms by Their Vulnerability to Manipulation

Next, we formally restate the concepts needed to compare mechanisms by their vulnerability to manipulation as formulated by Pathak and Sönmez (2013). In the following let φ and ψ be two mechanisms.

Definition 1 (Manipulability)**.** ψ is *manipulable by student i at problem* $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$ if there exists a preference order $P_i' \neq P_i$ such that $P_i : \psi_i\left(\left(P_i', P_{-i}\right), \boldsymbol{\Pi}\right) \ > \ \psi_i\left(\boldsymbol{P}, \boldsymbol{\Pi}\right)$. ψ is *manipulable at* $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$ if it is manipulable by some student $i \in I$ at $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$. ψ is *manipulable* if it is manipulable at some problem.

Definition 2 (As Manipulable As-relation)**.** ψ is *as manipulable as* φ if whenever φ is manipulable at some problem $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$, then ψ is also manipulable at $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$. ψ is *as strongly manipulable as* φ if whenever φ is manipulable by some student $i \in I$ at some problem $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$, then ψ is also manipulable by the student i at the problem $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$.

Definition 3 (More Manipulable Than-relation)**.** ψ is *more manipulable than* φ if ψ is as manipulable as φ and there exists a problem $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$ at which ψ is manipulable but φ is not. ψ is *more strongly manipulable than* φ if ψ is as strongly manipulable as φ and there exists a student i and a problem $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$ such that ψ is manipulable by i at $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$ but φ is not.

2 Failure of Strict Comparison

Theorem 1. *ABM is as manipulable as NBM.*

The proof is available on request.

Corollary 1. *ABM and NBM are in the same equivalence class with respect to the* as manipulable as-*relation.*

Proof. Dur (2015) has shown that NBM is as manipulable as ABM. With this, the statement follows directly from Theorem 1. □

3 Failure of Strong Comparison

Proposition 1. *NBM is not as strongly manipulable as ABM.*

Proof. Consider a problem $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$ with five students $I = \{1, \ldots, 5\}$ and five schools $S = \{a, b, c, d, e\}$ with a single seat each. Let the preference profile be given by

$$P_1 \ : \ a \succ \ldots, \tag{1}$$
$$P_2 \ : \ b \succ \ldots, \tag{2}$$
$$P_3 \ : \ d \succ \ldots, \tag{3}$$
$$P_4 \ : \ a \succ b \succ d \succ c \succ e, \tag{4}$$
$$P_5 \ : \ a \succ b \succ c \succ d \succ e, \tag{5}$$

and let the priorities be the same at all schools such that $\pi_s : 1 \succ \ldots \succ 5$ for all $s \in S$.

Under the naïve Boston mechanism and truthful reporting we have $\text{NBM}_5(\boldsymbol{P}, \boldsymbol{\Pi}) = c$. Since a and b are taken by students 1 and 2 in the first round and both students have priority over student 5, student 5 cannot improve her assignment by misreporting. Under the adaptive Boston mechanism and truthful reporting we have $\text{ABM}_5(\boldsymbol{P}, \boldsymbol{\Pi}) = e$. However, if student 5 ranks school c in first position (i.e., she reports $P'_5 : c \succ \ldots$), then $\text{ABM}_5((P'_5, P_{-5}), \boldsymbol{\Pi}) = c$. This represents a strict improvement for student 5. Thus, for the problem $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$, ABM is manipulable by student 5 but NBM is not. □

Corollary 2. *NBM and ABM are incomparable by the* as strongly manipulable as-*relation.*

Proof. Proposition 1 already shows that NBM is not as strongly manipulable as ABM. If the comparison was possible, then ABM would have to be more strongly manipulable than NBM. However, a simple example shows that this is not the case: consider a problem $(I, S, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{\Pi})$

with four students $I = \{1, \ldots, 4\}$ and four schools $S = \{a, b, c, d\}$ with a single seat each. Let the preference profile be given by

$$P_1 \;:\; a \succ \ldots, \tag{6}$$

$$P_2 \;:\; b \succ \ldots, \tag{7}$$

$$P_3 \;:\; a \succ b \succ c \succ d, \tag{8}$$

$$P_4 \;:\; a \succ c \succ \ldots, \tag{9}$$

and let the priorities be the same at all schools such that $\pi_s : 1 \succ \ldots \succ 4$ for all $s \in S$. Under the naïve Boston mechanism and truthful reporting we have $\mathrm{NBM}_3(\boldsymbol{P}, \boldsymbol{\Pi}) = d$, but if student 3 reports $P_3' : a \succ c \succ \ldots$ she will be assigned to c. Under the adaptive Boston mechanism and truthful reporting we have $\mathrm{ABM}_3(\boldsymbol{P}, \boldsymbol{\Pi}) = c$. Since a and b are taken by students 1 and 2 in the first round, student 3 cannot improve her assignment by misreporting under ABM. □

4 Failure of Weak Comparison for Random Priorities

4.1 Modeling Random Priorities in School Choice Mechanisms

In many school choice settings, priorities are not strict but coarse. This means that when two students apply for the same seat at some school, then ties between these students must be broken. Normally, this is done using a random tie-breaker. A *random assignment* of the students to the schools is represented by an $n \times m$-matrix $x = (x_{i,s})_{i \in I, s \in S}$ where $x_{i,s} \in [0, 1]$ denotes the probability that student i is assigned to school s. From the perspective of the students, reporting a different preference order leads to a different random assignment. Therefore, we need to extend their preferences, which we do by endowing them with vNM utility functions. Given a preference order P_i, a utility function $u_i : S \to \mathbb{R}^+$ is *consistent with* P_i if $u_i(s) > u_i(s')$ whenever $P_i : s \succ s'$, and we denote by U_{P_i} the set of all vNM utility functions that are consistent with P_i. We assume that students wish to maximize their expected utility.

To model the uncertainty from random tie-breaking, we assume that any mechanism first collects the preference orders of the students and then chooses a priority profile randomly from a distribution $\mathbb{P}$ over priority profiles. We denote the resulting mechanism by $\varphi^{\mathbb{P}}$. $\varphi_i^{\mathbb{P}}(\boldsymbol{P})$ denotes the random assignment vector to student i if the reported preference profile is $\boldsymbol{P}$. This is the ith row of the random assignment matrix. In particular, we denote by $\mathbb{U}$ the uniform distribution over all single priority profiles, that is the priority profiles $\boldsymbol{\Pi} = (\pi, \ldots, \pi)$ where the priority order $\pi_s = \pi$ is the same at all schools.

To study mechanisms in the presence of random priorities we consider *random problems* $(I, S, \boldsymbol{q}, \boldsymbol{u})$. Note that this problem contains no priority profile because determining this profile is now part of the mechanism. Furthermore, the preference profile $\boldsymbol{P}$ is replaced by a utility profile $\boldsymbol{u} = (u_1, \ldots, u_n)$. We consider ordinal mechanisms, which depend only on the ordinal preference profile induced by the utility profile $\boldsymbol{u}$. This means that for two different utility profiles $\boldsymbol{u}, \boldsymbol{u}'$ that are both consistent with the same preference profile $\boldsymbol{P}$, any mechanism $\varphi^{\mathbb{P}}$ has to select the same random assignment. Thus, it is without loss of generality that we consider mechanisms as functions of preference profiles (rather than utility profiles).

4.2 Comparing Random Mechanisms by Their Vulnerability to Manipulation

Since we changed the structure of the problem to accommodate random mechanisms, we also need to re-define the concepts for the comparison of mechanisms by their vulnerability to manipulation. Let $\mathbb{P}$ be a priority distribution and let $\varphi^{\mathbb{P}}$ and $\psi^{\mathbb{P}}$ be two random mechanisms.

Definition 4 (Manipulability (of Random Mechanisms))**.** $\psi^{\mathbb{P}}$ is *manipulable by student i at problem* $(I, S, \boldsymbol{q}, \boldsymbol{u})$ if there exists a preference order $P_i' \neq P_i$ such that $\mathbb{E}_{\psi_i^{\mathbb{P}}(P_i', P_{-i})}[u_i] > \mathbb{E}_{\psi_i^{\mathbb{P}}(\boldsymbol{P})}[u_i]$. $\psi^{\mathbb{P}}$ is *manipulable at* $(I, S, \boldsymbol{q}, \boldsymbol{u})$ if it is manipulable by some student $i \in I$ at the problem $(I, S, \boldsymbol{q}, \boldsymbol{u})$. $\psi^{\mathbb{P}}$ is *manipulable* if it is manipulable at some problem.

Definition 5 (As Manipulable As-relation (for Random Mechanisms))**.** $\psi^{\mathbb{P}}$ is *as manipulable as* $\varphi^{\mathbb{P}}$ if whenever $\varphi^{\mathbb{P}}$ is manipulable at some problem $(I, S, \boldsymbol{q}, \boldsymbol{u})$, then $\psi^{\mathbb{P}}$ is also manipulable at the problem $(I, S, \boldsymbol{q}, \boldsymbol{u})$. $\psi^{\mathbb{P}}$ is *as strongly manipulable as* $\varphi^{\mathbb{P}}$ if whenever $\varphi^{\mathbb{P}}$ is manipulable by some student i at some problem $(I, S, \boldsymbol{q}, \boldsymbol{u})$, then $\psi^{\mathbb{P}}$ is also manipulable by i at $(I, S, \boldsymbol{q}, \boldsymbol{u})$.

4.3 Failure of the Comparison of NBM$^{\mathbb{U}}$ and ABM$^{\mathbb{U}}$

Proposition 2.

- *NBM$^{\mathbb{U}}$ is not as manipulable as ABM$^{\mathbb{U}}$,*
- *ABM$^{\mathbb{U}}$ is not as manipulable as NBM$^{\mathbb{U}}$.*

Proof. We construct a problem $(I^{(1)}, S^{(1)}, \boldsymbol{q}^{(1)}, \boldsymbol{u}^{(1)})$ for which NBM$^{\mathbb{U}}$ is manipulable but ABM$^{\mathbb{U}}$ is not, and we construct a second problem $(I^{(2)}, S^{(2)}, \boldsymbol{q}^{(2)}, \boldsymbol{u}^{(2)})$ for which ABM$^{\mathbb{U}}$ is manipulable but NBM$^{\mathbb{U}}$ is not.

(1) Consider a problem $(I^{(1)}, S^{(1)}, \boldsymbol{q}^{(1)}, \boldsymbol{u}^{(1)})$ with four students $I^{(1)} = \{1, \ldots, 4\}$ and four schools $S^{(1)} = \{a, b, c, d\}$ with a single seat each. Let the preference profile be given by

$$P_1^{(1)} \ : \ a \succ b \succ c \succ d, \tag{10}$$
$$P_2^{(1)}, P_3^{(1)} \ : \ a \succ c \succ b \succ d, \tag{11}$$
$$P_4^{(1)} \ : \ b \succ \ldots. \tag{12}$$

Student 1's assignment vector under NBM$^{\mathbb{U}}$ is $\mathrm{NBM}_1^{\mathbb{U}}(\boldsymbol{P}^{(1)}) = (1/3, 0, 0, 2/3)$ for a, b, c, d, respectively. If student 1 swaps b and c in her report (i.e., she reports $P_1' : a \succ c \succ b \succ d$), her assignment vector changes to $\mathrm{NBM}_1^{\mathbb{U}}(P_1', P_{-1}^{(1)}) = (1/3, 0, 1/3, 1/3)$. Since $\mathrm{NBM}_1^{\mathbb{U}}(P_1', P_{-1}^{(1)})$ first order-stochastically dominates $\mathrm{NBM}_1^{\mathbb{U}}(\boldsymbol{P}^{(1)})$ at P_1, this misreport is an unambiguous improvement for student 1 with true preference order P_1 (independent of her vNM utility function $u_1^{(1)}$).

For settings with four students and four schools with a single seat each, ABM$^{\mathbb{U}}$ is 1/3-partially strategyproof (Mennle and Seuken, 2015). Thus, if all students have utilities $(9, 3, 1, 0)$ for their first, second, third, and last choices, respectively, truthful reporting is a dominant strategy for all of them. With the utility profile $\boldsymbol{u}^{(1)}$ defined in this way NBM$^{\mathbb{U}}$ is manipulable (by student 1) at the problem $(I^{(1)}, S^{(1)}, \boldsymbol{q}^{(1)}, \boldsymbol{u}^{(1)})$ but ABM$^{\mathbb{U}}$ is not manipulable (by any student).

(2) Consider a problem $(I^{(2)}, S^{(2)}, \boldsymbol{q}^{(2)}, \boldsymbol{u}^{(2)})$ with six students $I^{(2)} = \{1, \ldots, 6\}$ and six schools $S^{(2)} = \{a, b, c, d, e, f\}$ with a single seat each. Let the preference profile be given by

$$P_1^{(2)}, P_2^{(2)} \quad : \quad a \succ e \succ c \succ d \succ f \succ b, \tag{13}$$
$$P_3^{(2)}, P_4^{(2)} \quad : \quad a \succ e \succ d \succ c \succ f \succ b, \tag{14}$$
$$P_5^{(2)} \quad : \quad b \succ c \succ \ldots, \tag{15}$$
$$P_6^{(2)} \quad : \quad b \succ d \succ \ldots. \tag{16}$$

Suppose that all students have utilities $(120, 30, 19, 2, 1, 0)$ for their first through sixth choices, respectively.

First, we study the incentives to manipulate under $\mathrm{NBM}^{\mathbb{U}}$. Note that student 1 cannot improve her expected utility by ranking another school than a first. To see this, observe the following: under truthful reporting she obtains a with probability 1/4 but if she ranks a different school first she will at best obtain her second choice e with certainty. Since b is exhausted in the first round, she will not obtain b with any positive probability unless she ranks it first. Therefore, it is a weakly better response to rank b last. Since f is the worst school which she obtains with positive probability, it is a weakly better response to leave f in fifth position. Otherwise, she will only reduce her chances at more preferred schools. Thus, without loss of generality, any beneficial misreport only involves the order of the schools e, c, d. It is a simple exercise to compute the changes in expected utility for student 1 under any such misreport:

Report P_1'	Change in expected utility
$a \succ e \succ d \succ c \succ f \succ b$	-2.1
$a \succ c \succ e \succ d \succ f \succ b$	-0.4
$a \succ c \succ d \succ e \succ f \succ b$	-0.3
$a \succ d \succ e \succ c \succ f \succ b$	-9.5
$a \succ d \succ c \succ e \succ f \succ b$	-8.1

This shows that truthful reporting is a best response for student 1. The same is true for student 2, 3, and 4 by symmetry. Student 5 receives her first and second choice with probabilities 1/2 each. Thus, the only way she can improve her expected utility is by increasing her probability for her first choice but this is obviously impossible. The same is true for student 6 by symmetry. In combination this means that for the problem $(I^{(2)}, S^{(2)}, \boldsymbol{q}^{(2)}, \boldsymbol{u}^{(2)})$ the mechanism $\mathrm{NBM}^{\mathbb{U}}$ is not manipulable.

Second, we study the incentives to manipulate under $\mathrm{ABM}^{\mathbb{U}}$ for the same problem: under truthful reporting, student 1's assignment vector is $\mathrm{ABM}_1^{\mathbb{U}}(\boldsymbol{P}^{(2)}) = (1/4, 1/4, 1/8, 1/8, 1/4, 0)$ for the schools a, e, c, d, f, b, respectively. If student 1 reported $P_1' : a \succ c \succ d \succ e \succ f \succ b$ instead, her assignment vector would be $\mathrm{ABM}_1^{\mathbb{U}}(P_1', P_{-1}^{(2)}) = (1/4, 0, 71/120, 3/40, 1/12, 0)$. This means an increase in expected utility from 40.375 to 41.475, a strict improvement.

This concludes the proof. $\square$

References

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