Trade-offs in School Choice: Comparing Deferred Acceptance, the Naïve and the Adaptive Boston Mechanism*

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Abstract

So far, research on school choice mechanisms has focused primarily on two procedures: the Deferred Acceptance (DA) mechanism and the (naïve) Boston mechanism (NBM). A variant of the Boston mechanism, where students automatically skip exhausted schools in the application process, has been largely overlooked by researchers, but is nonetheless frequently used in practice. We formalize this adaptive Boston mechanism (ABM), and we study the three mechanisms DA, NBM, and ABM in the presence of coarse priorities and random tie-breaking. When priorities are determined by a single, uniform lottery, we find that the three mechanisms form two hierarchies: one with respect to strategyproofness, and one with respect to efficiency: DA is known to be strategyproof while NBM is not even weakly strategyproof, and we show in this paper that ABM satisfies the intermediate requirement of partial strategyproofness. Regarding efficiency, we show that NBM rank dominates DA whenever a comparison is possible. Furthermore, using new limit arguments and simulations, we establish that ABM has intermediate efficiency between NBM and DA. Our results reveal the inherent trade-off between strategyproofness and efficiency that market designers face when choosing between these three school choice mechanisms. While all of our results hold when priorities are determined by a single uniform lottery, we prove that many continue to hold for general priority structures and other tie-breakers as well.

Keywords: School Choice, Matching, Assignment, Boston Mechanism, Deferred Acceptance, Strategyproofness, Partial Strategyproofness, Rank Dominance, Rank Efficiency

JEL: C78 Bargaining Theory, Matching Theory; D47 Market Design; D78 Positive Analysis of Policy Formulation and Implementation

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1. Introduction

Each year, millions of children enter a new public school. However, the capacities of schools are limited and therefore, the students’ individual wishes can almost never be accommodated perfectly. When students are allowed to express preferences over schools, administrators face the challenge of designing a *market* with scarce resources (i.e., seats at public schools) on one side and agents with diverse preferences over these resources on the other side (i.e., students). In particular, administrators must devise a *school choice mechanism*, i.e., a procedure that determines an assignment of students to schools, taking into account the students’ preferences, but usually without any monetary transfers. Since the seminal paper by Abdulkadiroğlu and Sönmez (2003), school choice mechanisms have attracted the attention of economists, and a growing body of research has had a substantial impact on policy decisions.

1.1. Boston ‘versus’ Deferred Acceptance Mechanism

Two particular mechanisms have received the lion’s share of the attention: the *Boston mechanism* and the *Deferred Acceptance mechanism* (DA).\(^1\) Both mechanisms collect preference reports from the students and then assign students to seats in rounds.

Under the Boston mechanism, students apply to their favorite school in the first round. If a school has sufficient capacity to accommodate all applications in the first round, all applicants are accepted permanently. Otherwise, the school accepts applications following some priority order until its capacity is exhausted, and then it rejects all remaining applications. Students who were rejected in the first round apply to their second choice school in the second round. The process then repeats until all students have received a school or all schools have reached capacity. Variants of the Boston mechanism are ubiquitous in school choice settings around the world, e.g., in Spain (Calsamiglia and Güell, 2014), in Germany (Basteck, Huesmann and Nax, 2015), and in many school districts in the United States (Ergin and Sönmez, 2006).

The main motivation for letting parents choose the schools for their children through a school choice mechanism is *student welfare*. Popular “measures” for student welfare are the number of assigned first choices, number of students who get their second choice or better, etc. Intuitively, the Boston mechanism fares well on this criterion if parents submit their preferences truthfully: it assigns as many applicants as possible to their first choices, then does the same with second choices in the second round, and so on. The Boston mechanism owes much of its popularity to this intuitive way in which it attempts to increasing student welfare.

On the other hand, it is susceptibility to strategic manipulation by students. In particular, it was found to disadvantage honest participants, and the equilibria of the induced type revelation game may have undesirable welfare properties (Ergin and Sönmez, 2006). Concerns about the manipulability of the Boston mechanism led to its abandonment in some cities in the US and around the world; and in England, “first-preference-first” mechanisms (essentially the Boston mechanism) were even declared illegal in 2007, because they were believed to give unfair advantage to more sophisticated parents (Pathak and Sönmez, 2013).

\(^1\) As of 2005 the Boston mechanism is no longer used in Boston (Abdulkadiroğlu et al., 2006). However, the name stuck, even though *Immediate Acceptance* mechanism would more accurately differentiate it from DA.
The (Student Proposing) Deferred Acceptance mechanism (DA) has been considered as an alternative to the Boston mechanism. Under DA, students also apply to schools in rounds. However, the acceptance at any school is tentative rather than final. If in any subsequent round a student applies to a school with no free capacity, she is not automatically rejected. Instead, she will be accepted at that school if another student who has been tentatively accepted at the same school has lower priority. In this case, the student with lowest priority is rejected and enters the next round. In each round, previously rejected students continue applying to the school they prefer most out of all the schools that have not rejected them yet. When no more new applications are received by any school, all tentative assignments are finalized.

DA makes truthful reporting a dominant strategy for students, which alleviates concerns about strategic manipulation. However, this strategyproofness comes at a cost: unlike the Boston mechanism, DA does not maximize the assigned number of first choices, then subsequently the number of second choices, etc. In this paper, we capture this efficiency difference formally: we prove that the Boston mechanism rank dominates Deferred Acceptance whenever the assignments are comparable at a given preference profile. To the best of our knowledge, we are the first to give a result of this kind, and we show that it holds in general for coarse priorities and random tie-breaking.

1.2. The Adaptive Boston Mechanism

So far, research on the Boston mechanism has largely focused on the traditional, “naïve” Boston mechanism (NBM) described above, where students apply to their kth choice school in the kth round. However, the Boston mechanism is sometimes implemented in a subtly different fashion: instead of applying to their kth choice in the kth round, in each round students apply to their most preferred school that still has available capacity.

For example, in the city of Freiburg, Germany, approximately 1’000 students enter one of ten secondary schools each year. Initially, they are asked to apply to their first choice school. If this application is successful, their assignment is finalized. Students whose application was rejected receive a list of schools that still have seats available. They are then asked to apply to one of the schools from this list in the second round, and this process repeats in subsequent rounds. The procedure resembles the Boston mechanism, except that students are barred from applying to schools that have no more remaining open seats. This alteration leads to an adaptive Boston mechanism (ABM), which eliminates the risk of “wasting one round” by applying to an already exhausted school. In the German state of North Rhine-Westphalia, the school choice process also exhibits such an adaptive characteristic: parents receive a single slip of paper that they submit to their preferred school. If their application at that school is rejected, the application slip is returned and they can use it to apply to another school. However, before they apply to another school, parents are encouraged to call ahead and verify whether their next best choice school still has seats available. The adaptive variant of the Boston mechanism, where exhausted schools are automatically skipped in the application process, is also in use for admissions to secondary schools in Amsterdam (de Haan et al., 2015).

On the one hand, ABM removes some obvious opportunities for manipulation that exist under NBM. On the other hand, a student can obtain her third choice in the second round, which may
prevent another student from getting her second choice in that round. Consequently, in terms of strategyproofness and efficiency, one would expect the adaptive Boston mechanism to take an intermediate position between the Deferred Acceptance and the naïve Boston mechanism. In this paper, we formalize and prove this intuition, thereby establishing ABM as a median alternative when weighing strategyproofness and efficiency against each other in the design of school choice mechanisms.

1.3. Coarse Priorities in School Choice

The mechanisms DA, NBM, and ABM are essentially two-sided matching mechanisms, but they do not operate in a two-sided market. Typical examples of two-sided markets are entry-level labor markets for medical professionals, where residents must be matched to hospitals, or markets for higher education, where prospective university students must be matched to university placements. In these markets, each side has preferences over the respective other side, and both sides may misreport strategically. DA, NBM, and ABM process rank-order lists from both sides of the market to compute the assignment. However, in contrast to two-sided markets, schools in school choice markets often don’t express preferences over students, but priorities take the place of their preferences in the mechanism.²

In two-sided markets, stability of the matching is often essential to prevent unraveling, i.e., to prevent situations in which some participants can benefit by breaking away and matching outside the mechanism. While stability is certainly an intriguing property, the constraint loses significance in most school choice markets, as schools usually have no way of acting strategically. In particular, administrators have sufficient control over schools to ensure that they do not circumvent the procedure and match with students whom they prefer outside of the mechanism. In this paper, we put stability aside and focus on the properties strategyproofness and efficiency.

Most results about school choice mechanisms have been obtained under the assumption that priorities are fixed and strict at all schools, i.e., assuming some non-strategic input from the school-side of the market. However, as Kojima and Ünver (2014) pointed out, this assumption is almost always violated: priorities are typically coarse, e.g., based only on neighborhoods or siblings. Recently, the role of priorities has been further de-emphasized; in Boston, for example, walk-zone priorities were abandoned in 2013 (Dur et al., 2014). Coarse priorities put many students in the same priority class. This necessitates tie-breaking when two students with equal priority compete for a seat at some school, which introduces randomization into the mechanism. In this paper, we explicitly model this uncertainty by considering the probabilistic assignments that arise before the tie-breaking procedure has been implemented.

The “opposite” assumption to fixed, strict priorities is that all students are part of a single priority class and ties are broken randomly according to a single, uniform lottery (call this assumption U). While this is usually only approximately true, there also exist school choice markets where assumption U is satisfied completely: in 1999, all 15 neighborhoods of the Beijing Eastern City District used the naïve Boston mechanism to assign students to middle schools with priorities determined according to U (Lai, Sadoulet and de Janvry, 2009). Furthermore,

²One notable is the school choice market in New York City, where principles develop preferences over students, e.g., based on reading scores or attendance (Abdulkadiroğlu, Pathak and Roth, 2005).
the second phase of the school choice procedure in New York City used Deferred Acceptance with priorities determined by a single, uniform lottery (Pathak and Sethuraman, 2011). Finally, most cities in Estonia employ this procedure for the assignment of children to elementary schools (Lauri, Pöder and Veski, 2014). All findings in this paper hold at least under assumption \( U \), but most of them generalize to arbitrary priority structures with or without randomization.

1.4. A Motivating Example

In this paper we study a trio of popular school choice mechanisms: Deferred Acceptance, the naïve, and the adaptive Boston mechanism. We uncover their relationship in terms of strategyproofness and efficiency. To obtain an intuition about this relationship, consider a market with 4 students \( t_1, t_2, t_3, t_4 \) and 4 schools \( a, b, c, d \) with a single seat each. Suppose the preferences are

\[
P_1 : a > b > c > d, \tag{1} 
\]

\[
P_2, P_3 : a > c > b > d, \tag{2} 
\]

\[
P_4 : b > a > c > d, \tag{3} 
\]

where \( P_i : x > y \) indicates that student \( i \) prefers school \( x \) to school \( y \). Furthermore, suppose that there are no priorities, and ties are broken using a single uniform lottery. The interim assignments, i.e., the probabilities of each student obtaining each of the seats when applying the different mechanisms to the preference profile \( P = (P_1, P_2, P_3, P_4) \) are the following:

<table>
<thead>
<tr>
<th>NBM</th>
<th>a</th>
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Incentives for Truth-telling:

- DA is known to be strategyproof for students, and therefore, no student can improve her assignment by misreporting.

- Under NBM, when reporting truthfully, student 1 has no chance of obtaining her second choice \( b \) or third choice \( c \). By swapping \( b \) and \( c \) in her report, she would receive \( a, c \), or \( d \) with probability 1/3 each. This is an unambiguous improvement, independent of her preference intensities, i.e., by misreporting she can obtain an assignment that stochastically dominates her assignment under truthful reporting.

- Swapping \( b \) and \( c \) is no longer a beneficial manipulation for student 1 under ABM; the misreport would have no effect on her assignment, because ABM automatically skips the exhausted school \( b \) for student 1 anyways. However, by ranking school \( c \) in first position, she can receive \( c \) with certainty (holding the other students’ preferences fixed). If she
had utility of 15, 10, 9, 0 for $a, b, c, d$, respectively, her expected utility would improve from 8 to 9. However, if her utility for $c$ was 6 instead, the expected utility would decrease from 7 to 6. Thus, whether or not ranking $c$ first is a useful manipulation for student 1 depends on “how strongly” she prefers $a$ over $c$.

**Student Welfare:** Consider the rank distributions under the different mechanisms, i.e., the expected numbers of students, who receive there $k$th choice for $k = 1, 2, 3, 4$. These are:

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<th>Mechanism</th>
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<td>NBM</td>
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<td>ABM</td>
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<td>DA</td>
<td>5/3</td>
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Observe that ABM and NBM assign the same number of first choices, but ABM assigns strictly fewer second choices and strictly more third choices. Thus, the assignment under NBM is “more efficient” in the formal sense that its rank distribution dominates the rank distribution of ABM ((Featherstone, 2011)). Similarly, DA assigns a lower number of first choices and strictly more second choices than ABM, but for ranks 3 and 4 the rank distributions coincide. Thus, the rank distribution of ABM dominates the rank distribution of DA. Consequently, with the rank distribution as a criterion for student welfare, NBM is more efficient than ABM, which in turn is more efficient than DA.

The above example provides the intuition for our main result that NBM, ABM, and DA form a hierarchy with respect to strategyproofness with DA being fully strategyproof, NBM being manipulable in a stochastic dominance sense, and ABM taking an intermediate position; but at the same time they also form a hierarchy with respect to efficiency with NBM being the most efficient mechanism, DA being least efficient, and ABM again taking an intermediate position.

### 1.5. Overview of Contributions

In this paper we show that this intuition generalizes beyond the comparison at the particular preference profile chosen in the motivating example, but in a non-trivial way. Our contributions (graphically depicted in Figure 1) are the following:
1. **Strategyproofness Hierarchy:** It is well-known that DA is strategyproof while NBM is not even weakly strategyproof; by showing that ABM satisfies the intermediate concept of *partial strategyproofness*, we formally position this mechanism between NBM and DA with respect to incentives for truth-telling. This distinction is contrasted by the fact that a comparison of ABM and NBM by their vulnerability to manipulation is inconclusive when priorities are coarse.

2. **“Cost of Strategyproofness:”** Regarding efficiency, we prove that NBM rank dominates DA whenever the two mechanisms are comparable, which formally captures the intuition that NBM is more efficient than DA. This distinction holds for *any* priority structure and tie-breaker, while ex-post efficiency is not a separating property, e.g., under assumption $\mathcal{U}$.

3. **Intermediate Efficiency of ABM:** We demonstrate the surprising fact that a direct efficiency comparison of ABM to either of the other two mechanisms is ambiguous. Despite this incomparability, we recover the assertion that ABM has *intermediate* efficiency between DA and NBM via limit results and simulations.

From a broader perspective, our results yield the important take-home message that choosing between NBM, ABM, and DA for school choice markets remains a question of trading off strategyproofness and efficiency: if strategyproofness is a hard requirement, DA is the mechanism of choice. When the weaker partial strategyproofness is also acceptable, ABM can be employed to harness improvements in the rank distribution. Finally, if manipulability is not a concern, NBM offers further efficiency gains over both DA and ABM. We do not advocate superiority of either mechanism, but instead our insights allow mechanism designers to make a conscious and informed decision about this trade-off.

**Remark 1.** We would like to point out that the present paper differs substantially from two related papers (Dur, 2015; Harless, 2015), and we highlight the distinctions in our discussion of related work.

**Organization of this paper:** In Section 2 we discuss related work. In Section 3, we introduce our formal model and basic concepts, and in Section 4 we formally define the mechanisms DA, NBM, and ABM. In Sections 5 and 6 we compare them by their incentive and efficiency properties, respectively, and Section 7 concludes.

2. **Related Work**

The naïve Boston mechanism has received significant attention because it is frequently used for the assignment of students to public schools in many school districts around the world. The mechanism has been heavily criticized for its manipulability: for the case of strict priorities, Abdulkadiroğlu and Sönmez (2003) showed that NBM is neither strategyproof nor stable, and they suggested the Deferred Acceptance mechanism (Gale and Shapley, 1962) as an alternative that is stable and strategyproof for students. Ergin and Sönmez (2006) showed that with full information, the NBM has undesirable equilibrium outcomes. Experimental studies, such as those conducted by Chen and Sönmez (2006) and Pais and Pinter (2008), revealed that it is indeed manipulated more frequently by human subjects than strategyproof alternatives.
Kojima and Ünver (2014) provided an axiomatic characterization of the naïve Boston mechanism for the case of fixed, strict priorities. However, they also pointed out that the assumption of fixed, strict priorities is usually violated in school choice problems. Some recent work has considered coarse priorities, uncovering a number of surprising properties: Abdulkadiroğlu, Che and Yasuda (2015) demonstrated that in a setting with no priorities and perfectly correlated preferences, NBM can lead to higher ex-ante welfare than Deferred Acceptance in equilibrium. Similarly, simulations conducted by Miralles (2008) illustrated that with single uniform tie-breaking and no priorities, equilibria of the naïve Boston mechanism can yield higher welfare ex-ante. It has remained an open research question, if and how the Boston mechanism can be understood to have preferable efficiency for general priority structures. The present paper addresses this question: we show that it rank dominates the Deferred Acceptance mechanism whenever the two mechanisms are comparable at a given preference profile.

While the majority of prior work was focused on the naïve Boston mechanism, the idea of an adaptive adjustment has previously been discussed as well. Alcalde (1996) studied a “now-or-never” mechanism for two-sided marriage markets, where men propose to their most preferred available partner in each round. Miralles (2008) informally argued that an adaptive order of applications may improve the position of unsophisticated (i.e., truthful) students. For the case when priorities are strict and fixed, Dur (2015) provided an axiomatic characterization of the adaptive Boston mechanism and showed that it is less manipulable than NBM in the sense of (Pathak and Sönmez, 2013). Harless (2015) independently provided a similar manipulability comparison and gave additional insights about robustness and solidarity properties of ABM. The present paper differs from both papers (Dur, 2015) and (Harless, 2015) in three ways: first, we consider the more general problem where priorities can be coarse and ties may be broken randomly. Second, we demonstrate that in the more general domain, a comparison by vulnerability to manipulation remains inconclusive; instead, we successfully apply the new concepts of partial strategyproofness and comparable dominance to obtain a meaningful comparison. Third, we give more results, including the intermediate efficiency of ABM through limit results and simulations, the proofs that neither ABM nor NBM lie on the efficient frontier; and our counter-examples to illustrate in-comparabilities are in parts stronger than those in (Harless, 2015). Our findings yield a comprehensive understanding of the hierarchical relationships of all three mechanisms on both dimensions of strategyproof and efficiency. To the best of our knowledge, ours is the first paper to consider the adaptive Boston mechanism in the general domain. We are also the first to formally establish its intermediate position in terms of strategyproofness and efficiency between DA and NBM.

Severe impossibility results restrict the design of school choice mechanisms that are strategyproof, efficient, and fair at the same time. Under the assumption U, i.e., single uniform tie-breaking and no priorities, the school choice problem becomes formally equivalent to the random assignment problem (Hylland and Zeckhauser, 1979). Zhou (1990) showed that one cannot hope to design a mechanism for the random assignment problem that is strategyproof, ex-ante efficient, and anonymous. While Random Serial Dictatorship is at least strategyproof, anonymous, and ex-post efficient, it is conjectured to be the unique mechanism with these properties (Lee and Sethuraman, 2011; Bade, 2014). Bogomolnaia and Moulin (2001) introduced the Probabilistic Serial mechanism, which is ordinaly efficient, but only weakly strategyproof. Moreover, they showed that no strategyproof, symmetric mechanism can also be ordinaly
efficient. Finally, Featherstone (2011) formalized rank efficiency, which is a strict refinement of ordinal efficient. He presented Rank Value mechanisms, which are rank efficient, but he also showed that strategyproofness and rank efficiency are incompatible, even without additional fairness requirements. Since the random assignment problem is a special case of the school choice problem, these restrictive impossibility results also apply to school choice mechanisms. Thus, one cannot hope to design school choice mechanisms that achieve the optimum on all dimensions, but instead trade-offs are called for. Our finding that NBM, ABM, and DA form hierarchies with respect to strategyproofness and efficiency reveal the trade-offs that are implicit in any decision between these mechanisms.

3. Preliminaries

3.1. Basic Notation

Let there be $n$ students, denoted by the set $N$, and let there be $m$ schools, denoted by the set $M$. We will usually use $i$ to refer to particular students and $j$ or $a, b, c, \ldots$ to refer to particular schools. Each school $j$ has a capacity of $q_j$ seats, and we assume that there are enough seats to accommodate all students at some school, i.e., $\sum q_j = m$ (otherwise we can add a dummy school with capacity greater than $n$). Students have strict preferences $P_i$ over schools, where $P_i: a > b$ means that $i$ prefers school $a$ over school $b$. The set of all possible preference orders is denoted by $P$. A preference profile $P = (P_1, \ldots, P_n) \in \mathcal{P}^N$ is a collection of preferences of all students, and we denote by $P_{-i}$ the collection of preferences of all students except $i$, so that $P = (P_i, P_{-i})$.

3.2. Deterministic and Probabilistic Assignments

A deterministic assignment of students to schools is represented by an $n \times m$-matrix $x = (x_{i,j})_{i \in N, j \in M}$, where the entry $x_{i,j}$ is equal to 1 if student $i$ holds a seat at school $j$, and 0 otherwise. An assignment $x$ is feasible if all students receive a seat at some school and no school's seats are assigned beyond its capacity, i.e., $\sum_{j \in M} x_{i,j} = 1$ for all students $i \in N$ and $\sum_{i \in N} x_{i,j} \leq q_j$ for all schools $j \in M$; and we denote by $X$ the set of all feasible assignments.

By virtue of the Birkhoff-von Neumann Theorem (Birkhoff, 1946) and its extensions (Budish et al., 2013), it suffices to consider the matrix representation of probabilistic assignments, as they are always implementable. In fact, the way in which we construct DA, NBM, and ABM prescribe a canonical way of implementing the resulting probabilistic assignments.
3.3. Priorities and Random Tie-breaking

The mechanisms we consider in this paper are essentially two-sided matching mechanisms. However, in school choice markets, schools usually do not report preferences over students, but instead, priorities take the role of preferences on the school-side. Typically, some coarse priority requirements are imposed exogenously, e.g., based on neighborhoods or siblings, and the remaining ties are broken randomly. We model this structure explicitly by introducing admissible priority distributions.

A priority order $\pi$ is a strict ordering of the students, where $i \pi i'$ means that student $i$ has priority over student $i'$, and we denote by $\Pi$ the set of all possible priority orders. A priority profile is a collection of priority orders $\pi = (\pi_j)_{j \in M} \in \Pi^M$, where each priority order $\pi_j$ is associated with the school $j$. These priority profiles take the place of the preferences of the schools in the mechanisms that we consider.

We model exogenous priority requirements (such as neighborhood or sibling priorities) by restricting the set of admissible priority profiles $\Pi$. As an example, suppose that all students from the neighborhood of school $j$ ($N_j$, say) should have priority over any other students at $j$. In this case, the set of admissible priority profiles $\Pi$ would consist only of those $\pi = (\pi_j, \pi_{-j})$ for which $\pi_j$ gives preference to students from $N_j$, i.e., for all $i \in N_j, i' \in N \setminus N_j$, we have $i \pi_i j, i'$. While a particular set $\Pi$ can reflect coarse exogenous priority requirements, random tie-breaking introduces uncertainty about which priority profile is ultimately used by the mechanism. A priority distribution $P$ is a probability distribution over the set of priority profiles $\Pi$, and $P$ is said to be $\Pi$-admissible if the support of $P$ only contains admissible priority profiles, i.e.,

$$\text{supp}(P) = \{ \pi \in \Pi^M \mid P[\pi] > 0 \} \subseteq \Pi. \quad (4)$$

Throughout the paper, we will use admissible priority distributions to incorporate coarse priority requirements as well as random tie-breaking into the mechanisms we study.

As mentioned in the introduction, some notable school choice markets operate with no priority requirements and tie-breaking by a single, uniform lottery, while in many markets, this reflects the situation at least approximately. A single priority profile is a priority profile $\pi$ that selects the same priority order at all schools, i.e., $\pi = (\pi, \ldots, \pi)$ for some $\pi \in \Pi$ (otherwise $\pi$ is a multiple priority profile). $P$ is a single priority distribution if it only randomizes over single priority profiles, i.e., $\text{supp}(P)$ contains only single priority profiles. If in addition, ties are broken uniformly at random, we obtain the single uniform priority distribution, denoted $U$, which is the uniform distribution over all single priority profiles. Formally,

$$U[\pi] = \begin{cases} \frac{1}{m!}, & \text{if } \pi = (\pi, \ldots, \pi) \text{ with } \pi \in \Pi, \\ 0, & \text{else.} \end{cases} \quad (5)$$

Under $U$, all students are part of a single, large priority class, each student draws a unique random number, and conflicting preferences at any school are resolved using these numbers.
3.4. Construction of School Choice Mechanisms

The mechanisms we study in this paper arise by specifying the way in which they handle the students’ preferences for each priority profile: a deterministic school choice mechanism is a mapping

\[ \varphi : \Pi^M \times \mathcal{P}^N \rightarrow X. \] (6)

that receives as input a priority profile \( \pi \in \Pi^M \) and student preferences \( P \in \mathcal{P}^N \) and selects a deterministic assignment. For a given random priority distribution \( \mathbb{P} \) the corresponding probabilistic school choice mechanism (or just mechanism for short) is the mapping

\[ \varphi^\mathbb{P} : \mathcal{P}^N \rightarrow \Delta(X) \] (7)

which receives the students’ preferences \( P \) as input and selects the probabilistic assignment

\[ \varphi^\mathbb{P}(P) = \sum_{\pi \in \Pi^m} \varphi(\pi, P) \cdot \mathbb{P}[\pi]. \] (8)

In practice, this mechanism can easily be implemented by the following procedure: first, collect the preference reports \( P \) from the students and choose a priority profile \( \pi \) randomly according to \( \mathbb{P} \), but independently of \( P \). Then assign students to schools according to \( \varphi(\pi, P) \). All mechanisms defined in the next section are constructed in this way. With slight abuse of notation we will also write \( \varphi^\pi \) for the deterministic mechanism \( \varphi(\pi, \cdot) \). For the sake of readability, we sometimes omit the superscript \( \mathbb{P} \) and simply write \( \varphi \), when \( \mathbb{P} \) is arbitrary or clear from the context.

3.5. Incentive Constraints

We briefly review the two common incentive requirements strategyproofness and weak strategyproofness. Let student \( i \in N \) have preference order \( P_i \), and let \( x_i \) and \( y_i \) be two assignment vectors for \( i \). We say that \( x_i \) (first order)-stochastically dominates \( y_i \) at \( P_i \) if for all schools \( j \in M \), \( i \) is at least as likely to receive a school she prefers to \( j \) under \( x_i \) than under \( y_i \). Formally, for all \( j \in M \)

\[ \sum_{j' \in M \mid P_i, j' > j} x_{i} \geq \sum_{j' \in M \mid P_i, j' > j} y_{i}. \] (9)

\( x_i \) strictly stochastically dominates \( y_i \) at \( P_i \) if inequality (9) is strict for some \( j \).

**Definition 1** (Strategyproofness). A mechanism \( \varphi \) is strategyproof if misreporting ones preferences leads to an assignment vector that is stochastically dominated by the assignment vector obtained from truthful reporting. Formally, for all students \( i \in N \), all preference profiles \( P = (P_i, P_{-i}) \in \mathcal{P}^N \), and all misreports \( P_i' \in \mathcal{P} \), strategyproofness requires that \( \varphi_i(P_i, P_{-i}) \) (weakly) stochastically dominates \( \varphi_i(P'_i, P_{-i}) \) at \( P_i \).

Bogomolnaia and Moulin (2001) introduced weak strategyproofness to describe the incentives under the non-strategyproof Probabilistic Serial mechanism. While strategyproofness requires that students are unambiguously worse-off when misreporting, weak strategyproofness captures the idea that they may not want to manipulate as long as the misreport does not make them unambiguously better off.
Definition 2 (Weak Strategyproofness). A mechanism $\varphi$ is weakly strategyproof if no student can obtain a strictly stochastically dominant assignment vector by misreporting her preferences, i.e., $\varphi_i(P_i, P_{-i})$ is never strictly stochastically dominated by $\varphi_i(P_i', P_{-i})$ at $P_i$.

4. School Choice Mechanisms

We now introduce the three mechanisms that we study in this paper, the Deferred Acceptance mechanism and two variants of the Boston mechanism.

4.1. Deferred Acceptance

For any preference profile $P$ and priority profile $\pi$, the (Student Proposing) Deferred Acceptance mechanism selects the assignment $DA(\pi, P)$ as follows:

**Round 1:** All students apply to their first choice according to $P_i$. Each school $j$ then processes all applications it has received:
- If $j$ has sufficient capacity, all applications to $j$ are *tentatively* accepted.
- If $j$ has less capacity than applications, it *tentatively* accepts applications from students with highest priority according to $\pi_j$ until all seats are filled. All other applications are *permanently* rejected.

**Round $k$:** Students, who were not tentatively accepted at some school at the end of round $k - 1$ apply to the best school (according to $P_i$) that has not permanently rejected them so far. The set of candidates at school $j$ is comprised of the new applicants at $j$ as well as all students who were tentatively accepted at $j$ at the end of round $k - 1$.
- If $j$ has sufficient capacity, all candidates are tentatively accepted.
- If $j$ has less capacity than candidates, it tentatively accepts candidates with highest priority according to $\pi_j$ until all seats are filled. All other candidates are permanently rejected.

**Last Round:** When no school receives new applications, the current assignment given by the tentative acceptances is finalized.

An important aspect of DA is that students’ tentative acceptances can be revoked in subsequent rounds if this is necessary to accommodate applications from students with higher priority.

Remark 2. For a single priority distribution $P$, the mechanism $DA^P$ is equivalent to a Serial Dictatorship (SD) mechanism, where the order in which the students get to pick their schools is the (single) priority order drawn from $P$ (Erdil, 2014). This makes the school choice problem becomes formally equivalent to the random assignment problem (Hylland and Zeckhauser, 1979; Bogomolnaia and Moulin, 2001; Featherstone, 2011). For the special case of the single uniform priority distribution $U$, this yields the well-known Random Serial Dictatorship (RSD) mechanism, i.e., $DA^U = RSD$ (Morrill, 2013). Thus, our findings shed light on the trade-offs between strategyproofness and efficiency in the random assignment domain as well.
4.2. Naïve Boston Mechanism

For any preference profile $P$ and priority profile $\pi$, the naïve Boston mechanism selects the assignment $\text{NBM}(\pi, P)$ as follows:

**Round 1:** All students apply to their first choice according to $P_i$. Each school $j$ then processes all applications it has received:

- If $j$ has sufficient capacity, all applications to $j$ are *permanently* accepted.
- If $j$ has less capacity than applications, it *permanently* accepts applications from students with highest priority according to $\pi_j$ until all seats are filled. All other applications are *permanently* rejected.

**Round $k$:** All students who have not been permanently accepted at some school in rounds 1, . . . , $k - 1$ apply to their $k$th choice school according to $P_i$; analogous to round 1, applicants are permanently accepted into the unoccupied seats at each school $j$ according to $\pi_j$. When all seats at $j$ are filled, the remaining applicants are permanently rejected.

This mechanism works similarly to DA, but in contrast to DA, the acceptance of a student in any round is final and cannot be revoked in subsequent rounds. For a priority distribution $\mathcal{P}$, we denote this mechanism by $\text{NBM}^\mathcal{P}$.

4.3. Adaptive Boston Mechanism

Under the “traditional” naïve Boston mechanism, students may apply to schools that have no more unfilled seats. When applying to such an exhausted school, a student has no chance of obtaining a seat at that school, independent of her priority at that school. While the student is making her futile application, the capacities of the other schools are further reduced. Thus, the student effectively wastes one round in which she could have competed for other schools instead. Under the adaptive Boston mechanism (ABM), students automatically skip exhausted schools and instead apply to their most preferred available schools in each round. Thus, a student’s application may still be rejected, but the rejection depends on her priority.

For a given preference profile $P$ and priority profile $\pi$, the adaptive Boston mechanism selects the assignment $\text{ABM}(\pi, P)$ as follows:

**Round 1:** All students apply to their first choice according to $P_i$. Each school $j$ then processes all applications it has received:

- If $j$ has sufficient capacity, all its applications are *permanently* accepted.
- If $j$ has less capacity than applications, it *permanently* accepts applications from students with highest priority according to $\pi_j$ until all seats are filled. All other applications are *permanently* rejected.

**Round $k$:** All students who have not been permanently accepted at some school in rounds 1, . . . , $k - 1$ apply to their best choice according to $P_i$ out of those schools that have positive remaining capacity; analogous to round 1, applicants are permanently accepted into the unoccupied seats at each school $j$ according to $\pi_j$. When all seats at $j$ are filled, the remaining applicants are permanently rejected.
For a priority distribution $P$, we denote this mechanism by $\text{ABM}^P$.

5. Incentives for Truth-telling

In this section, we present our first main result that ABM has intermediate incentive properties: while it may be manipulable in general, it satisfies the novel concept of *partial strategyproofness* (Mennle and Seuken, 2014). Broadly speaking, this establishes a hierarchy of manipulability between DA, ABM, and NBM. Our finding contrasts the rather surprising fact that a comparison by *vulnerability to manipulation* (Pathak and Sönmez, 2013) works for fixed, strict priority profiles, but remains inconclusive for general priority distributions.

5.1. Failure of Comparison by Vulnerability to Manipulation

It is well-known that DA is strategyproof, while NBM is not even weakly strategyproof (see Proposition 3 in Appendix C.1). Even though ABM is not fully strategyproof, intuitively, it should have better incentive properties than NBM: under ABM students automatically skip exhausted schools, which removes some obvious opportunities for manipulation. This intuition is further supported by the motivating example in the introduction. Pathak and Sönmez (2013) have introduced a method for comparing mechanisms by their *vulnerability to manipulation*. For the case of strict, fixed priority profiles $\pi \in \Pi^M$, Dur (2015) and Harless (2015) showed independently that NBM$^\pi$ is more manipulable than ABM$^\pi$ in this sense. Unfortunately, when ties are broken randomly, as is common in school choice, this result is no longer true. In Appendix A, we give Examples 1 and 2, which show that the comparison by the *as manipulable as*-relation is inconclusive for NBM$^U$ and ABM$^U$. Specifically, Example 1 gives a preference profile (and consistent vNM utilities), such that truthful reporting is a best response for all students under ABM$^U$, but under NBM$^U$ some student can improve her expected utility by misreporting. Conversely, in Example 2 we show that the opposite case is also possible, i.e., at a particular preference profile (and consistent vNM utilities), NBM$^U$ makes truthful reporting a best response, but ABM$^U$ is manipulable. This incomparability highlights that a different approach must be taken to compare the incentive properties of ABM and NBM. To this end, we employ the intermediate partial strategyproofness concept, which we review in the next section.

5.2. Review of Partial Strategyproofness

Since NBM and ABM are incomparable by the vulnerability to manipulation comparison, except for fixed, strict priority profiles, we now briefly review the partial strategyproofness concept, which we have introduced in (Mennle and Seuken, 2014).

We have shown that strategyproofness can be decomposed into three simple axioms. These axioms restrict the way in which a mechanism may change the assignment of some student when that student changes her report by swapping two consecutive schools in her reported

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4In addition, we show that even in the simple case when $\pi$ is a single priority profile, the comparison result cannot be strengthened to the statement that “NBM$^\pi$ is as strongly manipulable as ABM$^\pi$” (Ex. 3 & 4).
preference order, e.g., from $P : a > b$ to $P' : b > a$. $\varphi$ is upper invariance if this swap leaves the student’s assignment unchanged for any school that she strictly prefers to $a$, and $\varphi$ is lower invariance if it leaves her assignment unchanged for any school that she likes strictly less than $b$. Finally, $\varphi$ is swap monotonicity if the swap either does not lead to a change of the student’s assignment at all, or if it induces any change, then her probability for $a$ must decrease strictly, and her probability for $b$ must increase strictly.

**Fact 1** (Theorem 1 in (Mennle and Seuken, 2014)). A mechanism is strategyproof if and only if it is upper invariant, swap monotonic, and lower invariant.

Now suppose that a student $i$ has a vNM utility function $u_i$ that is consistent with her preference order $P_i$. We say that $u_i$ satisfies uniformly relatively bounded indifference with respect to indifference bound $r \in [0, 1]$ (URBI$(r)$) if for any schools $a, b \in M$ with $P_i : a > b$ we have that

$$r(u_i(a) - \min u_i) \geq u_i(b) - \min u_i,$$

where $\min u_i$ is the utility that $i$ has for her last choice school. (10) implies that the factor difference between $i$’s (normalized) preference intensity for $a$ over $b$ is a least $1/r$. Lower $r$ means that the student differentiates more strongly, while higher $r$ allows her to be closer to indifferent between any two schools.

**Definition 3** (Partial Strategyproofness). For a given setting, i.e., set of student, set of schools, and school capacities, a mechanism $\varphi$ is $r$-partially strategyproof if truthful reporting is a dominant strategy for any student whose vNM utility $u_i$ satisfies URBI$(r)$. $\varphi$ is partially strategyproof if it is $r$-partially strategyproof for some $r > 0$.

**Fact 2** (Theorem 2 from (Mennle and Seuken, 2014)). For a given setting, a mechanism is partially strategyproof if and only if it is upper invariant and swap monotonic.

Partial strategyproofness is implied by strategyproofness, and it implies weak strategyproofness, convex strategyproofness (Balbuzanov, 2013), and strategyproofness for students with downward-lexicographic preferences (Cho, 2012). Thus, partial strategyproofness can be understood as an intermediate incentive requirement. We further discuss its implications in the context of our partial strategyproofness result for ABM in Section 5.3. A more detailed review of the partial strategyproofness concept is given in Appendix B.

### 5.3. Partial Strategyproofness of ABM

Our first main result in this paper formally establishes that the incentive properties of ABM are in fact intermediate between those of DA and NBM.

**Lemma 1.** $ABM^U$ is upper invariant and swap monotonic, but not lower invariant.

**Theorem 1.** $ABM^U$ is partially strategyproof, but not strategyproof.
Proof Outline (formal proof in Appendix C.2). To prove upper invariance of $ABM^U$, we first show that $ABM^\pi$ is upper invariant for any priority profile $\pi$, and then we show that this property is inherited by any mechanism that randomly selects the priority profile $\pi$ according to some priority distribution $P$. The more challenging proof is swap monotonicity: we first observe that $ABM^\pi$ is always weakly swap monotonic. Next, given any priority profile $\pi$ such that $ABM^\pi$ changes the manipulating student’s assignment under a swap of some schools $a, b$, we construct a single priority profile $\pi^*$ such that under $ABM^\pi^*$, the manipulating student receives either $a$ or $b$, depending on the relative ranking of $a$ and $b$ in her report. Thus, the change in probability for $a$ and $b$ is strict as $\pi^*$ is chosen with positive probability. Theorem 1 follows directly from Lemma 1 and the characterization of partial strategyproofness.

Theorem 1 has a number of interesting consequences. First, partial strategyproofness is the strongest incentive requirement (for finite markets) that has been shown to hold for the celebrated Probabilistic Serial (PS) mechanism (Mennle and Seuken, 2014). Thus, from an axiomatic perspective, Theorem 1 means that the incentive properties of $ABM^U$ are in the same class as those of the PS mechanism.

Second, partial strategyproofness implies weak strategyproofness, i.e., a student cannot obtain a stochastically dominant assignment by misreporting her preferences. Put differently, any manipulation will necessarily involve a trade-off on the part of the student between probabilities for different schools. This is illustrated by the example in the introduction: recall that student 1 could obtain schools $a, b, d$ with probability $1/3$ each under truthful reporting or her third choice school $c$ with certainty by ranking $c$ in first position. By misreporting, the student had to “sacrifice” all probability for her first choice $a$ in order to convert chances to obtain her last choice $d$ into chances to obtain $c$. Whether or not she would prefer this manipulation to reporting truthfully depends on her relative preference intensities for the different schools. Theorem 1 teaches us that any manipulation will take such a form, and no student can gain unambiguously from misreporting, i.e., obtain a stochastically dominant assignment.

Third, partial strategyproofness by Theorem 1 implies that $ABM^U$ makes truthful reporting a dominant strategy for all students who differentiate sufficiently between different schools. Formally, for any setting, there exists $r > 0$ such that any student whose vNM utility satisfies $URBI(r)$ will have a dominant strategy to be truthful.

Remark 3. It is worth noting that NBM also satisfies the upper invariance axiom, which is essentially equivalent to truncation robustness: students cannot improve their chances of obtaining a better school by “truncating” their preference reports, i.e., falsely claiming that some lower ranking schools are unacceptable (Hashimoto et al., 2014). However, NBM violates swap monotonicity (Proposition 3 in Appendix C.1), and therefore it cannot be partially strategyproof. Table 1 provides an overview of the properties that each of the mechanisms violate or satisfy.

Generality of Theorem 1: We have proven that Theorem 1 also holds for extended to general priority distributions. We say that a priority distribution $P$ supports all single priority profiles if any single priority profile is selected with positive probability, i.e., for all $\pi \in \Pi$ we have that $P[(\pi, \ldots, \pi)] > 0$, but multiple priority profiles may also be selected. With this definition in mind, Theorem 1 generalizes as follows:
Table 1: Incentive properties of mechanisms

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>UI</th>
<th>SM</th>
<th>LI</th>
<th>PSP</th>
<th>SP</th>
</tr>
</thead>
<tbody>
<tr>
<td>DA$^U$</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>ABM$^U$</td>
<td>✔</td>
<td>✔</td>
<td>✗</td>
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</tr>
<tr>
<td>NBM$^U$</td>
<td>✔</td>
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</table>

Theorem 2. For any priority distribution $P$ that supports all single priority profiles, $ABM^P$ is partially strategyproof.

The proof of Lemma 1 in Appendix C.2 covers this more general case.

6. Efficiency Comparison of DA, NBM, and ABM

As is common in the study of school choice mechanisms, we assess the welfare properties of school choice mechanisms via dominance and efficiency notions. Specifically, we compare the resulting assignments $DA^P(P)$, $NBM^P(P)$, and $ABM^P(P)$ via dominance relations at the preference profile $P$.

6.1. Decoupling Strategyproofness and Efficiency

Since DA is strategyproof, one can expect self-interested students to report their preferences truthfully. Therefore, it is meaningful to assess the efficiency of DA by analyzing the resulting assignments at the true preference profiles. However, NBM and ABM are not strategyproof, and thus self-interested students will not necessarily report truthfully. Assuming fully rational student behavior, the most accurate assessment of efficiency would be to consider the assignments that arise in equilibrium; and one mechanism would be considered to be more efficient if the resulting assignments dominate those of the other mechanism in equilibrium at the true preferences. However, for the general case with coarse priorities and random tie-breaking, the shape of the equilibria under NBM and ABM is an open research question and beyond the scope of this paper.

For the case of fixed, strict priority profiles, it is known that, the Nash equilibrium outcomes of $NBM^\pi$ are weakly dominated by those of $DA^\pi$ (Ergin and Sönmez, 2006). Conversely, for the single uniform priority distribution $U$, the equilibrium outcomes under $NBM^U$ at some vNM utility profiles are preferred by all students to those under $DA^U$ ex-ante (Miralles, 2008; Abdulkadiroğlu, Che and Yasuda, 2015). These “contradictory” findings illustrate that even if equilibria were known for the general case, a efficiency comparison “in equilibrium” might not necessarily indicate a preference for either of the mechanisms.

The fact that an unambiguous equilibrium analysis is presently unavailable (and beyond the scope of this paper) motivates the alternative approach we take in this paper: we decouple strategyproofness and efficiency by considering each dimension separately. Concretely, we compare the efficiency of DA, NBM, and ABM when each mechanism is applied to the true preference profile. This comparison is strait-forward and analogous to the assessment of
non-strategyproof mechanisms in prior work, e.g., the celebrated Probabilistic Serial mechanism (Bogomolnaia and Moulin, 2001) is ordinally efficient only when applied to the true preference profile, and the men-proposing Deferred Acceptance mechanism (Gale and Shapley, 1962) is men-optimal only applied to the true preferences profile, i.e., when women do not misreport. Similarly, characterization results for non-strategyproof mechanisms often rely on axioms that involve the true preferences, such as *ordinal fairness* (Hashimoto et al., 2014) and *respect of preference rankings* (Kojima and Ünver, 2014). The assignments resulting from the respective mechanisms satisfy the characterizing properties only when the mechanisms are applied to the true preferences. Until a more comprehensive equilibrium analysis becomes available, these (and our) insights serve as a useful second-best to inform decisions about school choice mechanisms.

6.2. Dominance and Efficiency Concepts

We now review the dominance and efficiency notions that we use for our analysis. In the following, $P$ is a preferences profile, $x$ and $y$ are assignments, and $\varphi$ is a probabilistic mechanism.

Definition 4 (Ex-post Dominance & Efficiency).

1. For deterministic $x, y$, $x$ ex-post dominates $y$ at $P$ if all students weakly prefer their school under $x$ to their school under $y$. This dominance is strict if the preference is strict for at least one student.

2. A deterministic $x$ is ex-post efficient at $P$ if it is not ex-post dominated by any other deterministic assignment at $P$.

3. A probabilistic $x$ is ex-post efficient at $P$ if it can be written a convex combination of ex-post efficient, deterministic assignments.

4. $\varphi$ is ex-post efficient if $\varphi(P)$ is ex-post efficient at $P$ for all $P \in \mathcal{P}^N$.

Ex-post efficiency can be viewed as a baseline requirement in the school choice problem. Conceivably, an administrator will be hard pressed to explain why two students received their particular placements if they would actually prefer the seats at the respective other schools.\(^5\)

Definition 5 (Ordinal Dominance & Efficiency).

1. $x$ ordinally dominates $y$ at $P$ if for all students $i \in N$ the respective assignment vector $x_i$ weakly stochastically dominates $y_i$ at $P_i$. This dominance is strict if $x_i$ strictly stochastically dominates $y_i$ for at least one student.

2. $x$ is ordinally efficient at $P$ if it is not strictly ordinally dominated at $P$ by any assignment.

3. $\varphi$ is ordinally efficient if $\varphi(P)$ is ordinally efficient at $P$ for all $P \in \mathcal{P}^N$.

\(^5\)Notably, parents of secondary school students in Amsterdam have recently gone to court over the fact that multiple tie-breaking led to an ex-post inefficient assignment of students.
Ordinal efficiency formalizes the idea of Pareto optimality for probabilistic assignments, i.e., the absence of trade cycles that can be shown to benefit all students (strict for some) when only the students' ordinal preferences are known. This efficiency notion has been used by Bogomolnaia and Moulin (2001) to describe the efficiency advantages of the non-strategyproof Probabilistic Serial mechanism over the strategyproof Random Serial Dictatorship mechanism.

**Definition 6 (Rank Dominance & Efficiency).**

1. Let \( r_P(i,j) \) denote the rank of school \( j \) in the preference order \( P_i \), i.e., the number of schools that \( i \) weakly prefers to \( j \). Let

\[
d_k^x = \sum_{i \in N} \sum_{j \in M : r_P(i,j) \leq k} x_{i,j}
\]

be the expected number of students who receive their \( k \)th choice under \( x \). Then the vector \( d^x = (d_1^x, \ldots, d_m^x) \) is called the rank distribution of \( x \) at \( P \).

2. \( x \) rank dominates \( y \) at \( P \) if the rank distribution \( d^x \) stochastically dominates \( d^y \), i.e., for all ranks \( k \in \{1, \ldots, m\} \) we have

\[
\sum_{r=1}^{k} d_r^x \geq \sum_{r=1}^{k} d_r^y.
\]

The rank dominance of \( x \) over \( y \) is strict if inequality (12) is strict for some \( k \).

3. \( x \) is rank efficient at \( P \) if it is not rank dominated by any other assignment at \( P \).

4. \( \varphi \) is rank efficient if \( \varphi(P) \) is rank efficient at \( P \) for all \( P \in \mathcal{P}^N \).

Rank efficiency represents a strict refinement of ordinal efficiency. To illustrate the difference, consider the trade-cycle interpretations of both efficiency concepts: an assignment is ordinarily efficient if and only if it does not admit trade cycles of probability shares that are unambiguously preferred by all students, given their ordinal preferences. However, it may still be possible to identify trade cycles that hurt one student but create a “much higher” benefit for another student, e.g., if assigning student \( i \) to her 2nd rather than 1st choice allows us to assign another student to her 1st rather than 3rd choice, this would improve overall rank distribution (from \((1,0,1)\) to \((1,1,0)\)), but it hurts student \( i \). Rank efficient assignments do not even admit improvements by these kinds of tough decisions. Focusing on the rank distribution closely resembles welfare criteria that are frequently used in practice: many school choice procedures in Germany have the express objective of assigning as many students to one of their top-3 or top-5 choices (Basteck, Huesmann and Nax, 2015). Similarly, rank efficiency resembles the informal objective of the matching procedure of the Teach-for-America program (Featherstone, 2011).

Finally, we define what it means for a mechanism to be on the efficient frontier.
Definition 7 (Efficient Frontier).

1. \( \phi \) ordinally (or rank) dominates another mechanism \( \phi' \) if the assignment \( \phi(P) \) ordinally (or rank) dominates \( \phi'(P) \) at \( P \) for all \( P \in \mathcal{P}^N \). This is strict if in addition, \( \phi(P) \) strictly ordinarily (or rank) dominates \( \phi'(P) \) at \( P \) for some preference profile \( P \in \mathcal{P}^N \).

2. For some set \( \Phi \) of mechanisms, \( \phi \in \Phi \) is on the efficient frontier with respect to ordinal (or rank), subject to \( \Phi \), if it is not strictly ordinarily (or rank) dominated by any other mechanism \( \phi' \) within \( \Phi \).

Intuitively, mechanisms on the efficient frontier are as efficient as possible while simultaneously satisfying other design requirements described by \( \Phi \), e.g., strategyproofness. Put differently, it may be possible to design a mechanism \( \phi' \) that outperforms \( \phi \) in terms of efficiency, but any such \( \phi' \) would necessarily violate other criteria, or equivalently lie outside \( \Phi \).

Some assignments are not comparable by any of the dominance concepts defined above. Consequently, some mechanisms may not be comparable at every preference profile, so that a strict extension of the dominance notions to mechanisms will yield inconclusive results. This in-comparability also occurs for NBM, ABM, and DA. To overcome this difficulty, we formalize the idea that a mechanism should be considered more efficient if it dominates another mechanism at least at those preference profile at which the two mechanisms are comparable.

Definition 8 (Comparable Dominance). \( \phi \) comparably ordinally (or rank) dominates another mechanism \( \phi' \) if \( \phi(P) \) ordinally (or rank) dominates \( \phi'(P) \) at \( P \) for all \( P \in \mathcal{P}^N \) whenever \( \phi(P) \) and \( \phi'(P) \) are comparable by ordinal (or rank) dominance.

Comparable dominance is less demanding than “perfect” dominance, as the decision in favor of one of the mechanisms as based solely on those profiles where a comparison is possible. The advantage of this approach is that more mechanisms become comparable but the distinction remains unambiguous.

6.3. Efficiency of NBM

Regarding the efficiency of NBM, we first establish that the traditional dominance and efficiency notions may not differentiate between NBM and DA. We then present our second main result (Theorem 3) that NBM comparably rank dominates DA. This formalizes our intuition from the motivating example in the introduction that “NBM is more efficient than DA.”

The following Fact 3 summarizes the known efficiency properties of DA.

Fact 3.

1. \( DA^P \) is not ex-post efficient in general (Roth and Sotomayor, 1990). For any single priority distribution \( P \), \( DA^P \) is ex-post efficient, but may not be ordinarily or rank efficient (Bogomolnaia and Moulin, 2001; Featherstone, 2011).

2. \( DA^U = RSD \) is not on the efficient frontier of strategyproof (symmetric) mechanisms with respect to ordinal dominance when there are more seats than students (Erdil, 2014).
In Proposition 1 we provide an analogous assessment of NBM.

Proposition 1.

1. For any priority distribution $\mathcal{P}$, $\text{NBM}^\mathcal{P}$ is ex-post efficient, but not ordinally or rank efficient in general.

2. Among upper invariant mechanisms, $\text{NBM}^U$ is not on the efficient frontier with respect to ordinal dominance, i.e., there exists an upper invariant mechanism that strictly ordinally dominates $\text{NBM}^U$.

Proof Outline (formal proof in Appendix C.3). We prove ex-post efficiency of NBM by showing that for each preference profile $\mathcal{P}$ and each priority profile $\pi$, we can construct a single priority profile $\sigma = (\sigma, \ldots, \sigma)$ such that $\text{NBM}^\pi(\mathcal{P}) = \text{SD}^\sigma(\mathcal{P})$. Since SD is ex-post efficient, $\text{NBM}^\pi(\mathcal{P})$ must be ex-post efficient at $\mathcal{P}$.\(^6\)

To see the second claim, we construct a mechanism $\text{NBM}^+$ that is essentially equal to $\text{NBM}^U$, except for a certain set of preference profiles. For these preference profiles, $\text{NBM}^+$ chooses the assignment selected by the Probabilistic Serial mechanism instead. It is easy to show that $\text{NBM}^+$ ordinally dominates $\text{NBM}^U$, and that this dominance may be strict. To show that $\text{NBM}^+$ satisfies upper invariance, we show that under any swap, the mechanism’s changes to the assignment are consistent with upper invariance. In particular, this is true for transitions between the special preference profiles where $\text{NBM}^+$ and $\text{NBM}^U$ choose different assignments and those preference profiles where the assignments of both mechanisms are equal. □

Juxtaposing Proposition 1 about the efficiency of NBM to Fact 3 about the efficiency of DA reveals that it may be difficult to differentiate between the two mechanisms: for single priority distributions, both mechanisms are ex-post efficient but fail more demanding efficiency requirements, and neither of them is on the efficient frontier subject to their respective incentive properties. Our second main result Theorem 3 resolves this ambiguity as it uncovers the efficiency advantage of NBM over DA in terms of comparable rank dominance.

Theorem 3. $\text{NBM}^U$ comparably rank dominates $\text{DA}^U$, i.e.,

1. $\text{NBM}^U(\mathcal{P})$ rank dominates $\text{DA}^U(\mathcal{P})$ at $\mathcal{P}$ for any $\mathcal{P} \in \mathcal{P}^N$ where $\text{NBM}^U(\mathcal{P})$ and $\text{DA}^U(\mathcal{P})$ are comparable by rank dominance, and

2. there exists $\mathcal{P} \in \mathcal{P}^N$ at which $\text{NBM}^U(\mathcal{P})$ strictly rank dominates $\text{DA}^U(\mathcal{P})$ at $\mathcal{P}$.

Proof Outline (formal proof in Appendix C.3). We must show that if $\text{DA}^U(\mathcal{P})$ and $\text{NBM}^U(\mathcal{P})$ are comparable by rank dominance, than this comparison favors NBM. For any single priority profile $\pi$ we show that if $\text{DA}^\pi$ and $\text{NBM}^\pi$ assign the same number of first choices, they will in fact assign these first choices to the same students. We can remove these students and the corresponding schools and proceed by induction, carefully handling the case when some school has capacity zero. Averaging over priority profiles, we find that if $\text{DA}$ weakly rank dominates NBM, then the assignment from both mechanism must be the same, and therefore DA never strictly rank dominates NBM. □

\(^6\)Even though ex-post efficiency of NBM appears straight-forward, we are not aware of any formal proof, and we therefore give a proof for completeness.
The rank distribution is frequently used as a welfare criterion by administrators and researchers (Budish and Cantillon, 2012; Featherstone, 2011). In line with this approach, Theorem 3 yields that \( NBM^U \) yields the more appealing assignments whenever the results of \( NBM^U \) and \( DA^U \) are comparable. This is the case, for instance, in the motivating example from the introduction, where \( DA^U \) assigns fewer first choices than \( NBM^U \), but also assigns third choices where \( NBM^U \) does not.

The significance of Theorem 3 is further emphasized by the fact that traditional efficiency notions are unable to differentiate between the two mechanisms (Fact 3 and Proposition 1). The clear preference for \( NBM^U \) in terms of efficiency from Theorem 3 contrasts with the clear preference for \( DA^U \) in terms of strategyproofness. In this sense, we have identified the “cost of strategyproofness” that a mechanism designer incurs when choosing Deferred Acceptance over the naïve Boston mechanism.

**Simulation Results:** In addition to the theoretical insights from Theorem 3, one may want to understand the strength of the dominance of \( NBM^U \) over \( DA^U \): if the assignments from both mechanisms are comparable, how often does the outcome of \( NBM^U \) have a strictly better rank distribution?

For settings with \( n = m \in \{3, \ldots, 10\} \) and \( q_j = 1 \) for all \( j \in M \), we sampled 100'000 preference profiles \( P \) uniformly at random for each value of \( n \). Whenever \( NBM^U(P) \) and \( DA^U(P) \) were comparable by rank dominance at \( P \), we determined whether the dominance was weak or strict. Figure 2 shows the results. For small \( n \), the rank distributions of both assignments are frequently equivalent, largely because both mechanisms produce the same assignments. As \( n \) increases, however, the efficiency advantage of \( NBM^U \) over \( DA^U \) quickly becomes apparent, e.g., for \( n = 8 \) or more students, the share of profiles where \( NBM^U \) does not produce a strictly better rank distribution is below 5\%, and it keeps decreasing as \( n \) grows (subject to comparability).

**Generality of Theorem 3:**

**Theorem 4.** \( NBM^P \) comparably rank dominates \( DA^P \) for any priority distribution \( P \).
Proof. For any fixed single priority profile, we first proved the following statement: if DA$^\pi$ rank dominates NBM$^\pi$ at some preference profile, then both mechanisms must select the same assignment at that preference profile. This insight allowed us to formulate an “extension”-argument and prove comparable rank dominance for any single priority distribution. For the case of fixed multiple priority profiles, Harless (2015) proved the statement about coincidence of the assignments. Observing that his proof is essentially analogous to ours, we can re-use our “extension”-argument to obtain comparable rank dominance of NBM$^P$ over DA$^P$ for arbitrary priority distributions $P$.

To see the significance of Theorem 4, recall that DA may be ex-post inefficient for multiple priority profiles (Fact 3), while NBM is always ex-post efficient. This difference can be interpreted as an efficiency advantage of NBM, but the distinction only holds for multiple priority profiles. The generalized comparison in Theorem 4 identifies the efficiency advantage of NBM, independent of the priority distribution.

In summary, we have formally established the efficiency advantage of NBM over DA, and we have shown via simulations that the difference is usually strict (in sufficiently large markets). When administrators face a decision on what mechanism to implement in a school choice setting, this difference in efficiency can be interpreted as the “cost of strategyproofness” that one incurs when choosing the strategyproof DA over the non-strategyproof NBM.

6.4. Efficiency of ABM

In this section we study the efficiency of the adaptive Boston mechanism, ABM. We first observe that (analogous to NBM) traditional efficiency notions do not differentiate between ABM and DA, nor between ABM and NBM.$^7$

Proposition 2.

1. For any priority distribution $P$, ABM$^P$ is ex-post efficient, but not ordinally or rank efficient in general.

2. Among all partially strategyproof mechanisms, ABM$^U$ is not on the efficient frontier with respect to ordinal dominance, i.e., there exists a partially strategyproof mechanism that strictly ordinally dominates ABM$^U$.

Proof Outline (formal proof in Appendix C.4). The proof is analogous to the proof of Proposition 1 for NBM. When constructing the mechanism ABM$^+$, we must ensure in addition that ABM$^+$ is also swap monotonic, which requires careful analysis of the transition cases, i.e., the change in the assignment when the preference profile changes from one where ABM$^+ = ABM^U$ to another where ABM$^+ = PS$.

Proposition 2 about ABM is analogous to Proposition 1 about NBM and Fact 3 about DA and leave all three mechanisms looking similar in terms of efficiency. The natural next step is

to attempt another comparison by comparable rank dominance. However, surprisingly, this turns out to be inconclusive between both pairs, i.e., ABM and DA as well as ABM and NBM (Section 6.4.1). Despite this in-comparability, limit arguments and simulations allow us to tease out the well-hidden efficiency differences between ABM and the other mechanisms (Section 6.4.2). This establishes our third main result, the intermediate efficiency properties of ABM between DA and NBM.

### 6.4.1. In-comparability of ABM by Comparable Rank Dominance

One would expect that ABM comparably rank dominates DA and is comparably rank dominated by NBM. Indeed, in the motivating example discussed in the introduction, NBM rank dominates ABM, which in turn rank dominates DA. Consequently, it is at least possible that such a comparison of NBM, ABM, and DA is strict and points in the expected direction.

Surprisingly, however, it can also point in the opposite direction: in Examples 6 and 7 (in Appendix C.4) we present preference profiles $P$ and $P'$ such that

i. $DA^U(P)$ strictly rank dominates $ABM^U(P)$ at $P$, and

ii. $ABM^U(P')$ strictly rank dominates $NBM^U(P')$ at $P'$.

**Remark 4.** Note that by Theorem 3, there cannot exist a one preference profile $P$ at which i. and ii. hold simultaneously. In the light of this restriction, our counter-examples are very general: in both examples, we give a single priority profile under which the respective dominance relation holds. Independently, Harless (2015) presented similar examples, which rely on multiplicity of the priority profiles. Since our examples do not make use of multiplicity, they show that comparability cannot be recovered, even when restricting attention to single priority profiles or the single uniform priority distribution.

These examples teach us that the identification of the efficiency differences between ABM and the other mechanisms requires a more subtle approach, since even the rather flexible requirement of comparable rank dominance does not enable a comparison.

### 6.4.2. Comparing ABM and DA: Comparable Rank Dominance in Large Markets

The surprising fact that $DA^U$ may strictly rank dominate $ABM^U$ at some preference profile (Example 6) raises the question how frequently this “unexpected” dominance relation occurs. Interestingly, complete enumeration (using a computer) has revealed that $DA^U$ does not dominate $ABM^U$ for any setting with less than 6 schools (assuming unit capacities). Furthermore, while such cases are possible for 6 or more schools, they turn out to be extremely rare. We now present our third main result that the share of preference profiles where $DA^U$ dominates $ABM^U$ vanishes in the limit as markets get large.

For our limit results, Theorems 5 and 6, we consider two independent notions of how market size increases: the first notion is adopted from (Kojima and Manea, 2010), where the number of schools is fixed, but the number of seats at each school grows as well as the number of students who demand them. This approximates *school choice* settings, where the capacity of public
schools frequently exceeds 100 seats per school. For the second notion, all schools have unit capacity, but the numbers of schools and students increase. This resembles house allocation problems, where every “house” is different and can only be assigned once.

**Theorem 5** (School Choice). Let \((N^k, M^k, q^k)_{k \geq 1}\) be a sequence of settings such that

- the set of schools does not change, i.e., \(M^k = M\) for all \(k\),
- the capacity of each school increases, i.e., \(\min_{j \in M} q_j^k \to \infty\) for \(k \to \infty\),
- the number of students equals the number of seats, i.e., \(|N^k| = \sum_{j \in M} q_j^k\).

Then the share of preference profiles where \(DA^U\) rank dominates \(ABM^U\) (even weakly) vanishes in the limit, i.e.,

\[
\lim_{k \to \infty} \frac{\#\{P \in \mathcal{P}^{N^k} : DA^U(P) \text{ rank dominates } ABM^U(P) \text{ at } P\}}{\#\{P \in \mathcal{P}^{N^k}\}} = 0.
\] (13)

**Theorem 6** (House Allocation). Let \((N^k, M^k, q^k)_{k \geq 1}\) be a sequence of settings such that

- the number of schools equals the number of students, i.e., \(|M^k| = |N^k| = k\),
- all schools have unit capacity, i.e., \(q_j^k = 1\) for \(j \in M^k\).

Then the share of preference profiles where \(DA^U\) rank dominates \(ABM^U\) (even weakly) vanishes in the limit, i.e.,

\[
\lim_{k \to \infty} \frac{\#\{P \in \mathcal{P}^{N^k} : DA^U(P) \text{ rank dominates } ABM^U(P) \text{ at } P\}}{\#\{P \in \mathcal{P}^{N^k}\}} = 0.
\] (14)

**Proof Outline** (formal proofs in Appendix C.4.) We prove the stronger statement that the share of profiles where \(DA^U\) assigns the same number of first choices as \(ABM^U\) converges to zero. We give a bound for this share in terms of multinomial coefficients. Here, we must separately treat the conditional probabilities of the different cases that schools are un-demanded, under-demanded, over-demanded, or exactly exhausted as first choices.

Theorem 6 follows in a similar fashion, but its proof is more demanding as it requires the new notion of overlap for preference profiles. An upper bound for the share of profiles can then be established using 2-associated Stirling numbers of the second kind and variants of the Stirling approximation.

The surprising finding that \(DA^U\) may strictly rank dominate \(ABM^U\) (Example 6) raised doubts about the idea that that \(ABM^U\) is intuitively the more efficient mechanism. However, our limit results, Theorems 5 and 6, show that Example 6 is pathological. As markets grow, preference profiles at which \(DA^U\) rank dominates \(ABM^P\) even weakly become rare. Consequently, for larger markets, we can be confident that \(ABM^U\) will produce the more efficient assignments whenever the results of both mechanisms are comparable. While our
theoretical results yield this confidence for sufficiently large markets, the following simulation results provide reassurance that convergence occurs quickly, i.e., the theoretical possibility of rank dominance of DA\textsuperscript{U} over ABM\textsuperscript{U} is not a relevant concern in markets of any size.

Simulation Results: As in Section 6.3, we conducted simulations to complement our theoretical results: for settings with $n = m \in \{3, \ldots, 10\}$ and $q_j = 1$ for all $j\in M$, we sampled 100'000 preference profiles uniformly at random for each value of $n$. We then determined the rank dominance relation between the resulting assignments under ABM\textsuperscript{U} and DA\textsuperscript{U} at the sampled preference profiles whenever they were comparable. Figure 3 shows the results.

The insights are very similar to those for NBM\textsuperscript{U} and DA\textsuperscript{U} (Figure 2). First, we observe that the share of profiles where ABM\textsuperscript{U} does not yield a strictly preferable rank distribution decreases rapidly and is under 5\% for 8 or more students (conditional on comparability). Second, despite the existence of Example 6, where DA\textsuperscript{U} strictly rank dominates ABM\textsuperscript{U}, this situation did not occur even once in the entire sample. This suggests that convergence in both limit results (Theorems 5 and 6) happens early, and there is no need to worry about exceptions in practice.

To summarize, we have found that concerns about cases where DA\textsuperscript{U} dominates ABM\textsuperscript{U} can be dismissed as pathological; ABM\textsuperscript{U} is essentially the more efficient mechanism.

6.4.3. Comparing ABM and NBM: the “Cost of Partial Strategyproofness”

In Section 6.3, we have identified a “cost of strategyproofness:” DA is comparably rank dominated by NBM, a price one must pay when full strategyproofness is achieved using DA. The motivating example suggests a similar relationship between ABM and NBM. This intuition is further supported by the observation that NBM considers only $k$th choices in the $k$th round, while ABM may also consider less preferred choices. However, surprisingly, Example 7 has shown that ABM\textsuperscript{U} may strictly rank dominate NBM\textsuperscript{U}, and consequently the “cost of partial strategyproofness” between ABM and NBM cannot be identified analogously in terms of comparable rank dominance.

Restricting attention to the single uniform priority distribution $U$, we now present evidence from simulations, which shows that NBM\textsuperscript{U} is usually the dominant mechanism: conditional on
comparability, dominance of \( \text{NBM}^U \) over \( \text{ABM}^U \) occurs much more frequently than the opposite case. Therefore, when choosing \( \text{ABM}^U \) over \( \text{NBM}^U \), the fact that \( \text{NBM}^U \) rank dominates \( \text{ABM}^U \) much more frequently can be considered a price that we pay for partial strategyproofness.

The setup for our simulation is analogous to those in Sections 6.3 and 6.4: for each \( n = m \in \{3, \ldots, 10\} \) we sampled 100,000 preference profiles uniformly at random in settings with unit capacities (\( q_j = 1 \) for all \( j \in M \)). Conditional on comparability, we determined which of the mechanisms had a better rank distribution at each profile. The results are given in Figure 4. First, we observe that share of preference profiles where \( \text{NBM}^U \) has a strict advantage over \( \text{ABM}^U \) grows for growing numbers of students (light grey bars); this value increases continuously and reaches \( \sim 75\% \) for \( n = 10 \). Second, the share of profiles where \( \text{ABM}^U \) strictly rank dominates \( \text{NBM}^U \) is small, i.e., always below 1%. However, recall that for DA and ABM the same simulation did not yield a single instance of a preference profile where \( \text{DA}^U \) strictly rank dominates \( \text{ABM}^U \). Thus, instances where \( \text{ABM}^U \) strictly rank dominates \( \text{NBM}^U \) are rare, but not as obviously negligible as was the case for the comparison with \( \text{DA}^U \). Thus, the efficiency advantage of \( \text{NBM}^U \) over \( \text{ABM}^U \) is less pronounced. Nonetheless, from the perspective of the market designer, the efficiency differences identified by our simulation constitute a “cost of partial strategyproofness” when choosing \( \text{NBM}^U \) over \( \text{ABM}^U \).

7. Conclusion

In this paper, we have studied the Deferred Acceptance mechanism, as well as the traditional (naïve) and a new (adaptive) variant of the Boston mechanism for school choice. These mechanisms are of particular interest as all three of them are used in practice, and in particular, ABM has been largely ignored in the literature. We have established that these mechanisms form two hierarchies, one with respect to strategyproofness, and one with respect to efficiency.

First, DA is strategyproof for students, while NBM is not even weakly strategyproof. We have proven that ABM satisfies the intermediate requirement of partial strategyproofness (subject to the technical condition that the priority distribution supports all single priority profiles). This result is contrasted by the fact that a comparison by vulnerability to manipulation (which
holds for the case of fixed, strict priority profiles) cannot be extended to the general case of random priority distributions.

Second, we have shown that NBM offers unambiguous efficiency gains over DA: the resulting assignments rank dominate those from DA whenever they are comparable, and this comparison holds independent of the priority distribution. Via simulations for the single uniform priority distribution we found that when the dominance relation holds, it is almost certainly strict, e.g., in 95% of the comparable cases for \( n = m \geq 8 \) or more schools and students. In the light of the manipulability of NBM, this efficiency difference can be interpreted as the “cost of strategyproofness” that a market designer incurs when choosing DA over NBM.

Third, we have found ABM to have intermediate efficiency between NBM and DA. This result required the use of limit arguments and simulation, because (surprisingly) a comparison by the rather flexible comparable rank dominance concept with either DA or NBM was inconclusive.

Throughout the paper, it has become apparent that traditional methods frequently fail to differentiate between the three popular school choice mechanisms: while a comparison by vulnerability to manipulation is inconclusive for NBM and ABM, except in the most basic case, partial strategyproofness provides a clear formal argument for the intuition that “ABM has better incentive properties than NBM.” Similarly, all mechanisms we considered are ex-post efficient (for single priority distributions), but neither ordinally nor rank efficient, and neither of them is on the efficient frontier subject to the respective strategyproofness properties. Nonetheless, a comparison using comparable rank dominance, limit arguments, and simulation revealed an efficiency hierarchy: NBM has the most appeal in terms of rank dominance, but ABM is still more appealing than DA. The failure of traditional methods to differentiate between the three mechanisms illustrates that the toolbox of concepts is not yet complete; by considering comparable dominance, limit results, and partial strategyproofness, we have applied new instruments to help market designers tackle the analysis of three common mechanisms and facilitate decisions in practice.

The general lesson to be learned from our results is that a decision between DA, NBM, and ABM requires a non-trivial trade-off between strategyproofness and efficiency: when partial (as opposed to full) strategyproof is acceptable, ABM can be used to obtain a preferable rank distribution; and if manipulability is not a concern, then NBM offers the most appealing efficiency advantages. Rather than indicating preference for any one of these mechanisms, our insights endow market designers with the means to make a conscious decision about this trade-off.
References


Appendix

A. Comparison by Vulnerability to Manipulation

In this section we compare NBM and ABM by their vulnerability to manipulation, considering the comparison “more manipulable” and “strongly more manipulable.” We show that this comparison is inconclusive, except in the simplest case. Our findings do not diminish the value of the vulnerability to manipulation concept, but they highlights the fact that there is no “one size fits all” solution, and that a wider array of concepts is required to perform a meaningful comparison in this case: for NBM and ABM, the partial strategyproofness concept is able to clearly differentiate between the two mechanisms.

A.1. Formalization of Vulnerability to Manipulation

We first review notions for the comparison of mechanisms by their vulnerability to manipulation, introduced by Pathak and Sönmez (2013). For a deterministic mechanisms \( \varphi \) and \( \varphi' \), these are as follows.

**Definition 9.** \( \varphi \) is manipulable by \( i \) at \( P \in \mathcal{P}^N \) if there exists a report \( P'_i \in \mathcal{P} \) such that \( i \) strictly prefers her assigned school under \( \varphi_i(P'_i, P_{-i}) \) to her school under \( \varphi_i(P'_i, P_{-i}) \). \( \varphi \) is manipulable at \( P \in \mathcal{P}^N \) if it is manipulable by some student \( i \) at \( P \).

**Definition 10.** \( \varphi \) is as manipulable as \( \varphi' \) if for any preference profile \( P \), \( \varphi \) is manipulable at \( P \) whenever \( \varphi' \) is manipulable at \( P \). \( \varphi \) is strongly as manipulable as \( \varphi' \) if for any preference profile \( P \), \( \varphi \) is manipulable by \( i \) at \( P \) whenever \( \varphi' \) is manipulable by \( i \) at \( P \).

\( \varphi \) is more manipulable than \( \varphi' \) if

- \( \varphi \) is as manipulable as \( \varphi' \),
- and there exists a preference profile \( P \) such that \( \varphi \) is manipulable at \( P \), but \( \varphi' \) is not.

Analogously, \( \varphi \) is strongly more manipulable than \( \varphi' \) if

- \( \varphi \) is strongly as manipulable as \( \varphi' \),
- and there exist a preference profile \( P \) and a student \( i \) such that \( \varphi \) is manipulable by \( i \) at \( P \), but \( \varphi' \) is not.

These original definitions require that the ordinal preference reports reflect all information that will explain the students’ strategic behavior. However, for probabilistic school choice mechanisms (again, denoted by \( \varphi \) and \( \varphi' \)), we need to extend the students’ preferences over schools to preferences over lotteries. This is achieved by assuming that students have vNM utility functions underlying their preferences, i.e., each student \( i \) with preference order \( P_i \) has a utility function \( u_i : M \rightarrow \mathbb{R} \) such that \( u_i(j) > u_i(j') \) whenever \( P_i : j > j' \) (denoted \( u_i \sim P_i \)).
She will choose a report that maximizes her expected utility with respect to \( u_i \), i.e., given reports from the other students \( P_{-i} \), \( i \) wants to report \( P_i' \in \mathcal{P}^N \) such that her expected utility
\[
\langle u_i, \varphi_i(P_i', P_{-i}) \rangle = \sum_{j \in M} u_i(j) \cdot \varphi_{i,j}(P_i', P_{-i})
\] (15)
is as high as possible. To study probabilistic mechanisms, we extend the definitions in such a way that they coincide with the Definitions 9 and 10 from (Pathak and Sönmez, 2013) in the deterministic case.\(^8\)

**Definition 11.** \( \varphi \) is manipulable by \( i \) at \( u = (u_i, u_{-i}) \) (where \( u_{i'} \sim P_{i'} \) for all \( i' \in N \), denoted \( u \sim P \)) if there exists a report \( P_i' \in \mathcal{P} \) such that
\[
\langle u_i, \varphi_i(P_i', P_{-i}) - \varphi_i(P_i, P_{-i}) \rangle > 0.
\] (16)
\( \varphi \) is manipulable at \( u = (u_1, \ldots, u_n) \sim P \) if it is manipulable by some student \( i \) at \( u \).

**Definition 12.** \( \varphi \) is as manipulable as \( \varphi' \) if for any utility profile \( u \), \( \varphi \) is manipulable at \( u \) whenever \( \varphi' \) is manipulable at \( u \). \( \varphi \) is strongly as manipulable as \( \varphi' \) if for any utility profile \( u \), \( \varphi \) is manipulable by \( i \) at \( u \) whenever \( \varphi' \) is manipulable by \( i \) at \( u \).
\( \varphi \) is more manipulable than \( \varphi' \) if
- \( \varphi \) is as manipulable as \( \varphi' \),
- and there exists a utility profile \( u \) such that \( \varphi \) is manipulable at \( u \), but \( \varphi' \) is not.

Analogously, \( \varphi \) is strongly more manipulable than \( \varphi' \) if
- \( \varphi \) is strongly as manipulable as \( \varphi' \),
- and there exist a utility profile \( u \) and a student \( i \) such that \( \varphi \) is manipulable by \( i \) at \( u \), but \( \varphi' \) is not.

The comparison concepts from Definitions 11 and 12 yield a best response notion of manipulability, i.e., if truthful reporting is a best response under a more manipulable mechanism, then truthful reporting is also a best response under the less manipulable mechanism.

In the case of a fixed strict priority profile \( \pi \), Dur (2015) showed that \( \text{NBM}^\pi \) is more manipulable than \( \text{ABM}^\pi \). One could hope to obtain a better understanding of the incentive properties of the two mechanisms in two ways: first, we can ask whether one of the mechanisms is as manipulable as the other under random priority distributions. Second, we can ask whether a stronger relation holds, namely whether \( \text{NBM}^\pi \) is strongly more manipulable than \( \text{ABM}^\pi \). The following Sections show that the answer to both questions is negative, i.e., comparability of \( \text{NBM} \) and \( \text{ABM} \) “ends” with the weaker notion for comparison and fixed priority profiles.

\(^8\)This is equivalent to assuming that types are cardinal, but the mechanism chooses the same outcome if the agents’ induced ordinal preference are the same.
A.2. Failure of Weak Comparison for Random Priority Distributions

First, we investigate whether comparability can be recovered through randomization, e.g., when $U$ is the uniform distribution over all single priority profiles. Recall that a probabilistic mechanism $\varphi$ is as manipulable as $\varphi'$ if it is manipulable at least at the same utility profiles as $\varphi$. Even though this is arguably the weakest way in which we can extend the least demanding concept from (Pathak and Sönmez, 2013) to probabilistic mechanisms, we find that $\text{NBM}^U$ and $\text{ABM}^U$ are in-comparable in this sense, i.e.,

- for some utility profile $u$, $\text{NBM}^U$ is manipulable, but $\text{ABM}^U$ is not (Example 1),
- for some other utility profile $u'$, $\text{ABM}^U$ is manipulable, but $\text{NBM}^U$ is not (Example 2).

Example 1. Consider the setting with 4 students $N = \{1, \ldots, 4\}$, 4 schools $M = \{a, b, c, d\}$, each with unit capacities, and the preference profile

\[ P_1 : a > b > c > d, \]
\[ P_2, P_3 : a > c > b > d, \]
\[ P_4 : b > \ldots. \]

Student 1’s assignment is $(1/3, 0, 0, 2/3)$ for the schools $a$ through $d$, respectively. If student 1 swaps $b$ and $c$ in her report, the assignment will be $(1/3, 0, 1/3, 1/3)$. The outcome from this manipulation stochastically dominates the outcome from truthful reporting. Thus, student 1 will want to misreport under $\text{NBM}^U$ (independent of her underlying utility).

Under $\text{ABM}^U$, the outcome for student 1 is $(1/3, 0, 1/3, 1/3)$, independent of whether or not she swaps $b$ and $c$. Now suppose that all students have utility $u = (9, 3, 1, 0)$ for their first, second, third, and fourth choice, respectively. Then no student has an incentive to deviate from truthful reporting under $\text{ABM}^U$. Therefore, at this utility profile, $\text{NBM}^U$ is vulnerable to manipulation, but $\text{ABM}^U$ is not.

The last example showed that $\text{ABM}^U$ is not as manipulable as $\text{NBM}^U$, while the next example shows that the converse also holds, i.e., $\text{NBM}^U$ is not as manipulable as $\text{ABM}^U$, which is surprising in the light of the Dur’s comparability result for any fixed priority profile.

Example 2. Consider the setting with 6 students $N = \{1, \ldots, 6\}$, 6 schools $M = \{a, \ldots, f\}$, each with unit capacities, and the preference profile

\[ P_1', P_2' : a > e > c > d > f > b, \]
\[ P_3', P_4' : a > e > d > c > f > b, \]
\[ P_5' : b > c > \ldots, \]
\[ P_6' : b > d > \ldots. \]

Suppose that all students have utility $u' = (120, 30, 19, 2, 1, 0)$ for their first through sixth choice, respectively.
Consider the incentives of the students under NBM: keeping the reports of all other students constant, student 5 has no incentive to deviate, and the same holds for student 6. If student 1 does not rank a in first position, she looses all chances at a and may at best get her second choice e for sure, which is not an improvement under the particular utility chosen. Also, it is easy to check that changing the position of f or b will never be beneficial for student 1. Thus, any beneficial manipulation for student 1 will involve only changes in the order of the schools e, c, d. However, none of these misreports are beneficial, which can be seen in Table 2 (middle column). Due to symmetry, none of the other student 2, 3, 4 have an incentive to misreport either.

Under ABM, however, student 1 does have an incentive to misreport, which can also be seen in Table 2 (bold values in right column).

### A.3. Failure of Strong Comparison for Fixed Priority Profiles

Now address the second question regarding the strong comparison. We show that NBM is not strongly more manipulable than ABM, i.e., there exist preference profiles $P, P'$, priority profiles $\pi, \pi'$, and students $i, i'$ such that

- NBM is manipulable by $i$ at $P$, but ABM is not (Example 3),
- ABM is manipulable by $i'$ at $P'$ by $i'$, but NBM is not (Example 4).

#### Example 3. Consider a setting with students $N = \{1, \ldots, 4\}$, schools $M = \{a, \ldots, d\}$, each with unit capacity, the preference profile

\begin{align*}
P_1 & : a > \ldots, \\
P_2 & : b > \ldots, \\
P_3 & : a > b > c > d, \\
P_4 & : a > c > \ldots,
\end{align*}

and the single priority profile $1 \ldots \pi 4$. Then student 3 will get d under NBM, but if student 3 reports

\begin{align*}
P'_3 & : a > c > \ldots,
\end{align*}

Table 2: Change in expected utility from misreports for student 1 in Example 2.
she will get c instead, a strict improvement. Under ABMπ and truthful reporting, b is exhausted by student 2 in the first round, and therefore, both students 3 and 4 apply for c in the second round, where 3 gets c. It is clear that due to her low priority, student 3 can not get a better school than c under ABMπ with any misreport. Thus, truthful reporting is a best response for student 3 under ABMπ, but not under NBMπ (when all other students report truthfully).

**Example 4.** Consider a setting with 5 students N = {1,...,5}, 5 schools M = {a,...,e}, each with unit capacities, the preference profile

\[
P_1' : a > \ldots,
\]
\[
P_2' : b > \ldots,
\]
\[
P_3' : d > \ldots,
\]
\[
P_4' : a > b > d > c > e,
\]
\[
P_5' : a > b > c > d > e,
\]

and the single priority profile 1 π' ... π' 5. Under NBMπ and truthful reporting, student 5 will get c, and there is no false report that will provide a better school, since a and b are exhausted in the first round (if all other students report truthfully). However, under ABMπ, student 5 will get e. By reporting c as her first choice instead, she can get c, which is better than e.

Note that in Examples 3 and 4, we considered a single priority profile, which is a special case of a general multiple priority profile. Therefore, the counter-examples also show the in-comparability of NBMπ and ABMπ by the strongly more manipulable-relation in general.

**Remark 5.** A natural next step to further understand the vulnerability of both mechanisms to manipulation is a quantitative analysis. This analysis should ask how often a mechanism is manipulable, i.e., given a prior over the students’ utility profiles, how likely is each mechanism manipulable. In (Mennle et al., 2015) we have studied NBMU, ABMU, and Probabilistic Serial in this way and find that under ABMU truth-telling is a best response for all students significantly more often than under NBMU. This result is robust to changes in the size of the setting, the correlation of the preferences, and the underlying distributions in the utility model.

**B. Review of Partial Strategyproofness**

In this section, we review partial strategyproofness, a relaxation of strategyproofness that we have introduced in Mennle and Seuken (2014). This relaxation induces a single-parameter measure for the degree of strategyproofness of a manipulable mechanism. The comparison by vulnerability to manipulation (Pathak and Sönmez, 2013) yields a best response notion of manipulability, while a comparison by the degree of strategyproofness yields a minimum guarantee for the share of utility profiles for which truthful reporting is a dominant strategy. The degree of strategyproofness measure is consistent with the vulnerability to manipulation-concept, but it has two further advantages: first, it can be used to compare any two partially strategyproof mechanisms, i.e., the comparison is never inconclusive. Second, we have presented
an algorithm that computes the degree of strategyproofness for any (computable) partially strategyproof mechanism.

In this sense, partial strategyproofness is a scalable requirement that bridges the gap between strategyproofness on one side and weak strategyproofness on the other side. Intuitively, partially strategyproof mechanisms are strategyproof, but only on a particular domain restriction, i.e., the students can still have any preference order, but their underlying vNM utility functions are constrained. First we define this domain restriction.

**Definition 13** (Uniformly Relatively Bounded Indifference). A utility function $u$ satisfies uniformly relatively bounded indifference with respect to bound $r \in [0, 1]$ (URBI($r$)) if for any two schools $a, b$ with $u(a) > u(b)$ we have

$$r \cdot (u(a) - \min(u)) \geq u(b) - \min(u).$$

We denote by URBI($r$) the set of all utility functions that satisfy uniformly relatively bounded indifference with respect to bound $r$.

A mechanism is $r$-partially strategyproof if it is strategyproofness in the domain restricted by the URBI($r$) constraint. Formally:

**Definition 14** ($r$-partial Strategyproofness). Given a setting $(N, M, q)$ and a bound $r \in [0, 1]$, mechanism $\varphi$ is $r$-partially strategyproof in the setting $(N, M, q)$ if for any student $i \in N$, any preference profile $P = (P_i, P_{-i}) \in \mathcal{P}^N$, any misreport $P'_i \in \mathcal{P}$, and any utility $u_i \in \text{URBI}(r)$, $u_i \sim P_i$, we have

$$\langle u_i, \varphi(P_i, P_{-i}) - \varphi(P'_i, P_{-i}) \rangle \geq 0.$$  

Sometimes, we simply want to state that a mechanism is $r$-partially strategyproof for some non-trivial $r > 0$ without explicitly naming the parameter $r$. In this case, we simply say that the mechanism is partially strategyproof.

Partial strategyproofness has an axiomatic characterization, which gives a good intuition about the gain obtained from using a strategyproof (rather than partially strategyproof) mechanism. In the following, we define the axioms and give our characterization results, including the key auxiliary concepts. A more detailed presentation of the axioms and results for the partial strategyproofness concept can be found in (Mennle and Seuken, 2014).

**Definition 15** (Neighborhood). The neighborhood of a preference order $P$ is the set $N_P$ of all preference orders $P'$ such that there exists $k \in \{1, \ldots, m-1\}$ with

$$P : a_1 > \ldots > a_k > a_{k+1} > \ldots > a_m,$$
$$P' : a_1 > \ldots > a_{k+1} > a_k > \ldots > a_m,$$

i.e., all the preference orders $P'$ where the corresponding preference order differs by just a swap of two adjacent schools.
**Definition 16** (Contour Sets). For a preference order \( P : a_1 > \ldots > a_k > \ldots > a_m \), the upper contour set \( U(a_k, P) \) and lower contour set \( L(a_k, P) \) of \( a_k \) at \( P \) are given by

\[
U(a_k, P) = \{a_1, \ldots, a_{k-1}\} = \{j \in M | P \rangle > a_k\},
\]

\[
L(a_k, P) = \{a_{k+1}, \ldots, a_m\} = \{j \in M | P \rangle < a_k\},
\]

i.e., the sets of schools that a student with preference order \( P \) strictly prefers to or likes strictly less than \( a_k \), respectively.

Next, we present three axioms, which in combination characterize strategyproofness. Each axiom restricts the way in which a misreport from the neighborhood of the student’s true preference order, i.e., a misreport involving only a single swap, can affect the assignment of the reporting student.

**Axiom 1** (Swap Monotonic). A mechanism \( \varphi \) is swap monotonic if for any student \( i \in N \), any preference profile \( P = (P_i, P_{-i}) \in \mathcal{P}^N \), and any preference order \( P'_i \in N P_i \) (i.e., in the neighborhood of \( P_i \)) with \( P_i : a_k > a_{k+1} \) and \( P'_i : a_{k+1} > a_k \), one of the following holds:

1) \( i \)'s assignment is unaffected by the swap, i.e., \( \varphi_i(P_i, P_{-i}) = \varphi_i(P'_i, P_{-i}) \), or

2) \( i \)'s assignment for \( a_k \) strictly decreases and her assignment for \( a_{k+1} \) strictly increases, i.e.,

\[
\varphi_{i,a_k}(P_i, P_{-i}) > \varphi_{i,a_k}(P'_i, P_{-i}) \quad \text{and} \quad \varphi_{i,a_{k+1}}(P_i, P_{-i}) < \varphi_{i,a_{k+1}}(P'_i, P_{-i}).
\]

**Axiom 2** (Upper Invariance). A mechanism \( \varphi \) is upper invariant if for any student \( i \in N \), any preference profile \( P = (P_i, P_{-i}) \in \mathcal{P}^N \), and any preference order \( P'_i \in N P_i \) with \( P_i : a_k > a_{k+1} \) and \( P'_i : a_{k+1} > a_k \), \( i \)'s assignment for the upper contour set \( U(a_k, P_i) \) is unaffected by the swap, i.e., \( \varphi_{i,j}(P_i, P_{-i}) = \varphi_{i,j}(P'_i, P_{-i}) \) for all \( j \in U(a_k, P_i) \).

**Axiom 3** (Lower Invariance). A mechanism \( \varphi \) is lower invariant if for any student \( i \in N \), any preference profile \( P = (P_i, P_{-i}) \in \mathcal{P}^N \), and any preference order \( P'_i \in N P_i \) with \( P_i : a_k > a_{k+1} \) and \( P'_i : a_{k+1} > a_k \), \( i \)'s assignment for the lower contour set \( L(a_k, P_i) \) is unaffected by the swap, i.e., \( \varphi_{i,j}(P_i, P_{-i}) = \varphi_{i,j}(P'_i, P_{-i}) \) for all \( j \in L(a_k, P_i) \).

The three axioms swap monotonicity, upper invariance, and lower invariance characterize strategyproof mechanisms.

**Fact 4** (Theorem 1 from (Mennle and Seuken, 2014)). A mechanism is strategyproof if and only if it is swap monotonic, upper invariant, and lower invariant.

Furthermore, by relaxing the least intuitive of the axioms, lower invariance, the class of partially strategyproof mechanisms emerges.

**Fact 5** (Theorem 2 from (Mennle and Seuken, 2014)). Given a setting \((N, M, q)\), a mechanism \( \varphi \) is \( r \)-partially strategyproof for some \( r \in (0, 1) \) if and only if \( \varphi \) is swap monotonic and upper invariant.
Finally, the URBI($r$) domain restriction is maximal, i.e., there is no systematically larger set of utilities for which a strategyproofness guarantee can also be provided (given $r$-partial strategyproofness).

**Fact 6** (Theorem 3 from (Mennle and Seuken, 2014)). Given a setting $(N,M,q)$, a bound $r \in (0,1)$, and a utility function $\tilde{u}_i \sim P_i$ that violates URBI($r$), there exists a mechanism $\tilde{\varphi}$ such that

- $\tilde{\varphi}$ is $r$-partially strategyproof,
- $\tilde{\varphi}$ is not $\{\tilde{u}_i\}$-partially strategyproof, i.e., there exists a preference order $P_i' \neq P_i$ and reports $P_{-i} \in \mathcal{P}^{n-1}$ such that
  \[
  \langle \tilde{u}_i, \tilde{\varphi}_i(P_i, P_{-i}) - \tilde{\varphi}_i(P_i', P_{-i}) \rangle < 0. \tag{23}
  \]

For any partially strategyproof mechanism, we can now consider the largest admissible indifference bound as a single-parameter measure for how strategyproof the mechanism is.

**Definition 17.** For a setting $(N,M,q)$, the degree of strategyproofness (DOSP) of a mechanism $\varphi$ is given by

$$\rho_{(N,M,q)}(\varphi) = \max\{r \in [0,1] : \varphi \text{ is } r\text{-partially strategyproof in } (N,M,q)\}. \tag{24}$$

In (Mennle and Seuken, 2014), we have shown that the DOSP is computable and consistent with the vulnerability to manipulation comparison (Pathak and Sönmez, 2013).

**C. Examples and Proofs**

**C.1. Incentives under the Na"ıve Boston Mechanism**

**Example 5.** Consider a setting with 4 students $N = \{1, \ldots, 4\}$, 4 schools $M = \{a, \ldots, d\}$, each with unit capacity, i.e., $q_j = 1$ for all $j \in M$, and preference profile

- $P_1 : a > b > c > d,$
- $P_2 : a > b > c > d,$
- $P_3 : b > c > a > d,$
- $P_4 : b > c > a > d.$

Student 2’s assignment vector is $\text{NBM}_U^U(P) = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$ for $a,b,c,d$, respectively. However, if she reports

$$P'_2 : a > c > b > d$$

instead, her assignment vector will be $\text{NBM}_U^U(P'_2, P_{-2}) = \left(\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4}\right)$, which strictly stochastically dominates $\text{NBM}_U^U(P)$. Thus, $\text{NBM}_U^U$ is not even weakly strategyproof. When we consider the single priority profile $\pi = (\pi, \ldots, \pi)$ with $1 \pi \ldots \pi 4$, we obtain in the same way that $\text{NBM}_U^\pi$ is manipulable and neither swap monotonic nor lower invariant.
Proposition 3. For any priority distribution $P$, $NBM^P$ is upper invariant, but neither swap monotonic, nor lower invariant.

Proof. Example 5 shows that NBM is neither swap monotonic, nor lower invariant: swapping $b$ and $c$ changed student 2’s assignment vector, but her probability for $b$ remained unchanged, which violates swap monotonicity. By swapping $b$ and $c$, student 1 managed to change her probability for $d$, which violates lower invariance.

To see upper invariance, first consider a fixed priority profile $\pi \in \Pi^M$ and let $a, b$ be two adjacent schools in student $i$'s preference order. Before applying to $a$ or $b$, $i$ applies to all the schools that she strictly prefers to $a$ and $b$ in previous rounds. The order of applications is same, independent of the order in which $a$ and $b$ appear in her preference report. Since her chances of being accepted at a better school only depend on these previous rounds, these chances do not change if she swaps $a$ and $b$. Thus, NBM$^\pi$ is upper invariant for any fixed priority profile.

Recall that NBM$^P$ randomly determines a priority profile $\pi$ according to $P$. Since NBM$^\pi$ is upper invariant for any fixed $\pi$ that may be drawn from $P$, the probabilities for the preferred schools are unaffected by the swap of $a$ and $b$. Thus, NBM$^P$ is upper invariant for any priority distribution $P$.

C.2. Incentives under the Adaptive Boston Mechanism

Proof of Lemma 1. $ABM^U$ is upper invariant and swap monotonic, but not lower invariant.

Proof. We prove the stronger statement that $ABM^P$ is upper invariant and swap monotonic for any priority distribution $P$ that supports all single priority profiles. The proof of upper invariance (for arbitrary priority profiles) is exactly the same as for NBM$^P$ (Proposition 3). Next, we show that for arbitrary priority profiles $\pi$, $ABM^\pi$ is weakly swap monotonic, i.e., if an student swaps the order of two adjacent schools $a$ and $b$, then her probability for acceptance at $b$ cannot strictly decrease, and her probability for acceptance at $a$ cannot strictly increase.

Claim 1. For any priority distribution $P$, $ABM^P$ and NBM$^P$ are weakly swap monotonic.

Proof of Claim 1. Recall that for some fixed $\pi$, $ABM^\pi$ is deterministic. If the student ranks $a$ below $b$, she will apply to $a$ in a later round. Thus, if she did not receive $a$ when under the original report, she will not receive it under the new report either. Similarly, she cannot decrease her chances at $b$ by applying in an earlier round. For the probabilistic mechanism $ABM^P$, this means weak swap monotonicity.

For NBM$^P$, the arguments are exactly the same.

Now suppose that $P$ supports all single priority profiles, i.e., $P[(\pi,...,\pi)] > 0$ for all priority orders $\pi \in \Pi$. We now have to show that if the swap of $a$ and $b$ changes the student's probability vector, then the change is in fact strict for $a$ and $b$. We show this by constructing a priority profile $\pi^* = (\pi^*,...,\pi^*)$, such that under $ABM^{\pi^*}$ she receives either school $a$ or school $b$, whichever she ranked first. Since $\pi^*$ is selected with positive probability, this implies swap monotonicity.
Claim 2. Let $P_i$ be student $i$’s preference order, where schools $a,b \in M$ are ranked consecutively and $P_i : a > b$, let $>_i$ be the same preference order, except that $P_i : b > a$, and let $P_{-i}$ be any collection of preference orders from the other students.

If there exists a priority profile $\pi$ such that

$$ABM_i^\pi(P_i, P_{-i}) \neq ABM_i^\pi(P_i', P_{-i})$$

then there exists a single priority profile $\pi^* = (\pi^*, \ldots, \pi^*)$ such that

$$ABM_{i,a}^{\pi^*}(P_i, P_{-i}) = ABM_{i,b}^{\pi^*}(P_i', P_{-i}) = 1,$$

i.e., student receives $a$ by reporting $P_i$ truthfully and $b$ by falsely reporting $P_i'$.

Proof of Claim 2. Let $j$ denote the school that $i$ receives when reporting $P_i$, and let $j'$ be the school that she receives when reporting $P_i'$, i.e., $ABM_{i,j}(P_i, P_{-i}) = 1$ and $ABM_{i,j'}(P_i', P_{-i}) = 1$. Consider the following cases:

- If $j = j'$, this violates the assumption that the assignment changes.
- If $P_i : j > a$, then $j = j'$ by upper invariance, a contradiction.
- If $j = b$, then $j' = b$ by weak swap monotonicity from Claim 1.
- If $j = c$ for some $c \in M$ with $P_i : b > c$, then $j' = c$. To see this, consider the order in which $i$ applies to the schools. If she applies to $a$ first (in round $k$, say), she is rejected, which implies that $a$ is exhausted by the end of round $k$. The same holds for $b$. Since the mechanism lets $i$ skip exhausted schools, she will not apply to $b$ or $a$, respectively, in round $k + 1$ after being rejected from $a$ or $b$, respectively. Therefore, the order of her applications remains unchanged after she was rejected in round $k$. This means that none of the other students are affected by this change of report, and therefore, the assignment does not change for $i$.

Thus, it must be the case that $j = a$ and $P_i : j' \succeq b$. Let $N_k$ be the students who received their $k$th choice school under $ABM^\pi$ and let $\pi'$ be some (i.e., any) priority order such that $i' \pi' i''$ whenever $i' \in N_k$ and $i'' \in N_{k'}$ for some $k < k'$. Observe that under $ABM^\pi$ with the single priority profile $\pi' = (\pi', \ldots, \pi')$ constructed from $\pi'$, all students from $N_k$ will receive their $k$th choice, i.e., $ABM^\pi(P) = ABM^\pi(P')$. If $j' = b$, set $\pi^* = \pi'$. Otherwise, let $\pi^*$ be the priority order that arises by taking $\pi'$ and inserting student $i$ just before the last student whose application to $b$ was accepted. Then, under $ABM^\pi^*(P')$, $i$ will get $b$.

This concludes the proof of Lemma 1.
C.3. Efficiency of the Naïve Boston Mechanism

Proof of Proposition 1.

1. For any priority distribution \( P \), \( \text{NBM}^P \) is ex-post efficient, but not ordinally or rank efficient in general.

2. Among upper invariant mechanisms, \( \text{NBM}^U \) is not on the efficient frontier with respect to ordinal dominance, i.e., there exists an upper invariant mechanism that strictly ordinally dominates \( \text{NBM}^U \).

Proof. Ex-post efficiency: We first prove that \( \text{NBM}^\pi(P) \) is ex-post efficient at \( P \) for any fixed priority profile \( \pi \). Let \( N_r \subseteq N, r = 1, \ldots, m \) be the set of students who receive their \( r \)th choice school under \( x = \text{NBM}^\pi(\pi) \). Let \( \sigma \in \Pi \) be some (any) priority order over the students such that all students in \( N_1 \) have priority over students in \( N_2 \), who in turn have priority over \( N_3 \), etc., i.e., for any \( r = 1, \ldots, m - 1 \) and any \( i_r \in N_r, i_{r+1} \in N_{r+1} \) we have that \( i_r \sigma i_{r+1} \).

Now, consider the assignment \( y = \text{SD}^\sigma(P) \). This is equivalent to an application of the serial dictatorship mechanism to the preference profile \( P \) with picking order \( \sigma \), and therefore, \( y \) is ex-post efficient at \( P \).

Under the serial dictatorship mechanism, all the students in \( N_1 \) receive their first choices, because (1) they are allowed to pick first and (2) there is sufficient capacity at there first choices (otherwise, they could not all have received these schools under the Boston mechanism). When the first student from \( N_2 \) gets to pick a school, the situation is exactly the same as at the beginning of the second round of the Boston mechanism, but where none of the first choice school of the remaining students have no more seats available. By the same argument as for first choices, all students from \( N_2 \) must get their second choice school. Inductively, we get that all students in \( N_r \) get their \( r \)th choice under the serial dictatorship mechanism with picking order \( \sigma \), and consequently the assignments \( x \) and \( y \) are the same. Hence, \( x \) is ex-post efficient.

Since \( \text{NBM}^P(P) \) is simply a lottery over deterministic assignments \( \text{NBM}^\pi(P) \), each of which is ex-post efficient by the previous arguments. Therefore, \( \text{NBM}^P \) is ex-post efficient at any preference profile and for any priority distribution.

Ordinal and Rank Inefficiency: Consider the setting with 4 students \( N = \{1, \ldots, 4\} \), 4 schools \( M = \{a, \ldots, d\} \), each with unit capacities, and the preference profile

\[
P_1 : a > b > c > d, \\
P_2 : a > b > d > c, \\
P_3 : b > a > c > d, \\
P_4 : b > a > d > c.
\]

The assignments is

\[
\text{NBM}^U(P) = \begin{pmatrix}
\frac{1}{2} & 0 & 3 & 1 \\
1 & 0 & 1 & 0 \\
0 & \frac{1}{2} & 0 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}.
\]
which is ordinally dominated at $\mathbf{P}$ by the assignment
\[
\begin{pmatrix}
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}.
\] (28)

Since ordinal dominance implies rank dominance, failure of ordinal efficiency implies failure of rank efficiency.

**Failure to be on the Efficient Frontier:** We construct a mechanism that is upper invariant and ordinally dominates $\text{NBM}^\mathbb{U}$. This mechanism, $\text{NBM}^+$, is essentially the same mechanism as $\text{NBM}^\mathbb{U}$, except that the assignment is altered at certain preference profiles. Again, consider the setting with 4 students and 4 schools in unit capacity. We say that a preference profile satisfies *separable wants* if the schools and students can be renamed such that

- students 1 and 2 have first choice $a$,
- students 3 and 4 have first choice $b$,
- students 1 and 3 prefer $c$ to $d$,
- and students 2 and 4 prefer $d$ to $c$.

Formally,

\[
P_1 : a > \{b, c, d\} \text{ and } c > d,
\]
\[
P_2 : a > \{b, c, d\} \text{ and } d > c,
\]
\[
P_3 : b > \{a, c, d\} \text{ and } c > d,
\]
\[
P_4 : b > \{a, c, d\} \text{ and } d > c.
\]

$\text{NBM}^+$ is the same as $\text{NBM}$, except that the outcome is adjusted for preference profiles with separable wants. Let

\[
\text{NBM}^+ (\mathbf{P}) = \begin{cases} 
\text{PS} (\mathbf{P}), & \text{if } \mathbf{P} \text{ satisfies separable wants}, \\
\text{NBM}^\mathbb{U} (\mathbf{P}), & \text{else},
\end{cases}
\] (29)

where $\text{PS}$ denotes the Probabilistic Serial mechanism (Bogomolnaia and Moulin, 2001). At some preference profile $\mathbf{P}$ that satisfies separable wants, the assignment under PS (after appropriately renaming of the students and schools) is

\[
\text{PS} (\mathbf{P}) = \begin{pmatrix}
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}.
\] (30)

Observe that under $\text{NBM}^\mathbb{U}$, $a$ is split equally between 1 and 2, and $b$ is split equally between 3 and 4. Consequently, students 1 and 2 get no share of $b$ and students 3 and 4 get no share.
of a, just as under PS. Among all assignments that distribute a and b in this way, student 1 prefers the ones that give her higher probability at c, i.e., \( \frac{1}{2} \), since she already receives a with probability \( \frac{1}{2} \). Similarly, students 2, 3, and 4 prefer their respective assignment under PS to any other assignment that splits a and b in the same way as PS and NBM\(^+\) \( N \). Therefore, NBM\(^+\) weakly ordinally dominates NBM\(^U\), and the dominance is strict for preference profiles \( P \) with separable wants.

It remains to be shown that NBM\(^+\) is upper invariant. To verify this, we only need to consider the change in assignment that the mechanism prescribes if some student swaps two adjacent schools in its reported preference ordering. Starting with any preference profile \( P \), the swap produces a new preference profile \( P' \). If neither \( P \) nor \( P' \) satisfy separable wants, the mechanism behave like NBM\(^U\).

For swaps where at least one of the preference profiles satisfies separable wants, we can assume without loss of generality that this is \( P \). Such a swap will lead to a new preference profile \( P' \) and one of the following three cases:

(i) The new preference profile satisfies separable wants.

(ii) The composition of the first choices has changed.

(iii) The preference profile no longer satisfies separable wants, but the composition of the first choices has not changed.

By symmetry, we can restrict our attention to student 1 whose preference order satisfies

\[
P_1: a > c > d \text{ and } a > b.
\]

Case (i) implies that \( P' \) still satisfies separable wants with respect to the same mappings \( \mu, \nu \). Thus, NBM\(^+\) will not change the assignment, i.e., upper invariance is not violated.

In case (ii), student 1 has a new first choice. If the new first choice is b, she will receive b with probability \( \frac{1}{3} \) and a with probability 0 under NBM\(^U(P') \). If the new first choice is c or d, the student will receive that that school with certainty under NBM\(^U(P') \). Both changes are consistent with upper invariance.

Finally, in case (iii), the swap must involve c and d, since this is the only way in which separable wants can be violated. Since \( P' \) violates separable wants, we have NBM\(^+\)(\( P' \)) = NBM\(^U(P') \), and therefore, a will still be split equally between the students who rank it first, and the same is true for b. Thus, student 1 will receive \( \frac{1}{2} \) of a and 0 of b, which is the same as under NBM\(^+\)(\( P \)) = PS(\( P \)). The only change can affect the assignment for the schools c and d. This is consistent with upper invariance.

Proof of Theorem 3. NBM\(^U\) comparably rank dominates DA\(^U\), i.e.,

1. NBM\(^U(P) \) rank dominates DA\(^U(P) \) at \( P \) for any \( P \in \mathcal{P}^N \) where NBM\(^U(P) \) and DA\(^U(P) \) are comparable by rank dominance, and

2. there exists \( P \in \mathcal{P}^N \) at which NBM\(^U(P) \) strictly rank dominates DA\(^U(P) \) at \( P \).
Proof. We first establish the following lemmas about rank dominance. Let \( x, x^1, \ldots, x^K \in \Delta(X) \) be assignments such that

\[
x = \sum_{k=1}^{K} x^k \cdot \alpha_k \tag{32}
\]

for some \( \alpha_1, \ldots, \alpha_K > 0 \) with \( \sum_{k=1}^{K} \alpha_k = 1 \), i.e., \( x \) is the convex combination of the assignments \( x^k, k = 1, \ldots, K \) with coefficients \( \alpha_k, k = 1, \ldots, K \).

**Lemma 2.** The rank distribution \( d^x \) of \( x \) (at some preference profile \( P \)) is equal to the convex combination of the rank distributions \( d^{x^k} \) of the \( x^k \) with respect to coefficients \( \alpha_1, \ldots, \alpha_K \), i.e.,

\[
d^x = \sum_{k=1}^{K} d^{x^k} \cdot \alpha_k. \tag{33}
\]

Lemma 2 is obvious from the definition of the rank distribution in Definition 6.

**Lemma 3.** Let \( y, y^1, \ldots, y^K \in \Delta(X) \) be assignments such that

\[
y = \sum_{k=1}^{K} y^k \cdot \alpha_k \tag{34}
\]

for the same coefficients \( \alpha_1, \ldots, \alpha_K \), and let there be \( r_k, k = 1, \ldots, K \) such that for all \( k = 1, \ldots, K \) and all \( r' < r_k \) we have

\[
d^{x^k}_{r'} = d^{y^k}_{r'}. \tag{35}
\]

Furthermore, if \( r_k \leq m \), then

\[
d^{x^k}_{r_k} > d^{y^k}_{r_k} \tag{36}
\]

(otherwise, \( d^{x^k} = d^{y^k} \)).

Then \( y \) does not even weakly rank dominate \( x \).

**Proof of Lemma 3:** Let \( r_{\text{min}} = \{r_k | k = 1, \ldots, K \} \) be the lowest rank (i.e., the best choice) at which inequality (36) holds strictly, and let \( k_{\text{min}} \in \{1, \ldots, K + 1\} \) be an index for which this is the case. Then for all \( r' < r_{\text{min}} \) and all \( k = 1, \ldots, K \) we have

\[
d^{x^k}_{r'} = d^{y^k}_{r'}. \tag{37}
\]

so that by Lemma 2

\[
d^{x^k}_{r_{\text{min}}} = d^{y^k}_{r_{\text{min}}}. \tag{38}
\]

In words, the rank distributions of \( x \) and \( y \) coincides for all ranks before \( r_{\text{min}} \). Furthermore,

\[
d^{x^k}_{r_{\text{min}}} \geq d^{y^k}_{r_{\text{min}}}, \tag{39}
\]

for all \( k \neq k_{\text{min}} \), and

\[
d^{x^k_{\text{min}}}_{r_{\text{min}}} > d^{y^k_{\text{min}}}_{r_{\text{min}}}. \tag{40}
\]
Thus, by Lemma 2 and the fact that $\alpha_{k_{\min}} > 0$,

$$d_{r_{\min}} = \sum_{k=1}^{K} d_{r_{\min}}^{k} \cdot \alpha_{k}$$

(41)

$$= d_{r_{\min}}^{y_{\min}} \cdot \alpha_{k_{\min}} + \sum_{k=1, k \neq k_{\min}}^{K} d_{r_{\min}}^{k} \cdot \alpha_{k}$$

(42)

$$> d_{r_{\min}}^{y_{\min}} \cdot \alpha_{k_{\min}} + \sum_{k=1, k \neq k_{\min}}^{K} d_{r_{\min}}^{k} \cdot \alpha_{k}$$

(43)

$$\geq \sum_{k=1}^{K} d_{r_{\min}}^{y_{k}} \cdot \alpha_{k} = d_{r_{\min}}^{y}.$$  

(44)

We now proceed to prove the Theorem in two steps.

**Step 1 (for any fixed single priority profile):** First, we show that for any fixed single priority profile $\pi = (\pi_{1}, \ldots, \pi_{q}) \in \Pi^{M}$ and any preference profile $P$, the assignment $y^{\pi} = DA^{\pi}(P)$ never strictly rank dominates the assignment $x^{\pi} = NBM^{\pi}(P)$ at $P$. In fact, we show something stronger, namely that the conditions of Lemma 3 are satisfied for $x^{\pi}$ and $y^{\pi}$, i.e., either $d_{r}^{x} = d_{r}^{y}$, or there exists some $r \in \{1, \ldots, m\}$ such that $d_{r}^{x} > d_{r}^{y}$, and $d_{r'}^{x} = d_{r'}^{y}$ for all $r' < r$.

In the proof, we consider the slightly larger domain, where schools can have zero capacity. Under $DA^{\pi}$, including additional empty schools does not make a difference for the resulting assignment. Under the $NBM^{\pi}$, it is easy to see that the assignments can be decomposed into two parts:

1. Run the first round of the mechanism, in which a set of students $N_{1}$ receives their first choice schools.

2. Remove the students $N_{1}$ from $N$, and also remove these students from all priority orders in the priority profile $\pi$. Reduce the capacities of the schools they received by the number of students who received each school. Change the preference orders of all remaining students by moving their first choice to the end of their ranking. Then, run the mechanism again on the reduced problem (which may include schools of capacity zero).

In the final assignment resulting from $NBM^{\pi}$, the students $N_{1}$ will receive their first choices, and the other students will receive the schools they got in the reduced setting.

**Claim 3.** $DA^{\pi}(P)$ assigns a weakly lower number of first choices than $NBM^{\pi}(P)$.

The claim is obvious from the observation that $NBM^{\pi}$ maximizes the number of assigned first choices.
Claim 4. If \( DA^\pi(P) \) assigns the same number of first choices as \( NBM^\pi(P) \), then the sets of students who get their first choices under both mechanism coincide.

Proof. By assumption, \( d^\pi_1 = d^\pi_1 \). Suppose towards contradiction that there exists some student \( i \in N \), who receives her first choice school \( j \in M \) under \( y^\pi \), but not under \( x^\pi \). That means that \( j \) was exhausted in the first round by other students, all of whom must have had higher priority than \( i \) (according to \( \pi \)). These students as well as \( i \) would also apply to \( j \) in the first round of \( DA^\pi \). But since \( j \) was already exhausted by the other students, \( i \) will also be rejected from \( j \) in the first round of \( DA^\pi \), a contradiction.

Observer that under \( DA^\pi \) students can only obtain their first choice school in the first round. By Claim 4, if \( d^\pi_1 = d^\pi_1 \), then \( NBM^\pi \) and \( DA^\pi \) assign the same students to their first choice schools, and therefore, none of the students who received their first choice school under \( DA^\pi \) (tentatively in the first round) was rejected in any subsequent round. Thus, we can also decompose the assignment from \( DA^\pi \) into two parts (as before for \( NBM^\pi \)):

1. The assignment from the first round.
2. The assignments from applying the mechanism to the reduced and altered setting.

We can now apply Claim 4 inductively to the reduced settings to show that \( d^\pi_r = d^\pi_r \) implies that the same students also got their \( r \)th choice under both mechanisms. Since \( d^\pi_r < d^\pi_r \) is impossible by Claim 3, we get that either the assignments from both mechanism coincide entirely, or \( d^\pi_r > d^\pi_r \) for some \( r \in \{1, \ldots, m\} \), i.e., the Boston mechanism assigns strictly more \( r \)th choices than Deferred Acceptance.

Step 2 (for any single priority distributions): For any single priority distribution \( P \), a single priority profile \( \pi \) is drawn at random according to \( P \). By construction

\[
\begin{align*}
x &= NBM^P(P) = \sum_{\pi} NBM^\pi(P) \cdot P[\pi], \\
y &= DA^P(P) = \sum_{\pi} DA^\pi(P) \cdot P[\pi],
\end{align*}
\]

i.e., both \( x \) and \( y \) can we written as convex combinations of assignments \( x^\pi = NBM^\pi(P) \) and \( y^\pi = DA^\pi(P) \), respectively, with the same coefficients \( \alpha_{\pi} = P[\pi] \). By Step 1, each pair \( x^\pi, y^\pi \) has the property that \( d^\pi_r = d^\pi_r \) for \( r' < r \leq m \) and \( d^\pi_r > d^\pi_r \) (or \( d^\pi_r = d^\pi_r \)). Thus, by Lemma 3, \( DA^P(P) \) never strictly rank dominates \( NBM^P(P) \).

Step 3 (for any priority distribution): Harless (2015) showed that Claim 4 also holds for multiple priority profiles. We can apply the same reasoning as in Step 2 to obtain comparable rank dominance of \( NBM^P \) over \( DA^P \) for any priority distribution \( P \). \( \square \)
C.4. Efficiency of the Adaptive Boston Mechanism

Proof of Proposition 2.

1. For any priority distribution \( P \), \( ABM^P \) is ex-post efficient, but not ordinally or rank efficient in general.

2. Among all partially strategyproof mechanisms, \( ABM^U \) is not on the efficient frontier with respect to ordinal dominance, i.e., there exists a partially strategyproof mechanism that strictly ordinally dominates \( ABM^U \).

Proof. Ex-post Efficiency & Ordinal and Rank Inefficiency: The proof of ex-post efficiency is completely analogous to the proof of ex-post efficiency for NBM. Furthermore, ordinal and rank inefficiency can be seen using the same example as for NBM, because the assignments from NBM and ABM at the particular preference profile coincide.

Failure to be on the Efficient Frontier: We construct the mechanism \( ABM^+ \) in the same way as \( NBM^+ \), i.e., we take \( ABM^U \) as a baseline mechanism, but replace the outcomes for preference profiles with separable wants by the outcomes chosen by the PS mechanism.

As for \( NBM^+ \), we consider a swap of two adjacent schools in the preference report of student 1, such that \( P \) satisfies separable wants.

In case (i), when the new profile also satisfies separable wants, the assignment does not change, which is consistent with upper invariance and swap monotonicity.

In case (ii), when the composition of first choices changes, student 1 must have ranked her second choice first. In this case, she will receive this new first choice with probability \( \frac{1}{3} \) and the prior first choice with probability 0. This is also consistent with upper invariance and swap monotonicity.

Finally, in case (iii), the swap must involve \( c \) and \( d \). She will still receive her first choice with probability \( \frac{1}{2} \) and her second choice with probability 0 (as in the proof for \( NBM^U \)). Therefore, her assignment for the school she brought down, can only decrease, and her assignment for the school she brought up can only increase, and both change by the same absolute value. This is consistent with upper invariance and swap monotonicity.

Example 6. Consider a setting with 6 students \( N = \{1, \ldots, 6\} \), 6 schools \( M = \{a, \ldots, f\} \), each with unit capacity, i.e., \( q_j = 1 \) for all \( j \in M \), and the preferences

\[
P_1, \ldots, P_4 : \quad a > b > c > d > e > f,
P_5, P_6 : \quad e > b > a > d > f > c.
\]

Consider the single priority profile given by the priority order 1 \( \pi \ldots \pi 5 \), which ranks the students according to their names. The deterministic mechanism \( SD^\pi \) will assign schools a through d to students 1 through 4. Students 5 and 6 will get schools e and f. Thus, \( SD^\pi \) assigns 2 first, 1 second, 1 third, 1 fourth, and 1 fifth choice. For the same priority profile, \( ABM^\pi \) will assign a, b, c to students 1, 2, 3, respectively. Students 5 and 6 will get e and d, which leaves student 4 with f. Observe that in this case, \( ABM^\pi \) assigns 2 first, 1 second, 1 third, 1 fourth,
no fifth, and 1 sixth choice. This is strictly rank dominated by the rank efficient assignment chosen by SD for this fixed priority profile.

For the probabilistic assignments that arise under the single uniform priority distribution we get that

\[
ABM^U(P) = \frac{1}{60} \begin{pmatrix}
15 & 12 & 15 & 3 & 0 & 15 \\
15 & 12 & 15 & 3 & 0 & 15 \\
15 & 12 & 15 & 3 & 0 & 15 \\
0 & 6 & 0 & 24 & 30 & 0 \\
0 & 6 & 0 & 24 & 30 & 0
\end{pmatrix},
\]

(47)

and

\[
DA^U(P) = \frac{1}{60} \begin{pmatrix}
15 & 12 & 15 & 8 & 0 & 10 \\
15 & 12 & 15 & 8 & 0 & 10 \\
15 & 12 & 15 & 8 & 0 & 10 \\
0 & 6 & 0 & 14 & 30 & 10 \\
0 & 6 & 0 & 14 & 30 & 10
\end{pmatrix}.
\]

(48)

Since the rank distribution \(d^{DA^U}(P) = (\ldots)\) strictly stochastically dominates \(d^{ABM^U}(P) = (\ldots)\), \(DA^U\) strictly rank dominates \(ABM^U\) at \(P\). Note that in this case in fact all assignments chosen by \(DA^U\) are rank inefficient, but some of those chosen by \(ABM^U\) are not.

**Example 7.** Consider a setting with 5 students \(N = \{1, \ldots, 5\}\), 5 schools \(M = \{a, \ldots, e\}\), each with unit capacity, i.e., \(q_j = 1\) for all \(j \in M\), and the preferences

\[
P_1, P_2 : a > b > c > d > e,
P_3, P_4 : a > d > c > e > b,
P_5 : b > \ldots.
\]

For the single priority profile \(\pi = (\pi, \ldots, \pi)\) with \(1 \pi 3 \pi 4 \pi 2 \pi 5\), \(NBM^\pi\) will assign 1 to a, 5 to b, 3 to d, 4 to c, and 2 to e, which yields 2 first, 1 second, 1 third, and 1 fifth choices. \(ABM^\pi\) will also assign 1 to a, 5 to b, and 3 to d. However, 2 will get c and 4 will get e, so that we get 2 first, 1 second, 1 third, and 1 fourth choices, a strictly rank dominant assignment.

The probabilistic assignments under \(NBM^U\) and \(ABM^U\) are

\[
NBM^U(P) = \frac{1}{60} \begin{pmatrix}
15 & 0 & 25 & 0 & 20 \\
15 & 0 & 25 & 0 & 20 \\
15 & 0 & 5 & 30 & 10 \\
15 & 0 & 5 & 30 & 10 \\
0 & 60 & 0 & 0 & 0
\end{pmatrix},
\]

(49)

\[
ABM^U(P) = \frac{1}{60} \begin{pmatrix}
15 & 0 & 30 & 0 & 15 \\
15 & 0 & 30 & 0 & 15 \\
15 & 0 & 0 & 30 & 15 \\
15 & 0 & 0 & 30 & 15 \\
0 & 60 & 0 & 0 & 0
\end{pmatrix}.
\]

(50)
The rank distribution under $NBM^U$ is $d^{NBM^U}(P) = (2, 1, 1, 1/3, 2/3)$, which is dominated by $d^{ABM^U}(P) = (2, 1, 1/2, 1/2)$, the rank distribution under $ABM^U$.

**Proof of Theorem 5.** Let $(N^k, M^k, q^k)_{k \geq 1}$ be a sequence of settings such that

- the set of schools does not change, i.e., $M^k = M$ for all $k$,
- the capacity of each school increases, i.e., $\min_{j \in M} q_j^k \to \infty$ for $k \to \infty$,
- the number of students equals the number of seats, i.e., $|N^k| = \sum_{j \in M} q_j^k$.

Then the share of preference profiles where $DA^U$ rank dominates $ABM^U$ (even weakly) vanishes in the limit, i.e.,

$$\lim_{k \to \infty} \frac{\#\{P \in P^{N^k} : DA^U(P) \text{ rank dominates } ABM^U(P) \text{ at } P\}}{\#\{P \in P^{N^k}\}} = 0. \quad (51)$$

**Proof.** An assignment $x$ is first-choice-maximizing at preference profile $P$ if it can be represented as a lottery over deterministic assignments that give the maximum number of first choices, i.e.,

$$d^x_i = \sum_{i \in N} x_{i,j} \mathbb{1}_{r_i(j) = 1} = \max_{y \in X} d^y_i. \quad (52)$$

Since any ex-post efficient assignment is supported by a serial dictatorship, $DA^U$ puts positive probability on all ex-post efficient, deterministic assignments. In contrast, $ABM^U$ assigns positive probabilities to only some ex-post efficient, deterministic assignments. In particular, $ABM^U$ is first choice maximizing, i.e., it gives no probability to any assignment that does not yield the maximum possible number of first choices. Consequently, if at some preference profile $P$ there exists at least one ex-post efficient, deterministic assignment that is not first-choice-maximizing, then $DA^U$ will assign strictly less first choices than $ABM^U$. At these preference profiles, $DA^U$ is guaranteed not to rank dominate $ABM^U$ (even weakly).

Using this observation, we can now prove the following Claim 5, which in turn yields the limit result, Theorem 5.

**Claim 5.** For any fixed number of schools $m \geq 3$ and any $\epsilon > 0$, there exists $q_{\min} \in \mathbb{N}$, such that for any capacities $q_1, \ldots, q_m$ with $q_j \geq q_{\min}$ for all $j \in M$ and $n = \sum_{j \in M} q_j$ students, and for $P$ chosen uniformly at random from $P^n$, the probability that $DA^U(P)$ is first-choice-maximizing is smaller than $\epsilon$.

For a given preference profile $P \in P^N$, the first choice profile $k^P = (k_j^P)_{j \in M}$ is the vector of non-negative integers, where $k_j^P$ represents the number of students whose first choice is $j$. For a fixed setting, i.e., the triple $(N, M, q)$, we consider a uniform distribution on the space of preference profiles $P^n$. As the preference profile $P$ is held fixed, we suppress the index and simply write $k_j$. We say that a school $j \in M$ is

- un-demanded if $k_j = 0$, [50]
• *under-demanded* if \( k_j \in \{1, \ldots, q_j - 1\} \),
• *exhaustively demanded* if \( k_j = q_j \),
• and *over-demanded* if \( k_j > q_j \).

For any first choice profile \( k^P \), one of the following cases must hold:

(I) There is at least one un-demanded school.

(II) All schools are exhaustively demanded.

(III) No school is un-demanded, at least one school is over-demanded, and at least one other school is exhaustively demanded.

(IV) No school is un-demanded, but at least two schools are over-demanded.

(V) There is exactly one over-demanded school, and all other schools are under-demanded.

We will show that for fixed \( m \) and increasing minimum capacity, the probabilities for cases (I) and (II) become arbitrarily small. We will further show that in cases (III), (IV), and (V), the probabilities that \( DA^U \) assigns the maximum number of first choices become arbitrarily small.

(I) The probability that under a randomly chosen preference profile at least one school is un-demanded is upper-bounded by

\[
\frac{\binom{m}{1}(m-1)^n}{m^n} = m \left( \frac{m-1}{m} \right)^n, \tag{53}
\]

which converges to 0 as \( n = \sum_{j \in M} q_j \geq mq_{min} \) becomes large (where \( m \) is fixed).

(II) Let \( \tilde{q} = \frac{n}{m} \). Without loss of generality, \( \tilde{q} \) can be chosen as a natural number (otherwise, we increase the capacity of the school with least capacity until \( n \) is divisible by \( m \)). The probability that under a randomly chosen preference profile all schools are exhaustively demanded is

\[
\frac{\binom{m}{1}}{m^n} \leq \left( \frac{n}{\tilde{q} \ldots \tilde{q}} \right) \leq \frac{(mq)^m \sqrt{mq}}{\sqrt{\tilde{q}^m}} = \sqrt{\frac{m}{q^m-1}}, \tag{54}
\]

which converges to 0 as \( \tilde{q} \geq q_{min} \) becomes large (where \( m \) is fixed).

(III) Suppose that \( DA^U \) is first choice maximizing. If one school \( a \) is over-demanded and another school \( b \) is exhaustively demanded, then no student with first choice \( a \) can have \( b \) as second choice. Otherwise, there exists an order of the student such that a student with first choice \( a \) will get \( b \). In that case, \( b \) is not assigned entirely to students with first choice \( b \), and hence, the assignment can not maximize the number of first choices. Thus, the probability that the \( k_a \) students who have first choice \( a \) all have a second choice different from \( b \) (conditional on the first choice profile) is

\[
\left( \frac{m-2}{m-1} \right)^{k_1} \leq \left( \frac{m-2}{m-1} \right)^{q_1} \leq \left( \frac{m-2}{m-1} \right)^{q_{min}}. \tag{55}
\]

51
This becomes arbitrarily small for increasing $q_{\text{min}}$. Thus, the probability that the maximum number of first choices is assigned by $DA^U$, conditional on case (III) becomes small.

(IV) This case is analogous to (III).

(V) Suppose that for some preference profile consistent with case (V), $DA^U$ assigns the maximum number of first choices. Let $a$ be the school that is over-demanded and let $j_2, \ldots, j_m$ be the under-demanded schools. Then the maximum number of first choices is assigned if and only if

- $q_a$ students with first choice $a$ receive $a$, and
- all students with first choices $j_2, \ldots, j_m$ receive their respective first choice.

If $DA^U$ maximizes the number of first choices, then for any ordering of the students, the maximum number of first choices must be assigned, i.e., the two conditions are true. If the students with first choice $a$ get to pick before all other students, then they exhaust $a$ and get at most $q_j - k_j$ of the schools $j \neq a$: otherwise, if they got more than $q_j - k_j$ of school $j$, then some student with first choice $j$ would get a worse choice, which violates first choice maximization.

After any $q_a$ of the $k_a$ students with first choice $a$ consume school $a$, there are $k_a - q_a$ students left, which will consume other schools. Since $n = \sum_{j \in M} q_j = \sum_{j \in M} k_j$, we get that

$$k_a - q_a = n - \sum_{j \neq a} k_j - (n - \sum_{j \neq a} q_j) = \sum_{j \neq a} q_j - k_j.$$  \hspace{1cm} (56)

Therefore, the second choice profile of these $k_a - q_a$ students must be $(l_2, \ldots, l_m)$, where $l_r = q_{j_r} - k_{j_r} \geq 1$. In addition, some student $i'$ who consumed $a$ has second choice $j'$, and some student $i''$ with first choice $a$ gets its second choice $j'' \neq j'$. If we exchange the place of $i'$ and $i''$ in the ordering, $i''$ will get $a$ and $i'$ will get $j'$. But then $q_{j'} - k_{j'} + 1$ students with first choice $a$ get their second choice $j'$. Therefore, when the students with first choice $j'$ get to pick their schools, there are only $k_{j'} - 1$ copies of $j'$ left, which is not sufficient. Thus, we have constructed an ordering of the students under which the number of assigned first choices is not maximized. This implies that for any preference profile with first choice profile satisfying case (V), $DA^U$ does not assign the maximum number of first choices.

Combining the arguments for all cases, we can find $q_{\text{min}}$ sufficiently high, such that we can
estimate the probability that $DA^U$ maximizes first choices ($DA^U \text{ mfc.}$) by

$$Q[DA^U \text{ mfc.}] = Q[DA^U \text{ mfc.}]Q[I] + Q[DA^U \text{ mfc.}]Q[II] + Q[DA^U \text{ mfc.}]Q[III] + Q[DA^U \text{ mfc.}]Q[IV] + Q[DA^U \text{ mfc.}]Q[V] \leq Q[I] + Q[II] + Q[III] + Q[IV] + Q[V] \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + 0 = \epsilon.$$  

Here, $Q$ is the probability measure induced by the random selection of a preference profile.  

**Proof of Theorem 6.** Let $(N^k, M^k, q^k)_{k \geq 1}$ be a sequence of settings such that

- the number of schools equals the number of students, i.e., $|M^k| = |N^k| = k$,
- all schools have unit capacity, i.e., $q^k_j = 1$ for $j \in M^k$.

Then the share of preference profiles where $DA^U$ rank dominates $ABM^U$ (even weakly) vanishes in the limit, i.e.,

$$\lim_{k \to \infty} \frac{\#\{P \in \mathcal{P}^{N^k} : DA^U(P) \text{ rank dominates } ABM^U(P) \text{ at } P\}}{\#\{P \in \mathcal{P}^{N^k}\}} = 0.  \tag{65}$$

**Proof.** As for the proof of Theorem 5, we establish that $DA^U$ is almost never first choice maximizing at a randomly selected preference profile.

**Claim 6.** For any $\epsilon > 0$, there exists $n \in \mathbb{N}$, such that for any setting $(N, M, q)$ with $|M| = |N| \geq n$ and $q_j = 1$ for all $j \in M$, and for $P$ chosen uniformly at random, the probability that $DA^U(P)$ is first-choice-maximizing is smaller than $\epsilon$.

Recall that for a fixed preference profile $P$, $DA^U$ is first-choice maximizing only if all ex-post efficient assignments are first-choice maximizing. We will introduce no overlap, a necessary condition on the preference profile that ensures that $DA^U$ assigns the maximum number of first choices. Conversely, if a preference profile violates no overlap, $DA^U$ will not assign the maximum number of first choices. To establish Claim 6, we show that the share of preference profiles that exhibit no overlap vanishes as $n$ becomes large.

The proof requires some more formal definitions: for convenience, we will enumerate the set $M$ of schools by the integers $\{1, \ldots, n\}$. As in the proof of Theorem 5, $k^P = (k_1^P, \ldots, k_n^P)$ is called the first choice profile of the type profile $P$, where $k_j^P$ is the number of students whose
first choice is school $j$. To reduce notation, we suppress the superscript $P$. For some first choice profile $k$ and school $j$ we define the following indicators:

$$w_k(j) := \begin{cases} 
1, & \text{if } k_j \geq 1 \\
0, & \text{else} 
\end{cases} \quad \text{and} \quad o_k(j) := \begin{cases} 
1, & \text{if } k_j \geq 2 \\
0, & \text{else}. 
\end{cases} \quad (66)$$

$w$ indicates whether $j$ is *demanded*, i.e., it is the first choice of at least one student, and $o$ indicates whether $j$ is *over-demanded*, i.e., it is the first choice of more than one student.

Further, we define

$$W_k := \sum_{j \in M} w_k(j), \quad O_k := \sum_{j \in M} o_k(j), \quad C_k = \sum_{j \in M} k_j \cdot o_k(j). \quad (67)$$

$W_k$ is the number of schools that are demanded by at least one student, $O_k$ is the number of over-demanded schools, and $C_k$ is the number of students competing for over-demanded schools. Finally, a preference profile $P$ *exhibits overlap* if there exists a student $i \in N$ with first choice $j_1$ and second choice $j_2$, such that $o_{kP}(j_1) = 1$ and $w_{kP}(j_2) = 1$, i.e., student $i$’s first choice is over-demanded and its second choice is demanded as a first choice by at least one other student. As an example consider a setting where three students have preferences

$$P_1 : a > \ldots, \quad \succ_2 : a > b > \ldots, \quad \succ_3 : b > \ldots. \quad (68)$$

The maximum number of first choices that can be assigned is 2, e.g., by giving $a$ to 1 and $b$ to 3. But for the priority order $1 \pi 2 \pi 3$, student 1 will get $a$ and student 2 will get $b$. Then student 3 cannot take $b$, and consequently DA$^U$ will not assign the maximum number of first choices. If a preference profile exhibits overlap, a situation as in (68) will arise for some priority order, and therefore, DA$^U$ will not assign the maximum number of first choices. Conversely, no overlap in $P$ is a necessary condition for DA$^U(P)$ to assign the maximum number of first choices. We will show in the following that the share of preference profiles exhibiting no overlap becomes small for increasing $n$.

Consider a uniform distribution (denoted $Q$) on the preference profiles, i.e., all students draw their preference order independently and uniformly at random from the space of all possible preference orders. Then the statement that *the share of preference profiles exhibiting no overlap becomes small* is equivalent to the statement that the *probability of selecting a preference profile with no overlap converges to 0*. The proof of the following Claim 7 is technical and requires involved combinatorial and asymptotic arguments.

**Claim 7.** $Q[P \text{ no overlap}] \to 0$ for $n \to \infty$.

**Proof of Claim 7.** Using conditional probability, we can write the probability that a preference profile is without overlap as

$$Q[P \text{ no overlap}] = \sum_k Q[k = k^P] \cdot Q[P \text{ no overlap} | k = k^P]. \quad (69)$$
The number of preference profiles that have first choice profile \( k = (k_1, \ldots, k_n) \) is proportional to the number of ways to distribute \( n \) unique balls (students) across \( n \) urns (first choices), such that \( k_j \) balls end up in urn \( j \). Thus,

\[
Q[k = k^P] = \frac{(k_1, \ldots, k_n)(n-1)^n}{(n!)^n} = \frac{(k_1, \ldots, k_n)}{n^n}.
\] (70)

In order to ensure no overlap, a student with an over-demanded first choice cannot have as her second choice a school that is the first choice of any other student. Students whose first choice is not over-demanded can have any school (except for their own first choice) as second choice. Thus, given a first choice profile \( k \), the conditional probability of no overlap is

\[
Q[P \text{ no overlap} | k = k^P] = \prod_{j \in M} \left( (1 - \alpha_k(j)) + \alpha_k(j) \left( \frac{n - W_k}{n-1} \right)^{k_j} \right)
\] (71)

\[
= \left( \frac{n - W_k}{n-1} \right)^{\sum_{j \in M} k_j \cdot \alpha_k(j)} = \left( \frac{n - W_k}{n-1} \right)^{C_k}
\] (72)

\[
= \left( \frac{C_k - O_k}{n-1} \right)^{C_k},
\] (73)

where the last equality holds, since \( n - W_k = n - (n - C_k + O_k) = C_k - O_k \). Thus, the probability of no overlap can be determined as

\[
Q[P \text{ no overlap}] = \frac{1}{n^n} \sum_{k} \left( \begin{array}{c} n \\ k_1, \ldots, k_n \end{array} \right) \left( \frac{C_k - O_k}{n-1} \right)^{C_k}.
\] (74)

\( C_k \) is either 0 or \( \geq 2 \), since a single student cannot be in competition. If no students compete \( (C_k = 0) \), all must have different first choices. Thus, for \( k = (1, \ldots, 1) \), the term in the sum in (74) is

\[
\left( \begin{array}{c} n \\ k_1, \ldots, k_n \end{array} \right) \left( \frac{C_k - O_k}{n-1} \right)^{C_k} = \left( \begin{array}{c} n \\ 1, \ldots, 1 \end{array} \right) \cdot 1 = n!.
\] (75)

Using this and sorting the terms for summation by \( c \) for \( C_k \) and \( o \) for \( O_k \), we get

\[
Q[P \text{ no overlap}] = \frac{1}{n^n} \left[ n! + \sum_{c=2}^{n} \sum_{o=1}^{\left\lfloor \frac{n}{c-1} \right\rfloor} \left( \frac{c - o}{n-1} \right)^{c} \sum_{k:C_k=c, O_k=o} \left( \begin{array}{c} n \\ k_1, \ldots, k_n \end{array} \right) \right].
\] (76)

Consider the inner sum

\[
\sum_{k:C_k=c, O_k=o} \left( \begin{array}{c} n \\ k_1, \ldots, k_n \end{array} \right)
\] (77)

in (76): with a first choice profile \( k \) that satisfies \( C_k = c \) and \( O_k = o \) there are exactly \( o \) over-demanded schools (i.e., schools \( j \) with \( k_j \geq 2 \)), \( n - c \) singly-demanded schools (with
\( k_j = 1 \), and \( c - o \) un-demanded schools (with \( k_j = 0 \)). Therefore,

\[
\sum_{\mathbf{k}: \mathbf{c}_k = c, \mathbf{o}_k = o} \binom{n}{k_1, \ldots, k_n} = \binom{n}{c - o} \sum_{\mathbf{k}' = (k_1', \ldots, k_{n-c+o}')} \binom{n}{k_1', \ldots, k_{n-c+o}'}
\]

\[
= \binom{n}{c - o} \binom{n - c + o}{n - c} \frac{n!}{c!} \sum_{\mathbf{k}' = (k_1', \ldots, k_{n-c+o}') : k_{n-c+o}' \geq 2} \binom{c}{k_1', \ldots, k_{n-c+o}'}. \quad (78)
\]

The first equality holds, because we simply choose \( c - o \) of the \( n \) schools to be un-demanded, and

\[
\binom{n}{k_1, \ldots, k_{r-1}, 0, k_{r+1}, \ldots, k_m} = \binom{n}{k_1, \ldots, k_{r-1}, k_{r+1}, \ldots, k_m}. \quad (80)
\]

The second equality holds, because we select the \( n - c \) singly-demanded schools from the remaining \( n - c + o \) schools as well as the \( n - c \) students to demand them. The sum (79) is equal to the number of ways to distribute \( c \) unique balls to \( o \) unique urns such that each urn contains at least 2 balls. This in turn is equal to

\[
o! \left\{ \left\{ \begin{array}{c} c \\ o \end{array} \right\} \right\}, \quad (81)
\]

where \( \left\{ \left\{ \vdots \right\} \right\} \) denotes the 2-associated Stirling number of the second kind. This number represents the number of ways to partition \( c \) unique balls such that each partition contains at least 2 balls. The factor \( o! \) in (81) is included to make the partitions unique. \( \left\{ \left\{ \vdots \right\} \right\} \) is upper-bounded by \( \{ \vdots \} \), the Stirling number of the second kind, which represents the number of ways to partition \( c \) unique balls such that no partition is empty. Furthermore, the Stirling number of the second kind has the upper bound

\[
\left\{ \begin{array}{c} c \\ o \end{array} \right\} \leq \binom{c}{o} o^{c-o}. \quad (82)
\]

Thus, the sum in (79) can be upper-bounded by

\[
\sum_{\mathbf{k}' = (k_1', \ldots, k_{n-c+o}') : k_{n-c+o}' \geq 2} \binom{c}{k_1', \ldots, k_{n-c+o}'} \leq o! \binom{c}{o} o^{c-o}. \quad (83)
\]

Combining all the previous observations, we can estimate the probability \( Q[\mathbf{P} \text{ no overlap}] \) from (76) by

\[
Q[\mathbf{P} \text{ no overlap}] \leq \frac{1}{n^n} \left[ n! + \sum_{c=2}^{\lfloor \frac{n}{2} \rfloor} \sum_{o=1}^{\lfloor \frac{c-o}{n-1} \rfloor} \binom{c-o}{o} \binom{n}{c-o} \binom{n-c+o}{n-c} \binom{c}{o} \frac{n! o!}{c!} o^{c-o} \right]. \quad (84)
\]

The Stirling approximation yields

\[
\sqrt{2\pi e} \frac{1}{12n+1} \leq \frac{n!}{\sqrt{n (\frac{n}{e})^n}} \leq \sqrt{2\pi e} \frac{1}{12n}, \quad (85)
\]

56
and therefore \( n! \approx \left( \frac{n}{e} \right)^n \sqrt{n} \) up to a constant factor. Using this, we observe that the first term in (84) converges to 0 as \( n \) increases, i.e.,

\[
\frac{n!}{n^n} \approx \frac{\sqrt{n}}{e^n} \to 0 \text{ for } n \to \infty.
\]  

(86)

Now we need to estimate the double sum in (84):

\[
\frac{1}{n^n} \sum_{c=2}^{n} \sum_{o=1}^{\lceil \frac{n}{2} \rceil} \left( \frac{c-o}{n-1} \right)^c \left( \frac{n}{c} \right) \left( \frac{n-c+o}{n-c} \right) \left( \frac{c}{o} \right) \frac{n!o!}{c!o^{c-o}}
\]  

\[
\approx \frac{n!}{n^n} \sum_{c=2}^{n} \sum_{o=1}^{\lceil \frac{n}{2} \rceil} \left( \frac{n-c+o}{n-c} \right) \left( \frac{n}{c} \right) \frac{1}{n^c} \cdot \left( \frac{c-o}{o} \right)^{c-o} \left( \frac{n-o}{n-1} \right) \frac{n!}{c!o^{c-o}}
\]  

\[
\leq \left[ e\sqrt{n} \left( \frac{n}{n-1} \right)^{n-1} \left( \frac{n}{n} \right) \right] \cdot \frac{1}{e^n} \sum_{c=2}^{n} \sum_{o=1}^{\lceil \frac{n}{2} \rceil} \left( \frac{n}{c} \right) \left( \frac{c}{o} \right) \frac{1}{n^c} \cdot \left( \frac{c-o}{o} \right)^{c-o} e^{c-o}
\]  

\[
\leq \left[ e\sqrt{n} \right] \frac{1}{e^n} \sum_{c=2}^{n} \sum_{o=1}^{\lceil \frac{n}{2} \rceil} \left( \frac{n}{c} \right) \left( \frac{c}{o} \right) \frac{1}{n^c} \cdot \left( \frac{c-o}{o} \right)^{c-o}
\]  

\[
\leq \frac{\sqrt{n}}{e^n} \sum_{c=2}^{n} \sum_{o=1}^{\lceil \frac{n}{2} \rceil} \left( \frac{n}{c} \right) \left( \frac{c}{o} \right) \frac{1}{n^c} \cdot \left( \frac{c-o}{o} \right)^{c-o}
\]  

(91)

where we use that \( (1 + \frac{x}{n})^n \leq e^x \). Using the binomial theorem and the fact that the function \( o \mapsto (c-o)^o o^{-o} \) is maximized by \( o = \frac{c}{2} \), we can further estimate (91) by

\[
\left[ e\sqrt{n} \right] \frac{1}{e^n} \sum_{c=2}^{n} \sum_{o=1}^{\lceil \frac{n}{2} \rceil} \left( \frac{n}{c} \right) \left( \frac{c}{o} \right) \frac{1}{n^c} \cdot \left( \frac{c-o}{o} \right)^{c-o} \leq \left[ e\sqrt{n} \right] \frac{1}{e^n} \sum_{c=2}^{n} \sum_{o=1}^{\lceil \frac{n}{2} \rceil} \left( \frac{n}{c} \right) \left( \frac{c}{o} \right) \frac{1}{n^c} \cdot \left( \frac{c-o}{o} \right)^{c-o}
\]  

\[
\leq \frac{\sqrt{n}}{e^n} \sum_{c=2}^{n} \sum_{o=1}^{\lceil \frac{n}{2} \rceil} \left( \frac{n}{c} \right) \left( \frac{c}{o} \right) \frac{1}{n^c} \cdot \left( \frac{c-o}{o} \right)^{c-o}
\]  

(92)

(93)

with \( \alpha = \frac{(1+e)}{2} \). To estimate the sum in (93), we first consider even \( n \) and note the following:

- \( \alpha = \frac{(1+e)}{2} \approx 1.85914 \ldots < e \), and therefore, the last term of the sum for \( c = n \) can be ignored as \( \binom{n}{n} \left( \frac{n}{e} \right)^n \to 0 \) for \( n \to \infty \).

- \( \binom{n}{c} = \binom{n}{n-c} \), and therefore, both terms \( \binom{n}{c} \left( \frac{c}{n} \right)^c \) and \( \binom{n}{c} \left( \frac{n-c}{n} \right)^{n-c} \) have the same binomial coefficient in the sum.

- The idea is to estimate the sum of both terms by an exponential function of the form \( c \mapsto e^{mc+b} \), where \( m \) and \( b \) depend only on \( n \) and \( \alpha \).
• Indeed, the log of the sum, the function $c \mapsto \log \left( (\alpha \frac{c}{n})^c + (\alpha \frac{n-c}{n})^{n-c} \right)$, is strictly convex and on the interval $[1, \frac{n}{2}]$ it is upper-bounded by the linear function

$$f(c) = \left( \frac{\log(4)}{n} - \log(2\alpha) \right) c + n \log(\alpha). \quad (94)$$

• Thus,

$$\binom{n}{c} \left( \frac{\alpha c}{n} \right)^c + \left( \frac{n}{n-c} \right) \left( \frac{\alpha (n-c)}{n} \right)^{n-c} \leq \binom{n}{c} e^{f(c)}. \quad (95)$$

We can bound (93) by

$$[e^{\sqrt{n}}] \frac{1}{e^n} \sum_{c=1}^{\frac{n}{2}} \binom{n}{c} e^{f(c)} = \left[ e^{\sqrt{n}} \right] \frac{1}{e^n} \sum_{c=1}^{\frac{n}{2}} \binom{n}{c} \frac{4\pi}{e^n} \alpha^n \left( \frac{1}{2\alpha} \right)^c \leq \left[ 4e^{\sqrt{n}} \right] \left( \frac{\alpha (1 + \frac{1}{2\alpha})}{e^n} \right)^n \approx \left[ 4e^{\sqrt{n}} \right] \left( \frac{2.35914 \ldots}{e} \right)^n. \quad (96)$$

Since $2.35914 < e$, the exponential convergence of the last term dominates the divergence of the first terms, which is of the order $\sqrt{n}$, and the expression converges to 0.

For odd $n$ the argument is essentially the same, except that we need to also consider the central term (for $c = \frac{n}{2} + 1$) separately.

$$\binom{n}{\frac{n}{2} + 1} \left( \frac{\alpha \frac{n}{2} + 1}{2} \right)^{\frac{n}{2} + 1} \leq 2^n \left( \sqrt{\alpha \left( \frac{1}{2} + \frac{1}{n} \right) \left( \frac{1}{2} + \frac{1}{n} \right)} \right)^n \cdot \alpha \left( \frac{1}{2} + \frac{1}{n} \right) \approx \left( \sqrt{\alpha \left( \frac{2}{n} \right) \left( \frac{4}{n} \right)} \right)^n \cdot \alpha \left( \frac{1}{2} + \frac{1}{n} \right). \quad (99)$$

With $\sqrt{\alpha \left( \frac{2}{n} + \frac{4}{n} \right)} \approx 1.92828 \ldots < e$, the result follows for odd $n$ as well.

This completes the proof of Theorem 6.