

An Axiomatic Approach to Characterizing and Relaxing Strategyproofness of One-sided Matching Mechanisms*

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Abstract

We study one-sided matching mechanisms where agents have vNM utility functions and report ordinal preferences. We first show that in this domain strategyproof mechanisms are characterized by three intuitive axioms: *swap monotonicity*, *upper invariance*, and *lower invariance*. Our second result is that dropping lower invariance leads to an interesting new relaxation of strategyproofness, which we call *partial strategyproofness*. In particular, we show that mechanisms are swap monotonic and upper invariant if and only if they are strategyproof on a restricted domain where agents have sufficiently different valuations for different objects. Furthermore, we show that this domain restriction is maximal and use it to define a single-parameter measure for the *degree of strategyproofness* of a manipulable mechanism. We also provide an algorithm that computes this measure. Our new partial strategyproofness concept finds applications in the incentive analysis of non-strategyproof mechanisms, such as the Probabilistic Serial mechanism, different variants of the Boston mechanism, and the construction of new hybrid mechanisms.

1. Introduction

The one-sided matching problem is concerned with the allocation of indivisible goods to self-interested agents with privately known preferences. Monetary transfers are not permitted, which makes this problem different from auctions and other settings with transferable utility.

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The first version of this problem was introduced by Hylland and Zeckhauser (1979), and has since attracted much attention from mechanism designers (e.g., Abdulkadiroğlu and Sönmez (1998); Bogomolnaia and Moulin (2001); Abdulkadiroğlu and Sönmez (2003)). In practice, such problems often arise in situations that are of great importance to peoples’ lives. For example, students must be matched to schools, teachers to training programs, houses to tenants, etc. (Niederle, Roth and Sönmez, 2008).

As mechanism designers, we care specifically about *efficiency*, *fairness*, and *strategyproofness*. Zhou (1990) showed that it is impossible to achieve the optimum on all three dimensions simultaneously, which makes the one-sided matching problem an interesting mechanism design challenge. Especially strategyproofness is a severe design constraint: the folklore Random Serial Dictatorship mechanism is strategyproof and anonymous, but only *ex-post efficient*. The more demanding *ordinal efficiency* is achieved by the Probabilistic Serial mechanism (PS), but any mechanism that guarantees ordinal efficiency and strategyproofness will violate symmetry (Bogomolnaia and Moulin, 2001). Finally, *rank efficiency*, an even stronger efficiency concept, can be achieved via Rank-value mechanisms (Featherstone, 2011), but is incompatible even with *weak strategyproofness*. Obviously, trade-offs are necessary, and researchers have called for useful relaxations (e.g., Azevedo and Budish (2012); Budish (2012)).

In recent years, some progress on *approximate strategyproofness* has been made for quasi-linear domains (see (Lubin and Parkes, 2012) for a survey). However, these approaches do not translate to the matching domain, where a relaxed notion of strategyproofness has remained elusive so far. In our view, a relaxed strategyproofness concept should address the following two questions that commonly arise in this domain:

- 1) What honest and useful strategic advice can we give to agents?
- 2) How can we measure “how strategyproof” a manipulable mechanism is, e.g., PS or the Boston mechanism?

In this paper, we take an axiomatic approach to the problem of characterizing and relaxing strategyproofness of one-sided matching mechanisms. Our new *partial strategyproofness* concept provides intriguing answers to both questions.

1.1. One Familiar and Two New Axioms

Suppose an agent considers swapping two adjacent objects in its reported preference order, e.g., from $a > b$ to $b > a$. Our axioms restrict the way in which the mechanism can react to this kind of change of report (i.e., how this can affect the allocation of the reporting agent). The first axiom, *swap monotonicity*, requires that either the allocation remains unchanged, or the allocation for b must strictly increase, and the allocation for a must strictly decrease. This means that the mechanism is responsive to the agent’s ranking of a and b and the swap affects at least the objects a and b , if any. The second axiom, *upper invariance*, requires that the allocation for all objects strictly preferred to a does not change under the swap. This essentially means that the mechanism is robust to manipulation by truncation, i.e., falsely claiming higher preference for an outside option. Upper invariance was introduced by Hashimoto et al. (2013)

(called *weak invariance* in their paper) as one of the axioms to characterize PS. Finally, we introduce *lower invariance*, which requires that the allocation does not change for any object that the agent ranks below b . It turns out that one-sided matching mechanisms are strategyproof if and only if they satisfy all three axioms, which is our first main result.

1.2. Partial Strategyproofness and Bounded Indifference

For our second main result we drop the lower invariance axiom. To understand incentives in mechanisms that are swap monotonic and upper invariant, we define a relaxed notion of strategyproofness. Intuitively, the mechanism must be strategyproof on a restricted domain, where agents have sufficiently different values for different objects.

A utility function satisfies *uniformly relatively bounded indifference* with respect to bound $r \in (0, 1]$ (URBI(r)) if, given $a > b$, the agent's normalized value for b is at least a factor r lower than its value for a , i.e., $r \cdot u(a) \geq u(b)$. We say that a mechanism is *r -partial strategyproof* if the mechanism is strategyproof for all agents whose utility functions are bounded away from indifference by the factor r . Our second main result is the following equivalence: for any setting (i.e., number of agents, number of objects, and object capacities) a mechanism is swap monotonic and upper invariant if and only if it is r -partially strategyproof for some $r \in (0, 1]$.

1.3. Overview of Contributions

The main contributions of this paper are an axiomatic characterization of strategyproofness and a characterization of r -partial strategyproofness, an intuitive relaxation of strategyproofness. We obtain the following results:

- 1) **Axiomatic Characterization of Strategyproof Mechanisms (Thm. 1):** we show that a mechanism is swap monotonic, upper invariant, and lower invariant *if and only if* it is strategyproof.
- 2) **Axiomatic Characterization of r -partially Strategyproof Mechanisms (Thm. 2):** we show that a mechanism is upper invariant and swap monotonic *if and only if* there exists $r \in (0, 1]$ such that it is r -partially strategyproof. Here, the bound r may depend on the setting.
- 3) **Maximality of the URBI(r) Domain Restriction (Thm. 3):** we show that given any setting with $m \geq 3$ objects, any bound $r \in (0, 1)$, and any utility function \tilde{u} that violates URBI(r), there exists a mechanism \tilde{f} that is r -partially strategyproof, but manipulable for \tilde{u} .
- 4) **Degree of Strategyproofness and Computability (Def. 7 & Prop. 2):** we introduce the maximum value of r as a measure for the degree of strategyproofness of a mechanism. We also show how r -partial strategyproofness can be algorithmically verified and how the degree of strategyproofness can be computed (although the algorithm we present has exponential complexity).

To the best of our knowledge, we present the first axiomatic characterization of strategyproof one-sided matching mechanisms in the vNM-utility domain. Based on this, our axiomatic treatment leads to a new way of thinking about how to relax strategyproofness. Furthermore, r -partial strategyproofness is the first parametric relaxation of strategyproofness in this domain. We also demonstrate how r -partial strategyproofness can be used to analyze the incentive properties of popular non-strategyproof mechanisms, like Probabilistic Serial or variants of the Boston mechanism.

2. Related Work

While the seminal paper on one-sided matching mechanisms by [Hylland and Zeckhauser \(1979\)](#) proposed a mechanism that elicits agents' *cardinal* utilities, this approach has proven problematic because it is difficult if not impossible to elicit cardinal utilities in settings without money. For this reason, recent work has focused on *ordinal mechanisms*, where agents submit ordinal preference reports, i.e., rankings over all objects (for an example see ([Abdulkadiroğlu, Pathak and Roth, 2005](#)), or ([Carroll, 2011a](#)) for a systematic argument). Throughout this paper, we only consider ordinal mechanisms.

For the deterministic case, strategyproofness of one-sided matching mechanisms has been studied extensively. [Papai \(2000\)](#) showed that the only group-strategyproof, ex-post efficient, reallocation-proof mechanisms are hierarchical exchanges. Characterizations of strategyproof, efficient, and reallocation-consistent ([Ehlers and Klaus, 2006](#)) or consistent ([Ehlers and Klaus, 2007](#)) mechanisms are also available. The only deterministic, strategyproof, ex-post efficient, non-bossy, and neutral mechanisms are known to be serial dictatorships ([Hatfield, 2009](#)). Furthermore, [Pycia and Ünver \(2014\)](#) showed that all group-strategyproof, ex-post efficient mechanisms are trading cycles mechanisms. [Barbera, Berga and Moreno \(2012\)](#) gave a characterization of strategyproofness that is similar in spirit to ours, but is restricted to deterministic social choice domains.

For non-deterministic mechanisms, [Abdulkadiroğlu and Sönmez \(1998\)](#) showed that Random Serial Dictatorship (RSD) is equivalent to the Core from Random Endowments mechanism for house allocation. [Bade \(2013\)](#) extended their result by showing that taking any ex-post efficient, strategyproof, non-bossy, deterministic mechanism and assigning agents to roles in the mechanism uniformly at random is equivalent to RSD. However, it is still an open conjecture whether RSD is the unique mechanism that is anonymous, ex-post efficient, and strategyproof.

The research community has also introduced stronger efficiency concepts, such as ordinal efficiency. The original Probabilistic Serial (PS) mechanism introduced by [Bogomolnaia and Moulin \(2001\)](#) was only defined for strict preferences. [Katta and Sethuraman \(2006\)](#) introduced an extension of the PS mechanism that allows agents to be indifferent between goods. Recently, [Hashimoto et al. \(2013\)](#) showed that the unique mechanism that is ordinally fair and non-wasteful is PS with uniform eating speeds. [Bogomolnaia and Moulin \(2001\)](#) had already shown that PS is not strategyproof, but [Kesten and Ekici \(2012\)](#) recently found that its Nash equilibria can lead to ordinally dominated outcomes. While incentive concerns for PS may be severe for small settings, they get less problematic for larger settings: [Kojima and Manea \(2010\)](#) showed

that for a fixed number of object types and a fixed agent, PS makes it a dominant strategy for that agent to be truthful if the number of copies of each object is sufficiently large.

While ex-post efficiency and ordinal efficiency are the most well-studied efficiency concepts for one-sided matching mechanisms, some mechanisms used in practice aim to maximize *rank efficiency*, which is a further refinement of ordinal efficiency (Featherstone, 2011). However, no rank efficient mechanism can be strategyproof in general. Other popular mechanisms, like the Boston Mechanism (Ergin and Sönmez, 2006; Miralles, 2009), are highly manipulable but nevertheless in frequent use. Budish and Cantillon (2012) found practical evidence from combinatorial course allocation, suggesting that using a non-strategyproof mechanism may lead to higher social welfare than using an ex-post efficient and strategyproof mechanism, such as RSD. This challenges whether strategyproofness should be a hard constraint for mechanism designers.

Given that strategyproofness is such a strong restriction, many researchers have tried to relax it, using various notions of *approximate strategyproofness*. For example, Carroll (2011b) took this approach in the voting domain and quantified the incentives to manipulate (for certain normalized utilities). Budish (2011) proposed the interesting Competitive Equilibrium from Approximately Equal Incomes mechanism for the domain of combinatorial assignments. For the single-object assignment domain, this reduces to RSD. Finally, Dütting et al. (2012) used a machine learning approach to design mechanisms with “good” incentive properties. Instead of requiring strategyproofness, they minimize the agents’ ex-post regret, i.e., the utility increase an agent could gain from manipulating. However, their notion of approximate strategyproofness only applies in quasi-linear domains and does not translate to the matching domain. Our axiomatic approach differs from these ideas, because instead of quantifying manipulation incentives (i.e., the potential utility gain from manipulation), we consider strategyproofness-like guarantees, which need only hold on a specified subset of the entire utility space.

Recently, Azevedo and Budish (2012) proposed a desideratum called *Strategyproof in the Large* (SP-L), which is applicable to the matching domain and formalizes the intuition that as the number of agents in the market gets large, the incentives for individual agents to misreport their preferences vanish in the limit. In contrast, our concepts presented in this paper apply to any problem size.

In order to compare different mechanisms by their vulnerability to manipulation, Pathak and Sönmez (2013) introduced a general framework. It is consistent with the measure for the degree of strategyproofness we propose in this paper. However, our concept has two advantages: it yields a parametric relaxation of strategyproofness and we show that it is computable. We discuss the connection in more detail in Section 8.2.

3. Model

A *setting* (N, M, \mathbf{q}) consists of a set N of n agents, a set M of m objects, and a vector $\mathbf{q} = (q_1, \dots, q_m)$ of *capacities*, i.e., there are q_j units of object j available. We assume that supply satisfies demand, i.e., $n \leq \sum_{j \in M} q_j$, since we can always add dummy objects. Agents are endowed with *von Neumann-Morgenstern* (vNM) utilities $u_i, i \in N$, over the objects. If

$u_i(a) > u_i(b)$, we say that agent i *prefers object a over object b* , which we denote by $a \succ_i b$. We work with the standard model, which excludes indifferences, i.e., $u_i(a) = u_i(b)$ implies $a = b$. A utility function u_i is *consistent with preference ordering \succ_i* if $a \succ_i b$ whenever $u_i(a) > u_i(b)$. Given a preference ordering \succ_i , the corresponding *type t_i* is the set of all utilities that are consistent with \succ_i , and T is the set of all types, called the *type space*. We use types and preference orderings synonymously.

An *allocation* is a (possibly probabilistic) assignment of objects to agents. It is represented by an $n \times m$ -matrix $x = (x_{i,j})_{i \in N, j \in M}$ satisfying the *fulfillment constraint* $\sum_{j \in M} x_{i,j} = 1$, the *capacity constraint* $\sum_{i \in N} x_{i,j} \leq q_j$, and $x_{i,j} \in [0, 1]$ for all $i \in N, j \in M$. The entries of the matrix are interpreted as probabilities, where i gets j with probability $x_{i,j}$. An allocation is *deterministic* if $x_{i,j} \in \{0, 1\}$ for all $i \in N, j \in M$. The Birkhoff-von Neumann Theorem and its extensions (Budish et al., 2013) ensure that given any allocation, we can always find a lottery over deterministic assignments that implements these marginal probabilities. Finally, let \mathcal{X} denote the space of all allocations.

A *mechanism* is a mapping $f : T^n \rightarrow \mathcal{X}$ that chooses an allocation based on a profile of reported types. We let $f_i(t_i, t_{-i})$ denote the allocation that agent i receives if it reports type t_i and the other agents report $t_{-i} = (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n) \in T^{n-1}$. Note that mechanisms only receive type profiles as input. Thus, we consider *ordinal* mechanisms, where the allocation is independent of the actual vNM utilities. If i and t_{-i} are clear from the context, we may abbreviate $f_i(t_i, t_{-i})$ by $f(t_i)$. The expected utility for i is given by the dot product $\langle u_i, f(t_i) \rangle$, i.e.,

$$\mathbb{E}_{f_i(t_i, t_{-i})}(u_i) = \sum_{j \in M} u_i(j) \cdot f_i(t_i, t_{-i})(j) = \langle u_i, f(t_i) \rangle. \quad (1)$$

For *strategyproof* mechanisms, reporting truthfully maximizes agents' expected utility:

Definition 1 (Strategyproofness). *A mechanism f is strategyproof if for any agent $i \in N$, any type profile $\mathbf{t} = (t_i, t_{-i}) \in T^n$, any misreport $t'_i \in T$, and any utility $u_i \in t_i$ we have*

$$\langle u_i, f(t_i) - f(t'_i) \rangle \geq 0. \quad (2)$$

Our model encompasses classical one-sided matching problems, such as house allocation and school choice markets, where only one side of the market has preferences. It is also straightforward to accommodate outside options. Priorities over the agents can be included implicitly in the mechanism.

4. The Axioms

We now define our axioms. A type t' that differs from another type t by just a swap of two adjacent objects in the corresponding preference orderings is said to be in the *neighborhood N_t* of t .

Definition 2 (Neighborhood). *The neighborhood of a type t is the set N_t of all types t' such that there exists $k \in \{1, \dots, m-1\}$ with*

$$\begin{aligned} t &: a_1 > \dots > a_k > a_{k+1} > \dots > a_m, \\ t' &: a_1 > \dots > a_{k+1} > a_k > \dots > a_m. \end{aligned}$$

The *upper contour set* of an object a with respect to some type t is the set of objects that are strictly preferred to a by an agent of type t , and the *lower contour set* are all the objects that the agent likes strictly less than a . Formally:

Definition 3 (Contour Sets). *For a type $t : a_1 > \dots > a_k > \dots > a_m$, the upper contour set $U(a_k, t)$ and lower contour set $L(a_k, t)$ are given by*

$$U(a_k, t) = \{a_1, \dots, a_{k-1}\} = \{j \in M \mid j > a_k\}, \quad (3)$$

$$L(a_k, t) = \{a_{k+1}, \dots, a_m\} = \{j \in M \mid a_k > j\}. \quad (4)$$

A swap of two adjacent objects is a basic manipulation. Our axioms limit the way in which a mechanism can change the allocation under this basic manipulation. This makes the axioms intuitive and simple.

Axiom 1 (Swap Monotonicity). *A mechanism f is swap monotonic if for any agent $i \in N$, any type profile $\mathbf{t} = (t_i, t_{-i}) \in T^n$, and any type $t'_i \in N_{t_i}$ (i.e., in the neighborhood of t_i) with $a_k > a_{k+1}$ under t_i and $a_{k+1} > a_k$ under t'_i , one of the following holds:*

- 1) *i 's allocation is unaffected by the swap, i.e., $f(t_i) = f(t'_i)$, or*
- 2) *i 's allocation for a_k strictly decreases and its allocation for a_{k+1} strictly increases, i.e.,*

$$f(t_i)(a_k) > f(t'_i)(a_k) \text{ and } f(t_i)(a_{k+1}) < f(t'_i)(a_{k+1}). \quad (5)$$

Swap monotonicity is an intuitive axiom, because it simply requires the mechanism to react to the swap in a *direct* and *responsive* way. The swap reveals information about the relative ranking of a_k and a_{k+1} for the agent; thus, if anything changes about the allocation for that agent, the objects a_k and a_{k+1} must be affected directly, or else no other object may be affected. In addition, the mechanism must respond to the agent's preferences by allocating more of the object the agent claims to like more and less of the object the agent claims to like less.

Consider a mechanism that gives you chocolate ice cream if you ask for vanilla, and gives you vanilla if you ask for chocolate. This mechanism is obviously extremely manipulable, and swap monotonicity prevents this kind of manipulability. Nevertheless, even swap monotonic mechanisms may be manipulable in a first order-stochastic dominance sense. However, the manipulations must involve changes of the allocation of other objects besides a_k and a_{k+1} as well. Example 1 presents such a mechanism.

Example 1. *Consider a mechanism where reporting $t_i : a > b > c > d$ leads to an allocation of $(0, 1/2, 0, 1/2)$ of a, b, c, d , respectively, and reporting $t'_i : a > c > b > d$ leads to an allocation of $(1/2, 0, 1/2, 0)$. This mechanism is swap monotonic, but the latter allocation first order-stochastically dominates the former for t_i . Thus, even under swap monotonicity, manipulations by some agent may produce first order-stochastically dominant outcomes for that agent.*

To ensure that this does not happen, we need an additional axiom.

Axiom 2 (Upper Invariance). *A mechanism f is upper invariant if for any agent $i \in N$, any type profile $\mathbf{t} = (t_i, t_{-i}) \in T^n$, and any type $t'_i \in N_{t_i}$ with $a_k > a_{k+1}$ under t_i and $a_{k+1} > a_k$ under t'_i , i 's allocation for the upper contour set $U(a_k, t_i)$ is unaffected by the swap, i.e., $f(t_i)(j) = f(t'_i)(j)$ for all $j \in U(a_k, t_i)$.*

Intuitively, under upper invariance, an agent cannot influence the allocation of one of its better choices by swapping two less preferred objects. Upper invariance was introduced by [Hashimoto et al. \(2013\)](#) as one of the central axioms to characterize the Probabilistic Serial mechanism. If a null object is present and the mechanism is individually rational, then upper invariance is equivalent to *truncation robustness*. Truncation robustness is a type of robustness to manipulation that is important in theory and application: it prevents that by bringing the null object up in its report, an agent can increase its chances of being allocated a more preferred object. Many mechanisms from the literature satisfy upper invariance, such as Random Serial Dictatorship, Probabilistic Serial, the Boston mechanism, and Student-proposing Deferred Acceptance.

Axiom 3 (Lower Invariance). *A mechanism f is lower invariant if for any agent $i \in N$, any type profile $\mathbf{t} = (t_i, t_{-i}) \in T^n$, and any type $t'_i \in N_{t_i}$ with $a_k > a_{k+1}$ under t_i and $a_{k+1} > a_k$ under t'_i , i 's allocation for the lower contour set $L(a_{k+1}, t_i)$ is unaffected by the swap, i.e., $f(t_i)(j) = f(t'_i)(j)$ for all $j \in L(a_{k+1}, t_i)$.*

Lower invariance complements upper invariance: it requires that an agent cannot influence the allocation for *less* preferred objects by swapping two more preferred objects. Lower invariance has a subtle effect on incentives: if agents were endowed with upward-lexicographic preferences ([Cho, 2012](#)), mechanisms that are not lower invariant will be manipulable for these agents, even if they are swap monotonic and upper invariant. Arguably, lower invariance is the least intuitive axiom, but it turns out to be the missing link to characterize strategyproof mechanisms. In [Section 6](#), we will drop lower invariance for the characterization of partially strategyproof mechanisms.

Note that our axioms describe the behavior of the mechanism from each agent's perspective individually. This is sufficient, since we only consider strategyproofness-like concepts, not best-responses to the strategies of other agents.

5. An Axiomatic Characterization of Strategyproofness

In this section, we present our first main result, an axiomatic characterization of strategyproof one-sided matching mechanisms.

Theorem 1. *A mechanism f is strategyproof if and only if it is swap monotonic, upper invariant, and lower invariant.*

Proof outline (formal proof in [Appendix A.1](#)). Assuming strategyproofness, consider a swap of two adjacent objects in the report of some agent. Towards contradiction, assume that f

violates either upper or lower invariance, and construct a utility function such that the agent finds a beneficial manipulation to f . The key idea is to make the agent almost indifferent between the two objects that are swapped, such that utility gains on other objects can be sufficiently high to make the manipulation attractive. With upper and lower invariance established, swap monotonicity follows as well.

If f satisfies the axioms, we show that no swap of two adjacent objects can ever be a beneficial manipulation. Using a result from [Carroll \(2012\)](#), this *local* strategyproofness can be extended to *global* strategyproofness. \square

Theorem 1 illustrates why strategyproofness is such a strong requirement. If an agent swaps two adjacent objects in its reported preference order, the only thing that a strategyproof mechanism can do (if anything) is to increase the allocation for the object that is brought forward and decrease the allocation for the object that is brought back by the same amount.

6. An Axiomatic Characterization of Partial Strategyproofness

In the previous section, we have seen that swap monotonicity, upper invariance, and lower invariance are necessary and sufficient conditions for strategyproofness. Example 1 has shown that swap monotonicity and upper invariance are essential to guarantee at least truncation robustness and the absence of manipulations in a first order-stochastic dominance sense. Lower invariance is the least intuitive and the least important of the axioms for good incentives. Obviously, mechanisms that violate lower invariance are not strategyproof. However, we will show that they are still strategyproof for a large subset of the utility functions. This will lead to a relaxed notion of strategyproofness, which we call *partial strategyproofness*: we will show that swap monotonicity and upper invariance are equivalent to partial strategyproofness on the subset of utility functions with *uniformly relatively bounded indifference*. Example 2 provides the intuition for this new concept.

Example 2. Consider the Probabilistic Serial mechanism in a setting with 3 agents and 3 objects with unit capacity. The agents have types

$$t_1 : a > b > c, \quad t_2 : b > a > c, \quad t_3 : b > c > a,$$

and agents 2 and 3 report truthfully. When reporting t_1 truthfully, agent 1 receives a, b, c with probabilities $(3/4, 0, 1/4)$, respectively. If it reports $t'_1 : b > a > c$, it will receive allocation $(1/2, 1/3, 1/6)$ instead. Suppose agent 1 has utility 0 from object c . Whether or not the misreport t'_1 increases agent 1's expected utility depends on its relative value for a over b : if $u_1(a)$ is close to $u_1(b)$, then agent 1 will find it beneficial to report t'_1 . If $u_1(a)$ is significantly larger than $u_1(b)$, then agent 1 will want to report truthfully. Precisely, the manipulation is not beneficial if $(\frac{3}{4} - \frac{1}{2}) u_1(a) \geq (\frac{1}{3} - 0) u_1(b)$, i.e., if $\frac{3}{4} u_1(a) \geq u_1(b)$. We observe that the incentive to manipulate hinges on the “degree of indifference” agent 1 exhibits between objects a and b : the closer to indifference the agent is, the higher the incentive to misreport.

6.1. Uniformly Relatively Bounded Indifference (URBI(r))

Generalizing the idea from Example 2, we introduce the concept of *uniformly relatively bounded indifference*: loosely speaking, an agent must value any object at least a factor r less than the next better object (after appropriate normalization).

Definition 4. A utility function u satisfies uniformly relatively bounded indifference with respect to bound $r \in [0, 1]$ (URBI(r)) if for any objects a, b with $u(a) > u(b)$

$$r \cdot (u(a) - \min(u)) \geq u(b) - \min(u). \quad (6)$$

If $\min(u) = 0$, uniformly relatively bounded indifference has an intuitive interpretation, because (6) simplifies to $r \cdot u(a) \geq u(b)$: given a choice between two objects a and b , the agents must value b at least a factor r less than a ($\frac{3}{4}$ in Example 2).

For a geometric interpretation of URBI(r), consider Figure 1: the condition means that the agent’s utility function, represented by the vector u , cannot be arbitrarily close to the indifference hyperplane $H(t, t')$ between the types t and t' , i.e., it must lie within the shaded area of type t , and v would violate URBI(r). For convenience we introduce the following convention: for a given setting, we denote by URBI(r) the set of all utility functions that satisfy uniformly relatively bounded indifference with respect to r for the number of objects defined by the setting.

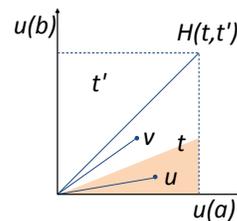


Figure 1: Geometric interpretation of uniformly relatively bounded indifference.

Remark 1. To gain some intuition about the “size” of the set URBI(r), consider a setting with $m = 3$ objects. Suppose $\min u = 0$ and the utilities for the first and second choice are determined by drawing a vector uniformly at random from $(0, 1)^2 \setminus H(t, t')$ (i.e., the open unit square excluding the indifference hyperplane). Then the share of utilities that satisfy URBI(r) is r , e.g., if $r = 0.4$, the probability of drawing a utility function from URBI(0.4) is 0.4. In Figure 1, this corresponds to the area of the shaded triangle over the area of the larger triangle formed by the x -axis, the diagonal, and the vertical dashed line on the right.

6.2. r -partial Strategyproofness

We now define a relaxed notion of strategyproofness. For some set A of utility functions, a mechanism f is A -*partially strategyproof* if for agents with utility functions in A it is a dominant strategy to report truthfully. In the following we will focus exclusively on r -partial strategyproofness. Therefore, we formally define:

Definition 5. Given a setting (N, M, \mathbf{q}) and a bound $r \in [0, 1]$, mechanism f is r -partially strategyproof in the setting (N, M, \mathbf{q}) if for any agent $i \in N$, any type profile $\mathbf{t} = (t_i, t_{-i}) \in T^n$, any misreport $t'_i \in T$, and any utility $u_i \in \text{URBI}(r) \cap t_i$, we have

$$\langle u_i, f(t_i) - f(t'_i) \rangle \geq 0. \quad (7)$$

When the setting is clear from the context, we simply write *r*-partially strategyproof or *r*-PSP without explicitly stating the setting. In Section 6.3, we will present our second main result, a characterization of *r*-PSP mechanisms by the axioms swap monotonicity and upper invariance. But first we explain the relation of *r*-PSP to the established concepts strategyproofness and weak strategyproofness.

6.2.1. Relation to Strategyproofness

Obviously, strategyproofness implies *r*-partial strategyproofness for any bound $r \in [0, 1]$ and any setting: for strategyproofness the incentive constraint (7) must hold for *all* possible utility functions. In contrast, *r*-partial strategyproofness requires (7) to hold only for *a subset* of the utility functions, namely $\text{URBI}(r)$. *r*-partial strategyproofness is equivalent to requiring that f is strategyproof for the restricted domain where agents' utility functions are uniformly relatively bounded away from indifference by a factor r . It inherits the strategic simplicity from strategyproofness, but with a caveat: a market designer can only give the honest advice that the agent need not deliberate about other agents' preferences and reports and that reporting truthfully is a dominant strategy *if* the agent's utility function lies within $\text{URBI}(r)$. But this advice is valid, even if other agents may have utilities outside $\text{URBI}(r)$.

6.2.2. Relation to Weak Strategyproofness

Weak strategyproofness is a relaxation of strategyproofness. It was employed by [Bogomolnaia and Moulin \(2001\)](#) to describe the incentive properties of the Probabilistic Serial mechanism. Under weakly strategyproof mechanisms, agents cannot attain a strictly first order-stochastically dominant outcome by manipulation.

Definition 6. *A mechanism is weakly strategyproof if for any type profile $\mathbf{t} = (t_i, t_{-i}) \in T^n$, the outcome from truthful reporting is never strictly first order-stochastically dominated by the outcome from any misreport for agent i .*

Weak strategyproofness is equivalent to requiring that for a given type profile $\mathbf{t} = (t_i, t_{-i})$ and a potential misreport t'_i , there exists a utility $u_i \in t_i$ such that $\langle u_i, f(t_i) - f(t'_i) \rangle \geq 0$. This turns out to be an extremely weak requirement: in particular, u_i can depend on t'_i . The mechanism might still offer a manipulation to the agent with utility u_i . The only guarantee given is that reporting t'_i will not increase its expected utility. To see just how weak the requirement is, consider Example 5 in Appendix A.2: even though the mechanism is weakly strategyproof, it is possible that an agent of a given type will find it beneficial to misreport, independent of its utility function. In contrast, *r*-partial strategyproofness provides an incentive guarantee for all agents with utilities in $\text{URBI}(r)$, and is therefore a strictly stronger condition, i.e., *r*-partial strategyproofness implies weak strategyproofness.

6.3. An Axiomatic Characterization of *r*-partial Strategyproofness

In this section, we show that dropping lower invariance as an axiom, but requiring swap monotonicity and upper invariance, leads to *r*-partially strategyproof mechanisms.

Theorem 2. *Given a setting (N, M, \mathbf{q}) , a mechanism f is r -partially strategyproof for some $r \in (0, 1]$ if and only if f is swap monotonic and upper invariant.*

Proof outline (formal proof in Appendix A.3). Suppose, an agent has true type $t : a_1 > \dots > a_K > a_{K+1} > \dots > a_m$ and is considering a misreport $t' : a_1 > \dots > a_K > a'_{K+1} > \dots > a'_m$, where only the positions of objects ranked below a_K are changed. We first show that under swap monotonicity and upper invariance, it suffices to consider misreports t' for which the allocation of a_{K+1} strictly decreases. The key insight comes from considering certain chains of swaps and their impact on the allocation (*canonical transitions* in Lemma 1). Then we show that for sufficiently small $r \in (0, 1]$, the decrease in expected utility that corresponds to the decrease in the allocation of a_{K+1} is sufficient to deter manipulation by any agent whose utility function satisfies $\text{URBI}(r)$, even though its allocation for less preferred objects a_{K+2}, \dots, a_m may improve. Finally, we show that a strictly positive r can be chosen uniformly for all type profiles and misreports. Thus, the bound r depends only on the mechanism and the setting.

To see the other direction, we assume towards contradiction that the mechanism is not upper invariant. For any $r \in (0, 1]$ we construct a utility function that satisfies $\text{URBI}(r)$, but for which the mechanism would be manipulable. The key idea is to make the agent almost indifferent between the two objects that are swapped, so that the value from attaining more of a better choice is sufficient to yield a beneficial manipulation. Finally, using upper invariance, swap monotonicity follows in a similar fashion. \square

Theorem 2 answers the first question raised in the introduction, because giving strategic advice to the agents is straightforward for r -partially strategyproof mechanisms: for any agent whose values for different objects differ by at least a factor r , it is a dominant strategy to report truthfully.

Remark 2. *For $0 < r < r' \leq 1$ we have $\text{URBI}(r) \subset \text{URBI}(r')$ by construction. Therefore, a mechanism that is r' -partially strategyproof will also be r -partially strategyproof. Furthermore, since the incentive constraint (7) is a weak inequality, the set of utilities for which a mechanism is partially strategyproof is topologically closed. Thus, there exists some maximal value $\rho \in (0, 1]$, for which the mechanism is ρ -partially strategyproof, but it is not r -partially strategyproof for any $r > \rho$.*

7. Maximality of the $\text{URBI}(r)$ Domain Restriction

Theorem 2 says that a mechanism f is swap monotonic and upper invariant if and only if it is r -partially strategyproof for *some* bound r . Despite this equivalence, this does not imply that the set of utility functions on which f is partially strategyproof is *exactly equal* to the set $\text{URBI}(r)$. Example 3 shows that we cannot hope for an exact equality: for some mechanism, the set of utilities where a mechanism is partially strategyproof may be strictly larger than any set $\text{URBI}(r)$ contained within.

However, in Theorem 3 we will show that the $\text{URBI}(r)$ domain restriction is maximal: consider a mechanism f that is r -partially strategyproof for some bound $r \in (0, 1)$, and hence swap monotonic and upper invariant. *Maximality* by Theorem 3 means that, unless we are

given additional structural information about f , $\text{URBI}(r)$ is in fact the largest set of utilities for which partial strategyproofness can be guaranteed.

Example 3. Consider a setting with 4 agents and 4 objects in unit capacity. In this setting, the adaptive Boston mechanism (Mennle and Seuken, 2014b) is r -partially strategyproof for any $r \leq \frac{1}{3}$, but not r -partially strategyproof for any $r > \frac{1}{3}$. However, an agent with utility function $\tilde{u} = (6, 2, 1, 0)$ will not find a beneficial manipulation for any report $t_- \in T^{n-1}$ from the other agents, i.e., ABM is $\{\tilde{u}\}$ -partially strategyproof in this setting. But $\tilde{u} \notin \text{URBI}(\frac{1}{3})$, since

$$\frac{\tilde{u}_3 - \min \tilde{u}}{\tilde{u}_2 - \min \tilde{u}} = \frac{1 - 0}{2 - 0} = \frac{1}{2} > \frac{1}{3}. \quad (8)$$

To verify this, we can compute $\rho_{(N, M, \mathbf{q})}(\text{ABM}) = \frac{1}{3}$, e.g., using Algorithm 2 in Section 9, and verify for any possible type profile $\mathbf{t} = (t, t_-) \in T^n$ and any misreport $t' \in T$ that the agent of type t with utility \tilde{u} will not find reporting t' beneficial.

We now show maximality of the $\text{URBI}(r)$ domain restriction.

Theorem 3. For any setting (N, M, \mathbf{q}) with $m \geq 3$, any bound $r \in (0, 1)$, and any utility function $\tilde{u} \in t$ that violates $\text{URBI}(r)$, there exists a mechanism \tilde{f} such that

- 1) \tilde{f} is r -partially strategyproof, but
- 2) there exists a type $t' \neq t$ and reports $t_- \in T^{n-1}$ such that

$$\langle \tilde{u}, \tilde{f}(t) - \tilde{f}(t') \rangle < 0. \quad (9)$$

Proof outline (formal proof in Appendix A.4). If \tilde{u} violates $\text{URBI}(r)$, there must be a pair of objects a, b such that

$$\frac{\tilde{u}(b) - \min \tilde{u}}{\tilde{u}(a) - \min \tilde{u}} = \tilde{r} > r. \quad (10)$$

We construct the mechanism \tilde{f} explicitly, considering a particular agent i . \tilde{f} allocates a constant vector to i , except when i reports type t' with $b > a$. In that case \tilde{f} allocates less of a , more of b , and less of i 's reported last choice (say, c) to i . Then \tilde{f} is swap monotonic and upper invariant. The re-allocation between a , b , and c must be constructed in such a way that i would want to manipulate if its utility is \tilde{u} , but would not want to manipulate if its utility satisfied $\text{URBI}(r)$. We show that this is possible. \square

Note that if some additional constraints are imposed on the space of possible mechanisms, the mechanism \tilde{f} constructed in the proof of Theorem 3 may no longer be feasible, such that the counterexample fails. However, as long as we know nothing more about the mechanism besides r -partial strategyproofness, we cannot rule out the possibility that an agent with some utility function outside $\text{URBI}(r)$ may want to manipulate. The following Corollary makes this argument precise.

Corollary 1. *For any setting (N, M, \mathbf{q}) with $m \geq 3$, we have*

$$\text{URBI}(r) = \bigcap_{f \text{ } r\text{-PSP in setting } (N, M, \mathbf{q})} \{u \mid f \text{ is } \{u\}\text{-PSP}\}. \quad (11)$$

This means that when considering the set of r -partially strategyproof mechanisms, the set of utilities for which they are *all* partially strategyproof is *exactly equal* to $\text{URBI}(r)$. Thus, there is no larger domain restriction for which all these mechanisms will also be strategyproof.

8. A New Measure for the Degree of Strategyproofness

Recall Theorem 2, which shows that a mechanism is r -partially strategyproof if and only if it is swap monotonic and upper invariant. This leads to a new, intuitive measure for the degree of strategyproofness of swap monotonic, upper invariant mechanisms: the largest possible relative indifference bound r for which the mechanism is still r -partially strategyproof.

Definition 7 (Degree of Strategyproofness). *Given a setting (N, M, \mathbf{q}) and a mechanism f that is upper invariant and swap monotonic, define the degree of strategyproofness (DOSP) of f by*

$$\rho_{(N, M, \mathbf{q})}(f) = \max \{r \in (0, 1] \mid f \text{ is } r\text{-PSP in the setting } (N, M, \mathbf{q})\}. \quad (12)$$

8.1. Interpretation of the Degree of Strategyproofness

By Remark 2, $\rho_{(N, M, \mathbf{q})}(f)$ is well-defined, and by Theorem 2 it is strictly positive. Maximality of the $\text{URBI}(r)$ domain restriction (Corollary 1) implies that when measuring the degree of strategyproofness of swap monotonic and upper invariant mechanisms using $\rho_{(N, M, \mathbf{q})}(f)$, no utility functions are omitted for which a guarantee could also be given.

DOSP also allows for the comparison of two mechanisms: $\rho_{(N, M, \mathbf{q})}(f) > \rho_{(N, M, \mathbf{q})}(g)$ means that f is partially strategyproof on a larger URBI domain restriction than g . And without any further information on the mechanisms, by Theorem 3, this comparison is the best that can be made for the sets of utility functions for which the mechanisms are partially strategyproof.

Remark 3. *From a quantitative perspective one might ask for “how many more” utility functions f is partially strategyproof compared to g . Recall Remark 1, where we considered $\text{URBI}(0.4)$ in a setting with 3 objects, $\min u = 0$, and the remaining utilities for the first and second choice were chosen uniformly at random for the unit square. Suppose that $\rho_{(N, M, \mathbf{q})}(f) = 0.8$ and $\rho_{(N, M, \mathbf{q})}(g) = 0.4$. Given this particular measure, the set $\text{URBI}(0.8)$ is twice the “size” of $\text{URBI}(0.4)$, i.e., the guarantee for f extends over twice as many utility functions as the guarantee for g . Thus, in some sense, f is “twice as strategyproof” as g .*

This answers the second question raised in the introduction as to how the degree of strategyproofness of a manipulable mechanism can be measured.

8.2. Relation of Degree of Strategyproofness to ‘‘Vulnerability to Manipulation’’

Pathak and Sönmez (2013) proposed an interesting method for comparing mechanisms by their vulnerability to manipulation. For the expected utility case their comparison states that g is *as intensely and strongly manipulable (ISM) as f* if whenever an agent with utility u finds a beneficial manipulation to f , the same agent in the same situation finds a manipulation for g that yields a weakly greater increase in expected utility. ISM and DOSP are *consistent* in the following sense:

Proposition 1. *For any setting (N, M, \mathbf{q}) ,*

- 1) *if g is as intensely and strongly manipulable as f , then $\rho_{(N,M,\mathbf{q})}(f) \geq \rho_{(N,M,\mathbf{q})}(g)$.*
- 2) *if $\rho_{(N,M,\mathbf{q})}(f) > \rho_{(N,M,\mathbf{q})}(g)$ and f and g are comparable by ISM, then g is as intensely and strongly manipulable as f .*

The proof is given in Appendix A.5. Despite this consistency, neither concept is always better at *strictly* differentiating mechanisms: ISM may be inconclusive when DOSP yields a strict winner, but DOSP may also indicate indifference (i.e., $\rho_{(N,M,\mathbf{q})}(f) = \rho_{(N,M,\mathbf{q})}(g)$) when one of the mechanisms is in fact intensely and strongly *more* manipulable.

An important difference between ISM and DOSP is that ISM considers each type profile separately, while the r -partial strategyproofness constraint must hold uniformly for *all* type profiles. Thus, ISM yields a best response notion while DOSP yields a dominant strategy notion of incentives. However, DOSP has two important advantages. First, Pathak and Sönmez (2013) do not present a method to perform the ISM comparison algorithmically, and the definition is such a method is not straightforward. In contrast, $\rho_{(N,M,\mathbf{q})}$ is computable, as we will show in Section 9. Second, and more importantly, DOSP is a parametric measure while ISM is not. A mechanism designer could easily express a minimal acceptable degree of strategyproofness and then consider only mechanisms satisfying this constraint, while this is not possible using ISM.

9. Computability of the Degree of Strategyproofness $\rho_{(N,M,\mathbf{q})}$

We now present an algorithm to determine whether a mechanism is r -partially strategyproof for given r . An algorithm to compute $\rho_{(N,M,\mathbf{q})}(f)$, i.e., the degree of strategyproofness, can also be derived from this procedure. Note that our main result is the *computability* of r -partial strategyproofness, *not* the development of *efficient* algorithms. Yet, we will briefly discuss the complexity of the algorithms and point out opportunities for improvement.

9.1. Computability Using Finite Constraint Sets

We first develop an equivalent condition for r -partial strategyproofness that does not involve uncountably many utility functions. To simplify notation, we mostly omit the arguments i, t_i, t_{-i} , and t'_i in the formulation of Proposition 2, even though constraint (13) is required to hold for any possible combination of these.

ALGORITHM 1: Verify r -partial strategyproofness

Input: setting (N, M, \mathbf{q}) , mechanism f , inverse bound $s = \frac{1}{r}$ **Variables:** agent i , type profile (t_i, t_{-i}) , type t'_i , vector δ , polynomial x , counter k , choice function ch **begin**

```
  for  $i \in N, (t_i, t_{-i}) \in T^n, t'_i \in T$  do
     $\forall j \in M : \delta_j \leftarrow f(t_i)(j) - f(t'_i)(j)$ 
     $x(s) \leftarrow \delta_{\text{ch}(t_i, 1)}$ 
    for  $k \in \{1, \dots, m-1\}$  do
      if  $x(s) < 0$  then
        return false
      end
       $x(s) \leftarrow x(s) \cdot s + \delta_{\text{ch}(t_i, k+1)}$ 
    end
  end
return true
end
```

Proposition 2. Given a setting (N, M, \mathbf{q}) and a mechanism f , for any agent $i \in N$, any type profile $\mathbf{t} = (t_i, t_{-i}) \in T^n$, any misreport $t'_i \in T$, and

- 1) for any object $j \in M$ let $\delta_j = f(t_i)(j) - f(t'_i)(j)$ be the change in the allocation of j to i as i changes its report between t_i and t'_i while the other agents report t_{-i} , and
- 2) for $k \in \{1, \dots, m-1\}$, define polynomials (in s) recursively by $x_1(s) = \delta_{\text{ch}(t_i, 1)}$ and $x_k(s) = s \cdot x_{k-1}(s) + \delta_{\text{ch}(t_i, k)}$, where $\text{ch}(t_i, k)$ is the k th choice of an agent of type t_i .

Then f is r -partially strategyproof if and only if for all agents $i \in N$, type profiles $\mathbf{t} = (t_i, t_{-i}) \in T^n$, misreports $t'_i \in T$, ranks $k \in \{1, \dots, m-1\}$, and $s = \frac{1}{r}$ we have

$$x_k(s) \geq 0. \tag{13}$$

Proof outline (formal proof in Appendix A.6). To see that r -partial strategyproofness implies (13), assume towards contradiction that (13) is violated for some set of arguments i, t_i, t_{-i}, t'_i , and \tilde{k} . Consider a utility function with relative utility differences exactly equal to r , except $\tilde{r} \cdot u(a_{\tilde{k}+1}) = u(a_{\tilde{k}})$ with $\tilde{r} < r$. This utility function satisfies URBI(r). However, exploiting the violation of (13), an agent with this utility will want to manipulate the mechanism if \tilde{r} is chose sufficiently small.

For the other direction, consider an agent with any utility function in URBI(r). Using Horner's method, we decompose the agent's incentive. If (13) holds, this incentive is positive, i.e., any agent with that utility function will not want to misreport. \square

Proposition 2 yields a method for verifying or falsifying that a given mechanism f is r -partially strategyproof in a given setting. Algorithm 1 implements this method. It iterates through all agents, type profiles, and possible misreports. For each combination it constructs

ALGORITHM 2: Compute $\rho_{(N,M,\mathbf{q})}(f)$

Input: setting (N, M, \mathbf{q}) , mechanism f (upper invariant, swap monotonic)

Variables: agent i , type profile (t_i, t_{-i}) , type t'_i , vector δ , reals ρ, s , polynomial x , counter k , choice function ch

```
begin
   $\rho \leftarrow 1$ 
  for  $i \in N, (t_i, t_{-i}) \in T^n, t'_i \in T$  do
     $\forall j \in M : \delta_j \leftarrow f(t_i)(j) - f(t'_i)(j)$ 
     $x(s) \leftarrow \delta_{\text{ch}(t_i, 1)}$ 
    for  $k \in \{1, \dots, m-1\}$  do
       $x(s) \leftarrow x(s) \cdot s + \delta_{\text{ch}(t_i, k+1)}$ 
       $\rho \leftarrow \min\left(\rho, (\max\{s \mid x(s) = 0\})^{-1}\right)$ 
    end
  end
  return  $\rho$ 
end
```

the vector δ of changes in the allocation and checks whether constraint (13) is violated. If this is ever the case, f is not r -partially strategyproof by Proposition 2. Otherwise, if the iterations finish without a violation, f is r -partially strategyproof.

We can also derive a method to compute the degree of strategyproofness, $\rho_{(N,M,\mathbf{q})}(f)$.

Corollary 2. *The degree of strategyproofness is given by*

$$\rho_{(N,M,\mathbf{q})}(f) = \max \left\{ r \in (0, 1] \mid \forall i \in N, (t_i, t_{-i}) \in T^n, t'_i \in T, k \in \{1, \dots, m-1\} : \right. \\ \left. x_k\left(\frac{1}{r}, i, t_i, t'_i, t_{-i}\right) \geq 0 \right\}. \quad (14)$$

Algorithm 2 implements this calculation. Initially, the guess for ρ is optimistically set to 1. Like Algorithm 1, the algorithm then iterates through all combinations of agents, type profiles, and possible misreports and finds the largest root s_0^{\max} of the polynomials $x_k(s)$ for each combination. If at any iteration the current ρ is higher than the inverse of the largest root $1/s_0^{\max}$, ρ is updated to $1/s_0^{\max}$. Thus, at termination, ρ is equal to the largest bound for which the constraints (13) are all satisfied.

9.2. Complexity

The computational complexity of Algorithm 1 is $O(m \cdot n \cdot (m!)^n \cdot O(f))$, i.e., exponential in the number of objects and agents. $O(f)$ is the computational complexity for determining a single outcome of the mechanism.¹ Computing $\rho_{(N,M,\mathbf{q})}(f)$ using Algorithm 2 has computational complexity $O(m \cdot n \cdot (m!)^n (O(f) + O(R)))$, where $O(R)$ is the complexity of finding the largest root of the polynomial. This root can be found in polynomial time via the LLL-algorithm (Lenstra, Lenstra and Lovasz, 1982).

¹Note that determining the probabilistic allocation of a mechanism may be computationally hard, even if implementing the mechanism is easy (e.g., see Aziz, Brandt and Brill (2013)).

Mechanism	Swap Monotonicity	Upper Invariance	Lower Invariance	Strategy-proofness	r -partial strategyproofness
RSD	✓	✓	✓	✓	✓
PS	✓	✓	✗	✗	✓
Naïve Boston mechanism	✗	✓	✗	✗	✗
Adaptive Boston mechanism	✓	✓	✗	✗	✓
Rank-Value mechanisms	✗	✗	✗	✗	✗
Hybrids of RSD & PS	✓	✓	✗	✗	✓
Hybrids of RSD & ABM	✓	✓	✗	✗	✓

Table 1: Comparison of mechanisms by axioms, strategyproofness, and r -partial strategyproofness.

9.2.1. Lower Bound for Complexity

In the most general case (without any additional structure), a mechanism is specified in terms of a set of allocation matrices $\{f(\mathbf{t}), \mathbf{t} \in T^n\}$. This set will contain $(m!)^n$ matrices of dimension $n \times m$. Consequently, the size of the problem is $S = (m!)^n \cdot n \cdot m$. In terms of S , Algorithm 1 has complexity $O(S \sqrt[n]{S})$. Thus, for the general case, there is not much room for improvement, since any correct and complete algorithm will have complexity $O(S)$ (we must consider all type profiles).

9.2.2. Improvements

If additional structure is available, faster algorithms may be possible: for anonymous mechanisms, the number of type profiles to consider reduces from $(m!)^n$ to $\binom{m! + n - 1}{n}$. This can be further reduced to $\binom{m! + n - 2}{n - 1}$, if the mechanism is neutral as well. These reductions apply to both algorithms. Of course, even with these reductions, the computational effort to run Algorithms 1 and 2 is prohibitively high for large settings. However, it is likely that more efficient algorithms exist for mechanisms with additional restrictions, and bounds may be derived analytically for interesting mechanisms, such as PS. Having shown computability, we leave questions regarding the design of efficient algorithms for specific mechanisms to future research.

10. r -partial Strategyproofness of Popular and New Mechanisms

We now apply our new r -partial strategyproofness concept to a number of popular and new mechanisms. Table 1 provides an overview how the different mechanisms fare on swap monotonicity, upper invariance, and lower invariance, and consequently on strategyproofness and r -partial strategyproofness.

10.1. Random Serial Dictatorship

Random Serial Dictatorship is strategyproof (see [Niederle, Roth and Sönmez \(2008\)](#)). Thus, it satisfies all three axioms and is 1-partially strategyproof for any setting.

10.2. Probabilistic Serial

Upper invariance of PS follows from Theorem 2 of [Hashimoto et al. \(2013\)](#). Proposition 3 yields swap monotonicity.

Proposition 3. *PS is swap monotonic.*

Proof outline (formal proof in Appendix A.7). We consider the times at which objects are exhausted under the Simultaneous Eating algorithm. Suppose an agent swaps two objects, e.g., from $a > b$ to $b > a$. If anything changes about that agent’s allocation, the agent will now spend *strictly more* time consuming b . By the time b is exhausted, there will be *strictly less of a available* or there will be *strictly more competition at a* (relative to reporting $a > b$). \square

Since PS is manipulable (see Example 2), it is not strategyproof, and hence by Theorem 1 it cannot be lower invariant in general. However, since it is swap monotonic and upper invariant, it is r -partially strategyproof by Theorem 2. This is a much stronger property than weak strategyproofness (from [Bogomolnaia and Moulin \(2001\)](#)). This generalizes earlier findings by [Balbuzanov \(2013\)](#), who shows that Probabilistic Serial is *convex strategyproof*.

[Kojima and Manea \(2010\)](#) have shown that for a fixed number of objects m and an agent i with a fixed utility function over these objects, i will not want to misreport if there are sufficiently many copies of each object. Note that this does not mean that the mechanism becomes strategyproof in some finite setting. However, we conjecture that the result of [Kojima and Manea \(2010\)](#) can be strengthened in the following sense: for m constant and $\min q_j \rightarrow \infty$: $\rho_{(N,M,q)}(\text{PS}) \rightarrow 1$. Numerical results presented in ([Mennle and Seuken, 2014a](#)) support this hypothesis, but we leave a proof of this hypothesis to future research.

10.3. “Naïve” Boston Mechanism

We consider the Boston mechanism with single tie-breaking and no priorities ([Miralles, 2009](#)). Intuitively, this mechanism is upper invariant, because the object to which an agent applies in the k th round has no effect on the applications or allocations in previous rounds (see ([Mennle and Seuken, 2014b](#)) for a formal proof). The Boston mechanism is, however, neither swap monotonic nor lower invariant, as Example 4 shows.

Example 4. *Consider the setting with $N = \{1, \dots, 4\}$, $M = \{a, b, c, d\}$, unit capacities, and the type profile*

$$\begin{aligned} t_1 & : a > b > c > d, \\ t_2 & : a > c > b > d, \\ t_3, t_4 & : b > c > a > d. \end{aligned}$$

Agent 1’s allocation is $(1/2, 0, 0, 1/2)$ for the objects a through d , respectively. If agent 1 swaps b and c in its report, the allocation will be $(1/2, 0, 1/4, 1/4)$. First, note that the allocation for b has not changed, but the overall allocation has, which violates swap monotonicity. Second, the allocation of d has changed, even though it is in the lower contour set of c , which violates lower invariance.

The Boston mechanism is “naïve”, since it lets agents apply at their second, third, etc. choices, even if these were already exhausted in previous rounds, such that agents “waste” rounds. Therefore, we will refer to it as the *naïve* Boston mechanism (*NBM*).

10.4. Adaptive Boston Mechanism

Obvious manipulation strategies arise from this naïve approach of NBM: an agent who knows that its second choice will already be exhausted in the first round is better off ranking its third choice second, because this will increase its chances at all remaining objects in a first order-stochastic dominance sense without forgoing any chances at its second choice object. If instead, the agent automatically “skipped” exhausted objects in the application process, this manipulation strategy would no longer be effective.

In (Mennle and Seuken, 2014b) we have shown that such an *adaptive* Boston mechanism (ABM) is swap monotonic and upper invariant, and thus r -partially strategyproof. Miralles (2009) used simulations to study how unsophisticated (truthful) agents are disadvantaged under NBM and finds evidence that such an adaptive correction may be attractive. Indeed, in Mennle and Seuken (2014b) we have also shown that ABM retains imperfect rank dominance over RSD in the limit.² This makes ABM an interesting alternative to the widely used NBM, as it is less manipulable than NBM, yet in a sense more efficient than RSD. Finally, since ABM is not strategyproof, it cannot be lower invariant, which completes the row in Table 1.

10.5. Rank Efficient Mechanisms

Featherstone (2011) introduced rank efficiency, a strict refinement of ex-post and ordinal efficiency. Rank efficient mechanisms are often considered in practical applications. However, no rank efficient mechanism is even weakly strategyproof (Theorem 3 in Featherstone (2011)). Furthermore, any rank efficient mechanism will be neither swap monotonic, nor upper invariant, nor lower invariant (Examples 6 and 7 in Appendix A.8), and thus not r -partially strategyproof for any bound $r \in (0, 1]$. This suggests that despite the attractive efficiency properties, manipulability must be a serious concern when considering rank efficient mechanisms.

10.6. Hybrid Mechanisms

In Mennle and Seuken (2014a), we show how hybrid mechanisms can facilitate the trade-off between strategyproofness and efficiency for one-sided matching mechanisms. The main idea is to consider convex combinations of two mechanisms, one of which has good incentives while

²A mechanism g *imperfectly rank dominates* another mechanism f if the resulting allocation from g is never rank dominated, but sometimes rank dominates the allocation from f . “Limit” here means for $n = m \rightarrow \infty$.

the other brings good efficiency properties. Under certain technical assumptions, these hybrid mechanisms are r -partially strategyproof, but can also improve efficiency beyond the ex-post efficiency of RSD. Furthermore, the trade-off is scalable in the sense that the mechanism designer can accept a lower degree of strategyproofness in exchange for more efficiency. The construction of hybrids can be shown to work with RSD and PS as well as with RSD and ABM. Note that prior to the introduction of r -partial strategyproofness, no measure existed to evaluate the degree of strategyproofness of such hybrid mechanisms.

11. Conclusion

In this paper, we have presented a new axiomatic approach to characterizing and relaxing strategyproofness of one-sided matching mechanisms in the vNM utility domain. First, we have shown that a mechanism is strategyproof if and only if it satisfies *swap monotonicity*, *upper invariance*, and *lower invariance*. This illustrates why strategyproofness is such a strong requirement: if an agent swaps two adjacent objects, e.g., from $a > b$ to $b > a$, in its reported preference order, the only thing that a strategyproof mechanism can do (if anything) is to increase the allocation of b and decrease the allocation of a by the same amount.

Second, we have shown that by dropping the least intuitive axiom, lower invariance, the class of r -partially strategyproof mechanisms emerges. These mechanisms are strategyproof for agents with sufficiently different values for different objects. We have also shown that the URBI(r) domain restriction is maximal. This implies that URBI(r) is the largest set of utility functions for which partial strategyproofness can be guaranteed without knowledge of further properties of the mechanism.

Finally, the characterization via uniformly relatively bounded utilities has allowed us to define a measure for the degree of strategyproofness of a mechanism. This measure is simple and consistent with the method of comparing mechanisms by their vulnerability to manipulation recently proposed by [Pathak and Sönmez \(2013\)](#). Furthermore, this measure is parametric, and we have shown that it is computable.

The r -partial strategyproofness concept can be applied to gain a better understanding of the incentives of many popular, non-strategyproof mechanisms. We have shown that the Probabilistic Serial mechanism is r -partially strategyproof, which is a significantly better description of the incentive properties than weak strategyproofness (from [\(Bogomolnaia and Moulin, 2001\)](#)) and gives insights into the incentives of PS for settings of any size (in contrast to large settings, as in [Kojima and Manea \(2010\)](#)). While the Boston mechanism in its naïve form is not even weakly strategyproof, an adaptive variant (ABM) is in fact r -partially strategyproof. Finally, r -partial strategyproofness can be used to measure the incentive properties of new hybrid mechanisms, as constructed in [\(Mennle and Seuken, 2014a\)](#), which enable a parametric trade-off between strategyproofness and efficiency.

Our new r -partial strategyproofness concept has an axiomatic motivation. It differs from prior approaches to relaxing strategyproofness in that it is parametric, computable, and applies not only in the limit, but in settings of any size. We believe this will lead to new insights in the analysis of existing non-strategyproof matching mechanisms and facilitate the design of new ones.

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A. Appendix

A.1. Proof of Theorem 1

Proof of Theorem 1. A mechanism f is strategyproof if and only if it is swap monotonic, upper invariant, and lower invariant.

SP \Rightarrow upper invariance & lower invariance & swap monotonicity First we show that a strategyproof mechanism must be upper invariant and lower invariant, then we use this to obtain swap monotonicity as well.

SP \Rightarrow upper invariance Suppose a mechanism f is strategyproof, but is not upper invariant. Then we can find $\mathbf{t} = (t_i, t_{-i}) \in T^n$ and $t'_i \in N_{t_i}$ such that

$$\begin{aligned} t_i &: a_1 > \dots > a_k > a_{k+1} > \dots > a_m \\ t'_i &: a_1 > \dots > a_{k+1} > a_k > \dots > a_m. \end{aligned}$$

By assumption there exists a smallest index $K < k$ such that

$$\delta := f_i(t'_i, t_{-i})(a_K) - f_i(t_i, t_{-i})(a_K) \neq 0. \quad (15)$$

a_K is the most preferred object for which the allocation changes. Without loss of generality $\delta > 0$, since otherwise we reverse the roles of t_i and t'_i . Suppose, i has a utility function

$$u_i = (C + (K - 1) \cdot c, \dots, C + c, C, (m - K) \cdot c, \dots, c), \quad (16)$$

where in particular $u_i(a_K) = C$ and $u_i(a_{K+1}) = (m - K) \cdot c$. From this misreport, i will gain some probability for a_K , but may lose all of its probability for a_{K+1} . Since the allocation is unchanged for the objects a_1, \dots, a_{K-1} , the change in utility is lower-bounded by

$$\delta \cdot C - (m - K) \cdot c, \quad (17)$$

which is positive for sufficiently large C and small $c > 0$. Thus, the mechanism would not be strategyproof, a contradiction.

SP \Rightarrow lower invariance. This is analogous to the previous argument for upper invariance: suppose a mechanism f is strategyproof, but not lower invariant. Similar to the previous case, we find the type profile $\mathbf{t} = (t_i, t_{-i}) \in T^n$ and $t'_i \in N_{t_i}$, such that there exists a largest index $K > k + 1$ with

$$f_i(t'_i, t_{-i})(a_K) - f_i(t_i, t_{-i})(a_K) =: \delta > 0. \quad (18)$$

Suppose, i has a utility function

$$u_i = (C + (K - 2) \cdot c, \dots, C + c, C, (m - K + 1) \cdot c, \dots, c), \quad (19)$$

where in particular $u_i(a_{K-1}) = C$ and $u_i(a_K) = (m - K + 1) \cdot c$. From this misreport, i will lose some probability for a_K , but will also gain probability for some object it prefers to a_K , since the allocation is unchanged for the a_{K+1}, \dots, a_m . Furthermore, all probability for objects a_1, \dots, a_{K-1} may be converted to probability just for a_{K-1} . The change in utility is lower-bounded by

$$\delta \cdot (C - (m - K + 1) \cdot c) - (K - 2) \cdot c, \quad (20)$$

which is positive for sufficiently large C and small $c > 0$. Thus, the mechanism would not be strategyproof, a contradiction.

SP \Rightarrow swap monotonicity Suppose a mechanism f is strategyproof, then we have already established that it is upper and lower invariant. Thus, when an agent swaps two consecutive objects in its preference ordering, the allocation can only change for these two objects. Suppose, the agent changes $a_k > a_{k+1}$ to $a_{k+1} > a_k$. If the allocation of a_k increases, the allocation of a_{k+1} must decrease by the same amount (otherwise the result is not a valid allocation). If the true preference order is $a_k > a_{k+1}$, then this swap is a beneficial manipulation, which contradicts strategyproofness.

Upper invariance & lower invariance & swap monotonicity \Rightarrow SP Consider any type profile $\mathbf{t} = (t_i, t_{-i}) \in T^n$ and some misreport $t'_i \in N_{t_i}$ by i in the neighborhood of the true

type t_i of i . From upper invariance, lower invariance, and swap monotonicity it follows that t'_i is not a beneficial manipulation: either the allocation remains unchanged, or i trades some probability at an object it prefers for probability at an object it likes less. Hence, the mechanism is not manipulable by swaps. By Proposition 1 from [Carroll \(2012\)](#) this implies strategyproofness. □

A.2. Example 5 from Section 6.2.2

Example 5. (Adapted from [Balbuzanov \(2013\)](#)) Suppose, an agent of type $t : a > b > c$ can either report truthfully or misreport as t' or t'' , where $t' : a > c > b$ and $t'' : b > a > c$. The resulting allocations are as follows:

$$f(t) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), f(t') = \left(\frac{5}{9}, 0, \frac{4}{9}\right), f(t'') = \left(\frac{2}{9}, \frac{7}{9}, 0\right). \quad (21)$$

Let $>_{SD}^{(t)}$ denote strict first order-stochastic dominance with respect to the preference order of type t . Then

$$f(t') \not>_{SD}^{(t)} f(t), f(t'') \not>_{SD}^{(t)} f(t), \quad (22)$$

$$f(t) \not>_{SD}^{(t')} f(t'), f(t'') \not>_{SD}^{(t')} f(t'), \quad (23)$$

$$f(t) \not>_{SD}^{(t'')} f(t''), f(t') \not>_{SD}^{(t'')} f(t''), \quad (24)$$

i.e., the allocations do not dominate each other for any type. In fact, if the agent had type t' or t'' , it would have a dominant strategy to report truthfully, i.e.,

$$f(t') >_{SD}^{(t')} f(t), f(t''), \quad f(t'') >_{SD}^{(t'')} f(t), f(t'). \quad (25)$$

However, if the agent reports t' and t'' with probability $\frac{1}{2}$ each, the allocation is

$$\frac{1}{2} (f(t') + f(t'')) = \left(\frac{3.5}{9}, \frac{3.5}{9}, \frac{2}{9}\right), \quad (26)$$

which strictly first order-stochastically dominates $f(t)$ for t . As a consequence, the agent would definitely want to manipulate the mechanism, but whether t' or t'' is the best misreport depends on the agent's utility function.

A.3. Proof of Theorem 2

Proof of Theorem 2. Given a setting (N, M, \mathbf{q}) , a mechanism f is r -partially strategyproof for some $r \in (0, 1]$ if and only if f is swap monotonic and upper invariant.

Throughout the proof, we fix a setting (N, M, \mathbf{q}) and use the abbreviated notation $f(t_i)$, for $f_i(t_i, t_{-i})$. Define

$$\epsilon = \min \left\{ |f(t_i)(j) - f(t_i^*)(j)| \mid \forall i \in N, (t_i, t_{-i}) \in T^m, t_i^* \in T, j \in M : |f(t_i)(j) - f(t_i^*)(j)| > 0 \right\}. \quad (27)$$

This is the smallest non-vanishing value by which the allocation of any object changes between two different types any agent could report. Since N, M, T are finite, ϵ must be strictly positive (otherwise f is constant).

Upper invariance & swap monotonicity $\Rightarrow r$ -PSP We must show that there exists $r \in (0, 1]$ such that no agent with utility in $\text{URBI}(r)$ can benefit from submitting a false report. Suppose, agent i of type t_i with

$$t_i : a_1 > \dots > a_K > b > c_1 > \dots c_L \quad (28)$$

is considering to misrepresent its type as t_i^* . Let b be the most preferred object for which the allocation changes, i.e., for all $k = 1, \dots, K$

$$f(t_i)(a_k) = f(t_i^*)(a_k), \quad (29)$$

$$f(t_i)(b) \neq f(t_i^*)(b). \quad (30)$$

Such an object must exist, because otherwise the allocations would be equal under both reports and t_i^* would not be a beneficial manipulation. Lemma 1 yields that the allocation for b weakly decreases. Since the allocation for b must change by assumption, a weak decrease implies a strict decrease. Thus, reporting t_i^* instead of t_i will necessarily decrease the probability of i getting b by at least ϵ . Non of the probabilities for the objects a_1, \dots, a_K are affected. Hence, in the best case (for the agent), all remaining probability is concentrated on c_1 . The maximum utility gain for i is upper bounded by $-\epsilon u_i(b) + u_i(c_1) - (1 - \epsilon) \min u$. The manipulation is guaranteed not to be beneficial if

$$u_i(c_1) - \min u_i - \epsilon(u_i(b) - \min u_i) < 0 \Leftrightarrow u_i(c_1) - \min u_i < \epsilon(u_i(b) - \min(u_i)). \quad (31)$$

This sufficient condition is satisfied by all utilities in $\text{URBI}(r)$ with the choice of $r = \epsilon$.

r -PSP \Rightarrow implies swap monotonicity & upper invariance

Upper invariance Suppose f is r -partially strategyproof for some fixed $r \in (0, 1]$, i.e., no agent with a utility function satisfying $\text{URBI}(r)$ can benefit from misrepresenting its type. We want to show that f is upper invariant. Suppose a type t with

$$t : \dots > a > b > \dots > c > d > \dots \quad (32)$$

Suppose further that a swap of c and d changes the allocation of some object ranked before c , and let a be the most preferred such object. Define ϵ as in (27), then without loss of generality the allocation of a increases by at least ϵ due to this swap (if it decreases, consider the reverse swap). This means that by swapping c and d , an agent of type t could gain at least probability ϵ for object a . Because a was the highest ranking object for which the allocation changed, the worst thing that can happen from the agent's perspective is that it loses all of its chances to get b and gets its last choice instead. Hence,

$$\epsilon u(a) - u(b) + (1 - \epsilon) \min u \quad (33)$$

is a lower bound for the benefit an agent of type t can have from swapping c and d in its report. The manipulation is guaranteed to be strictly beneficial if

$$\epsilon u(a) - u(b) + (1 - \epsilon) \min u > 0 \Leftrightarrow u(b) - \min u < \epsilon(u(a) - \min u). \quad (34)$$

But for any $r \in (0, 1]$, the set $\text{URBI}(r)$ will contain a utility function satisfying this condition. This is a contradiction to the assumption that no agent with a utility function in $\text{URBI}(r)$ will have a strictly beneficial manipulation. Consequently, f must be upper invariant.

Swap monotonicity Suppose f is r -partially strategyproof for some fixed $r \in (0, 1]$. We know already that f must be upper invariant. Towards contradiction, assume that upon a swap of two adjacent objects a and b (from $a > b$ to $b > a$) by a type t agent, the mechanism violates swap monotonicity. Say that

$$t : \dots > a > b > c > \dots > d > d' > \dots,$$

then one of the following holds:

1. the allocation of a increases,
2. or the allocation of a remains constant, and the allocation of b increases,
3. or the allocation of a remains constant, and the allocation of b decreases,
4. or the allocations of a and b remain constant, but its allocation changes for some object $d \neq a, b$,
5. or the allocations of both a and b decrease.

Because of upper invariance, we know that the allocation of objects ranking above a cannot be affected. Therefore, in case 1, the agent can gain at least ϵ probability of getting a , with ϵ defined as in (27). Then, the worst thing (for the agent) that could happen is that it loses all its chances of getting anything but its least preferred object. Hence,

$$\epsilon u(a) - u(b) + (1 - \epsilon) \min u \quad (35)$$

is a lower bound for the benefit the agent can have from swapping a and b . But as in the proof of upper invariance, this leads to a contradiction.

In case 2, the agent gains at least ϵ probability for b , but may lose shares in the next lower ranking object c . Again, the lower bound for the benefit is

$$\epsilon u(b) - u(c) + (1 - \epsilon) \min u \quad (36)$$

which leads to a contradiction. Note that if b is the lowest ranking object, this case is impossible.

Case 3 is symmetric to case 2, and we can consider the reverse swap instead.

In case 4, let d be the highest ranking object for which the allocation changes, which must lie after b because of upper invariance. Then without loss of generality, the

agent can increase its chances of getting d by at least ϵ , but potentially loses all chances for the next lower ranking object d' . This again leads to a contradiction.

For case 5, we consider the reverse swap, which is covered by case 1.

We have shown that none of the cases 1 through 5 can occur under a mechanism that is r -partially strategyproof. Therefore, the mechanism must satisfy strict swap monotonicity. □

Lemma 1. *Given a setting (N, M, \mathbf{q}) , a upper invariant and swap monotonic mechanism f (in this setting), and $(t_i, t_{-i}) \in T^n, t_i^* \in T$ such that (28) and (29) from the proof of Theorem 2 hold. Then the allocation for b must weakly decrease, i.e.,*

$$f(t_i)(b) > f(t_i^*)(b). \quad (37)$$

Proof. Consider two types t and t' . A *transition* from t to t' is a sequence of types $\tau(t, t') = (t^0, \dots, t^S)$ such that

- $t^0 = t$ and $t^S = t'$,
- $t^{k+1} \in N_{t_k}$ for all $k \in \{0, \dots, S-1\}$.

A transition can be interpreted as a sequence of swaps of adjacent objects that transform one type into another if applied in order. Suppose,

$$t' : a_1 > a_2 > \dots > a_m.$$

Then the *canonical transition* is the transition that results from starting at t and swapping a_1 (which may not be in first position for t) up until it is in first position. Then do the same for a_2 , until it is in second position, and so on, until t' is obtained.

To prove the Lemma, consider the first part of canonical transition from t_i^* to t_i : a_1 is swapped with its predecessors until it reaches its final position at the front of the ranking. With each swap the share of a_1 allocated to i can only increase or stay constant, because the mechanism is swap monotonic. On the other hand, once a_1 is at the front of the ranking, the allocation of a_1 will remain unchanged during the rest of the transition. This is because f is upper invariant, i.e., no change of order below the first position can affect the allocation of the first ranking object. But $f(t_i)(a_1) = f(t_i^*)(a_1)$, and hence non of the swaps involving a_1 will have any effect on the allocation of a_1 . But by swap monotonicity this means that non of the swaps will have any effect on the allocation at all. Next consider the second part of transition, where a_2 is brought into second position by swapping it upwards. The same argument applies to show that the overall allocation must remain unchanged. The same is true for a_3, \dots, a_K . Thus, we arrive at a type

$$t'_i : a_1 > \dots > a_K > c'_1 > \dots c'_{L'} > b > c'_{L'+1} > \dots > c'_L.$$

Under t'_i all of the a_k are in the same positions are for type t_i , b holds some position below its rank for type t_i , and some of the c_l are ranking above b (possibly in a different order). From the previous argument we know that

$$f(t'_i) = f(t_i^*). \quad (38)$$

As a consequence, without loss of generality, we can consider a misreport t_i^* , for which the order of the objects ranking above b under t_i (the a_k) remains unchanged. Assume towards contradiction that

$$f(t_i)(b) < f(t_i^*)(b). \quad (39)$$

By a similar argument as above, we can consider swapping b up to the position directly after a_K to get

$$t''_i : a_1 > \dots > a_K > b > c''_1 > \dots > c''_L,$$

which differs from t_i only beyond the position of b . Swap monotonicity yields that each swap will weakly increase the probability that i gets b . Then by assumption

$$f(t''_i)(b) \geq f(t'_i)(b) = f(t_i^*)(b) > f(t_i)(b). \quad (40)$$

However, upper invariance implies

$$f(t''_i)(b) = f(t_i)(b), \quad (41)$$

since the orderings of t_i and t''_i coincide up to and including the position of b . This is a contradiction, and hence the probability of i getting b must weakly decrease under t^* . \square

A.4. Proof of Theorem 3

Proof of Theorem 3. For any setting (N, M, \mathbf{q}) with $m \geq 3$, any bound $r \in (0, 1)$, and any utility function $\tilde{u} \in t$ that violates URBI(r), there exists a mechanism \tilde{f} such that

- 1) \tilde{f} is r -partially strategyproof, but
- 2) there exists a type $t' \neq t$ and reports $t_- \in T^{n-1}$ such that

$$\langle \tilde{u}, \tilde{f}(t) - \tilde{f}(t') \rangle < 0. \quad (42)$$

By assumption, \tilde{u} violates URBI(r). Thus, for some pair a, b of adjacent objects in the preference order corresponding to \tilde{u} we have

$$\frac{\tilde{u}(b) - \min \tilde{u}}{\tilde{u}(a) - \min \tilde{u}} = \tilde{r} > r. \quad (43)$$

Additionally, b is not the last choice of i , since the constraint $\frac{0}{\tilde{u}(a) - \min \tilde{u}} \leq r$ is trivially satisfied. We now need to define the mechanism \tilde{f} that offers a manipulation to an agent with utility \tilde{u} , but would not offer any manipulation to an agent whose utility satisfies URBI(r). For partial

strategyproofness, an agent should not have a beneficial manipulation for any set of reports from the other agents. Thus, it suffices to specify \tilde{f} for a single set of reports t_{-i} , where only agent i can vary its report. The allocation for i must then be specified for any possible report \hat{t}_i from i . In order to prove the theorem, this specification must be consistent with upper invariance and swap monotonicity.

We define $\tilde{f}_i(\cdot, t_{-i})$ as follows:

- For a report \hat{t}_i with $a > b$,

$$\tilde{f}_i(\hat{t}_i, t_{-i}) = \left(\frac{1}{m}, \dots, \frac{1}{m} \right). \quad (44)$$

- For a report \hat{t}_i with $b > a$, we adjust the original allocation by

$$\tilde{f}_i(\hat{t}_i, t_{-i})(a) = \frac{1}{m} + \delta_a, \quad (45)$$

$$\tilde{f}_i(\hat{t}_i, t_{-i})(b) = \frac{1}{m} + \delta_b, \quad (46)$$

$$\tilde{f}_i(\hat{t}_i, t_{-i})(d) = \frac{1}{m} + \delta_d, \quad (47)$$

where $\delta_a < 0$, $\delta_b > -\delta_a$, $\delta_d = -\delta_a - \delta_b < 0$. Here d denotes the last choice. In case $a = d$, both δ_a and δ_d are added. Note that if the last object changes, the allocation for the new last object is decreased (by adding δ_d), and the allocation of the previous last object is increased (by adding δ_d).

This mechanism is upper invariant: swapping the order of a and b induces a change in the allocation of a, b , and the last object d . Therefore no higher ranking object is affected. Swapping the last and the second to last object also only changes the allocation for these two object.

This mechanism is also swap monotonic: swapping a and b changes the allocation for both objects in the correct way, since $\delta_a < 0$, $\delta_b > 0$. Swapping the last to objects also changes the allocation appropriately, since $\delta_d < 0$. No other change of report changes the allocation.

Now we analyze the incentives for the different possible utility functions i could have:

Case $u_i = \tilde{u}$: In this case, the true preference order is $a > b$. Swapping a and b in its order is beneficial for i if

$$\delta_a \tilde{u}(a) + \delta_b \tilde{u}(b) + \delta_d \tilde{u}(d) = \delta_a (\tilde{u}(a) - \min \tilde{u}) + \delta_b (\tilde{u}(b) - \min \tilde{u}) > 0 \quad (48)$$

$$\Leftrightarrow \delta_a > -\delta_b \frac{\tilde{u}(b) - \min \tilde{u}}{\tilde{u}(a) - \min \tilde{u}}. \quad (49)$$

(49) is satisfied if

$$\delta_a > -\delta_b \cdot \tilde{r}, \quad (50)$$

since $\frac{\tilde{u}(b) - \min \tilde{u}}{\tilde{u}(a) - \min \tilde{u}} = \tilde{r}$ by construction.

Case $u_i \in \text{URBI}(r), a > b$: Swapping a and b should no longer be beneficial for i . This is the case if

$$\delta_a u_i(a) + \delta_b u_i(b) + \delta_d u_i(d) = \delta_a (u_i(a) - \min u_i) + \delta_b (u_i(b) - \min u_i) \leq 0 \quad (51)$$

$$\Leftrightarrow \delta_a \leq -\delta_b \frac{u_i(b) - \min u_i}{u_i(a) - \min u_i}. \quad (52)$$

(52) is satisfied if

$$\delta_a \leq -\cdot r, \quad (53)$$

since $\frac{u_i(b) - \min u_i}{u_i(a) - \min u_i} \leq r$ by construction.

Case $u_i \in \text{URBI}(r), b > a$: Swapping b and a to $a > b$ should not be beneficial for i . This is the case if

$$\delta_b u_i(b) + \delta_a u_i(a) + \delta_d u_i(d) = \delta_a (u_i(a) - \min u_i) + \delta_b (u_i(b) - \min u_i) \geq 0 \quad (54)$$

$$\Leftrightarrow \delta_a \geq -\delta_b \frac{u_i(b) - \min u_i}{u_i(a) - \min u_i}. \quad (55)$$

(55) is satisfied if

$$\delta_a \geq -\frac{\delta_b}{\tilde{r}}, \quad (56)$$

since $\frac{u_i(a) - \min u_i}{u_i(b) - \min u_i} \leq r$ for agents with $b > a$ by construction.

This means that if δ_a and δ_b satisfy (50), (53), and (56), the mechanism \tilde{f} is in fact what we are looking for. Given some $\delta_b > 0$, we can choose δ_a appropriately, since $r < 1$, $r < \tilde{r}$, and

$$-\delta_b \cdot \tilde{r} < -\delta_b \cdot r \Leftrightarrow r < \tilde{r}, \quad -\delta_b r > \frac{-\delta_b}{r} \Leftrightarrow r^2 < r. \quad (57)$$

Since i and t_{-i} could be chosen arbitrarily, anonymity is not a significant constraint. \square

A.5. Proof of Proposition 1

Proof of Proposition 1. For any setting (N, M, \mathbf{q}) ,

- 1) if g is as intensely and strongly manipulable as f , then $\rho_{(N, M, \mathbf{q})}(f) \geq \rho_{(N, M, \mathbf{q})}(g)$.
- 2) if $\rho_{(N, M, \mathbf{q})}(f) > \rho_{(N, M, \mathbf{q})}(g)$ and f and g are comparable by ISM, then g is as intensely and strongly manipulable as f .

To see 1), note that if f is as intensely and strongly manipulable as g , then any agent who can manipulate g also finds a manipulation to f . Thus, the set of utilities on which g is partially strategyproof cannot be larger than the set of utilities on which f is partially strategyproof. This in turn implies $\rho_{(N, M, \mathbf{q})}(f) \geq \rho_{(N, M, \mathbf{q})}(g)$.

For 2), observe that if $\rho_{(N, M, \mathbf{q})}(f) > \rho_{(N, M, \mathbf{q})}(g)$, then there exists a utility function \tilde{u} in $\text{URBI}(\rho_{(N, M, \mathbf{q})}(f))$, which is not in $\text{URBI}(\rho_{(N, M, \mathbf{q})}(g))$, and for which g is manipulable, but f is not. Thus, f cannot be as intensely and strongly manipulable as g , but the reverse is possible. \square

A.6. Proof of Proposition 2

Proof of Proposition 2. Given a setting (N, M, \mathbf{q}) and a mechanism f , for any agent $i \in N$, any type profile $\mathbf{t} = (t_i, t_{-i}) \in T^n$, any misreport $t'_i \in T$, and

- 1) for any object $j \in M$ let $\delta_j = f(t_i)(j) - f(t'_i)(j)$ be the change in the allocation of j to i as i changes its report between t_i and t'_i while the other agents report t_{-i} , and
- 2) for $k \in \{1, \dots, m-1\}$, define polynomials (in s) recursively by $x_1(s) = \delta_{\text{ch}(t_i, 1)}$ and $x_k(s) = s \cdot x_{k-1}(s) + \delta_{\text{ch}(t_i, k)}$, where $\text{ch}(t_i, k)$ is the k th choice of an agent of type t_i .

Then f is r -partially strategyproof if and only if for all agents $i \in N$, type profiles $\mathbf{t} = (t_i, t_{-i}) \in T^n$, misreports $t'_i \in T$, ranks $k \in \{1, \dots, m-1\}$, and $s = \frac{1}{r}$ we have

$$x_k(s) \geq 0. \quad (58)$$

r -PSP \Rightarrow (13) satisfied Let $s = \frac{1}{r}$. Assume towards contradiction that (13) is not satisfied for some $i \in N$, $(t_i, t_{-i}) \in T^n$, $t'_i \in T$, and $\tilde{k} \in \{1, \dots, m-1\}$, i.e.,

$$x_{\tilde{k}}(s, i, t_i, t'_i, t_{-i}) < 0. \quad (59)$$

Let $t_i : a_1 > \dots > a_m$, and consider the utility function u_i with

- $u_i(a_m) = 0, u_i(a_{m-1}) = 1$,
- $u_i(a_k) = u_i(a_{k+1}) \cdot s$ if $k \geq m-2, k \neq \tilde{k}$,
- $u_i(a_{\tilde{k}}) = u_i(a_{\tilde{k}+1}) \cdot s \cdot S$,

for some large S . Then u_i satisfies URBI(r). Thus, if f is r -partially strategyproof, i with utility function u_i should not find a beneficial manipulation.

The utility gain from reporting truthfully over reporting t'_i for i is

$$\langle u_i, f_i(t_i, t_{-i}) - f_i(t'_i, t_{-i}) \rangle \quad (60)$$

$$= \delta(i, t_i, t'_i, t_{-i})_{a_1} u_i(a_1) + \dots + \delta(i, t_i, t'_i, t_{-i})_{a_{\tilde{k}}} u_i(a_{\tilde{k}}) + \delta(i, t_i, t'_i, t_{-i})_{a_{\tilde{k}+1}} u_i(a_{\tilde{k}+1}) + \dots + \delta(i, t_i, t'_i, t_{-i})_{a_m} u_i(a_m) \quad (61)$$

$$= x_{\tilde{k}}(s, i, t_i, t'_i, t_{-i}) \cdot S \cdot s^{m-\tilde{k}-1} + \delta(i, t_i, t'_i, t_{-i})_{a_{\tilde{k}+1}} s^{m-\tilde{k}-2} + \dots + \delta(i, t_i, t'_i, t_{-i})_{a_m} \cdot 0 \quad (62)$$

$$\leq x_{\tilde{k}}(s, i, t_i, t'_i, t_{-i}) \cdot S \cdot s^{m-\tilde{k}-1} + s^{m-\tilde{k}-2} \cdot (m - \tilde{k}). \quad (63)$$

Since $x_{\tilde{k}}(s, i, t_i, t'_i, t_{-i}) < 0$ by assumption and S can be chosen arbitrarily large, (63) is negative. Thus, the incentive constraint is violated, and i finds a beneficial misreport t'_i , a contradiction.

(13) satisfied $\Rightarrow r$ -PSP Towards contradiction, assume that f is not r -partially strategyproof, i.e., there exist $i \in N$, $(t_i, t_{-i}) \in T^n$, $t'_i \in T$ and utility function $u_i \in \text{URBI}(r) \cap t_i$ such that

$$\langle u_i, f_i(t_i, t_{-i}) - f_i(t'_i, t_{-i}) \rangle < 0 \quad (64)$$

(without loss of generality assume $\min u_i = u_i(a_m) = 0$). We can re-write this term using Horner's method and get

$$0 > \langle u_i, f(t_i) - f(t'_i) \rangle \quad (65)$$

$$= \sum_{k=0}^m \delta_{a_k} u_i(a_k) \quad (66)$$

$$= \left(\left(\dots \left(\delta_{a_1} \frac{u_i(a_1)}{u_i(a_2)} + \delta_{a_2} \right) \frac{u_i(a_2)}{u_i(a_3)} + \dots \right) \frac{u_i(a_{m-2})}{u_i(a_{m-1})} + \delta_{m-1} \right) u_i(a_{m-1}), \quad (67)$$

dropping the arguments i, t_i, t'_i, t_{-i} for the sake of brevity. This term is definitely negative, but we can also find the smallest $K \in \{1, \dots, m-1\}$ for which

$$\left(\left(\dots \left(\delta_{a_1} \frac{u_i(a_1)}{u_i(a_2)} + \delta_{a_2} \right) \frac{u_i(a_2)}{u_i(a_3)} + \dots \right) \frac{u_i(a_{K-1})}{u_i(a_K)} + \delta_K \right) u_i(a_K) \quad (68)$$

is negative, and for all $k < K$,

$$\left(\left(\dots \left(\delta_{a_1} \frac{u_i(a_1)}{u_i(a_2)} + \delta_{a_2} \right) \frac{u_i(a_2)}{u_i(a_3)} + \dots \right) \frac{u_i(a_{k-1})}{u_i(a_k)} + \delta_k \right) u_i(a_k) \geq 0. \quad (69)$$

Thus, we can consecutively replace the $\frac{u_i(a_{k-1})}{u_i(a_k)}$ by s , since u_i satisfies URBI(r) and only make the term smaller, i.e.,

$$0 > \left(\left(\dots \left(\delta_{a_1} \frac{u_i(a_1)}{u_i(a_2)} + \delta_{a_2} \right) \frac{u_i(a_2)}{u_i(a_3)} + \dots \right) \frac{u_i(a_{K-1})}{u_i(a_K)} + \delta_K \right) u_i(a_K) \quad (70)$$

$$\geq \left(\left(\dots \left(\delta_{a_1} \frac{u_i(a_1)}{u_i(a_2)} + \delta_{a_2} \right) \frac{u_i(a_2)}{u_i(a_3)} + \dots \right) s + \delta_K \right) u_i(a_K) \quad (71)$$

$$\geq \dots \quad (72)$$

$$\geq \left(\left(\dots \left(\delta_{a_1} \frac{u_i(a_1)}{u_i(a_2)} + \delta_{a_2} \right) s + \dots \right) s + \delta_K \right) u_i(a_K) \quad (73)$$

$$\geq ((\dots (\delta_{a_1} s + \delta_{a_2}) s + \dots) s + \delta_K) u_i(a_K) \quad (74)$$

$$= x_K \cdot u_i(a_K). \quad (75)$$

But since $u_i(a_K) > 0$, it follows that $x_K = x_K(s, i, t_i, t'_i, t_{-i}) < 0$, and hence constraint (13) is not satisfied, a contradiction. □

A.7. Proof of Proposition 3

Proof of Proposition 3. PS is swap monotonic.

Suppose agent i is considering the following two reports that only differ by the ordering of x and y :

$$t_i : a_1 > \dots > a_K > x > y > b_1 > \dots > b_L,$$

$$t'_i : a_1 > \dots > a_K > y > x > b_1 > \dots > b_L.$$

The Probabilistic Serial mechanism is implemented via the Simultaneous Eating algorithm, objects are continuously consumed as time progresses. Let τ_j be the time when object j is exhausted under report t_i , and τ'_j the time when j is exhausted under report t'_i .

If $\tau_A = \max(\tau_{a_k}, k \leq K) \geq \min(\tau_x, \tau_y)$, the last of the objects a_k is exhausted only after the first of x and y is exhausted. By upper invariance, $\tau_A = \tau'_A$. This means that by the time i arrives at x (under report t_i) or at y under report t'_i , one of them is already exhausted. Thus, i will proceed directly to the respective other object. The consumption pattern does not differ between the two reports, i.e., the allocation does not change.

Now suppose that $\tau_A < \tau_y \leq \tau_x$. Then i received no shares of y under t_i . But under t'_i , it consumes shares of y from τ_A until $\tau'_y > \tau_A$. Thus, i 's share in y strictly increases. Furthermore, i consumed shares of x from τ_A until τ_x under report t_i . Under report t'_i , i arrives at x only later at $\tau'_y > \tau_A$. The same agents that consumed x under report t_i will also be consuming x under report t'_i and at the same times. In addition, there may be some agents who arrive together with i from y . Thus, under report t'_i agent i faces strictly more competition for weakly less capacity of x , implying that its share of x will strictly decrease. Note that if i faced no competition at y , it was the only agent at y , and thus consumes it until time 1. In this case the allocation will also decrease, because i arrived later under report t'_i .

Finally, suppose that $\tau_A < \tau_x < \tau_y$. Under report t'_i , agent i will arrive strictly earlier at y , i.e., the competing agents will be the same and arrive at the same times or later (if they arrived from x). Thus, the allocation for y will strictly increase under report t'_i . Furthermore, i might not receive any shares of x under report t'_i , a strict decrease. Otherwise, the argument why i receives strictly less shares of x under t'_i is the same as for the case “ $\tau_A < \tau_y \leq \tau_x$ ”. \square

A.8. Examples from Section 10.5

Example 6. Consider the setting $N = \{1, \dots, 4\}$, $M = \{a, b, c, d\}$, $q_j = 1$, and the type profile

$$\begin{aligned} t_1 & : a > d > c > b, \\ t_2 & : a > b > d > c, \\ t_3 & : b > c > d > a, \\ t_4 & : c > a > b > b. \end{aligned}$$

The unique rank efficient allocation is $d \rightarrow 1, a \rightarrow 2, b \rightarrow 3, c \rightarrow 4$. Suppose agent 1 changes its report to

$$t''_1 : a > c > b > d.$$

Now the only rank efficient allocation is $a \rightarrow 1, d \rightarrow 2, b \rightarrow 3, c \rightarrow 4$. The reports $>_1$ and $>''_1$ differ by two swaps: $d \leftrightarrow c$ and $d \leftrightarrow b$. Thus, at least one of these swaps must have increased the likelihood of getting object a for agent 1. This contradicts upper invariance. Also, under no report out of $>_1, >'_1: a > c > d > b, >''_1$ did agent 1 have any probability of getting objects b or c . Hence, the swap that changes the allocation involved a change of position of either object b or c , but the probability for each remained zero, a contradiction to swap monotonicity.

Example 7. Again, consider the setting $N = \{1, \dots, 5\}$, $M = \{a, b, c, d, e\}$, $q_j = 1$, and the type profile

$$\begin{aligned}
 t_1 & : a > c > b > d > e, \\
 t_2 & : c > b > a > d > e, \\
 t_3 & : c > a > b > e > d, \\
 t_4 & : a > c > b > e > d, \\
 t_5 & : e > a > b > c > d.
 \end{aligned}$$

The unique rank efficient allocation is $d \rightarrow 1, b \rightarrow 2, c \rightarrow 3, a \rightarrow 4, e \rightarrow 5$.

Agent 1 could change its report to

$$t_1'' : b > a > c > d > e,$$

in which case $b \rightarrow 1, d \rightarrow 2, c \rightarrow 3, a \rightarrow 4, e \rightarrow 5$ is the unique rank efficient allocation. Hence, either the swap $c \leftrightarrow b$ or the swap $a \leftrightarrow b$ changed the allocation for d , a contradiction to lower invariance.